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FAMILIES OF TRANSITIVE MAPS ON $\mathbb{R}$ WITH HORIZONTAL ASYMPTOTES

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Abstract. We will prove the existence of a class of transitive maps on the real line $\mathbb{R}$, with a discontinuity and horizontal asymptotes, whose set of periodic orbits is dense in $\mathbb{R}$; that is, a class of chaotic families. In addition, we will show a rare phenomenon: the existence of periodic orbits of period three prevents the existence of transitivity.

1. Introduction

The notion of transitivity is a fundamental tool in the study of maps with chaotic dynamics; see [19, 5, 3]. On boundaryless compact manifolds there is a well established theory about transitive diffeomorphisms; see [1, 15, 17, 18]. On compact intervals of $\mathbb{R}$ there exists a large class of examples and characterizations leading to such dynamic property; see [14, 16, 7] and references therein.

In the non-compact cases the situation is very different, since there is no known notion of hyperbolicity producing persistently chaotic dynamics and, regarding characterizations of transitivity, there are not many references. For example, in [9] it is proved that any Anosov diffeomorphism defined in the plane $\mathbb{R}^2$ onto $\mathbb{R}^2$ ($\mathbb{R}^2 \longrightarrow \mathbb{R}^2$) cannot be transitive; on the other hand, there are transitive diffeomorphisms of the plane $\mathbb{R}^2$ minus a line of discontinuities (see [4, 10]). In the case of non-bounded intervals the situation is similar to the case of non-compact manifolds; in this direction, in [12] it is shown that continuous transitive maps from $\mathbb{R}$ to $\mathbb{R}$ ($\mathbb{R} \longrightarrow \mathbb{R}$) must have infinite critical points; also in [13] there is a large class of examples of transitive maps on $\mathbb{R}$. In any case, a characterization for transitivity is not shown and, in the context of such articles, small perturbations of those maps lose the transitive property. Recently, [11] shows the existence of a class of maps, leading to a geometric model of the well known Boole transformation $T(x) = x - \frac{1}{x}$ (see [2]), where a characterization for transitivity is possible; also, it is shown that, in the space of continuously differentiable maps (from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R}$) the maps belonging to such geometric model are persistently transitive, with respect to the C^1 uniform topology. Such geometric model consists of maps with

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domain $\mathbb{R} \setminus \{0\}$ onto $\mathbb{R}$, increasing on each connected component of $\mathbb{R} \setminus \{0\}$ and non-bounded on $\mathbb{R} \setminus K$, where K is a compact interval with $0 \notin K$.

In the attempt to extend the study of this type of maps and expand the space of transitive maps over unbounded intervals, we ask ourselves: What is the dynamics of the maps bounded in $\mathbb{R} \setminus K$, where K is a compact interval? Have these maps a non-empty intersection with the world of chaotic dynamics? The purpose of this article is to answer such questions positively, in addition to expanding the geometric model of the Boole maps, treated in [11], to a class with horizontal asymptotes; also to show a couple of characterizations of transitivity. Specifically, let $f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ be a continuous map; we say that f *is transitive* if there is a point $x \in \mathbb{R} \setminus \{0\}$ such that its positive orbit (with respect to f), that is $O_f^+(x) = \{x, f(x), f^2(x), \dots\}$, is dense in $\mathbb{R}$.

Definition 1. *A continuous function $f : \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ is an* increasing alternating system with asimptotes relative to its first pre-image *if:*

(1a) *f is strictly increasing in $(-\infty, 0)$ and $(0, +\infty)$.*
(1b) *$\lim_{x\to 0^+} f(x) = -\infty$ and $\lim_{x\to 0^-} f(x) = +\infty$.*
(1c) *There exist $x_0 < 0$ and $x_1 > 0$ such that $f^{-1}(0) = \{x_0, x_1\}$, $\lim_{x\to+\infty} f(x) = x_1$ and $\lim_{x\to-\infty} f(x) = x_0$.*
(1d) *$f(x) \neq x$ for all $x \in \mathbb{R} \setminus \{0\}$.*

If $f : \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ is as in Definition 1, we say that f is a SACAH.

Theorem 2 (**Main Theorem**). *Let $f : \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ be a SACAH. Then f is transitive if and only if $\bigcup_{n\geq 0} f^{-n}(0)$ is dense in $\mathbb{R}$.*

Corollary 3. *Let $f : \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ be a SACAH. If $\bigcup_{n\geq 0} f^{-n}(0)$ is dense in $\mathbb{R}$, then the set of periodic orbits of f is dense in $\mathbb{R} \setminus \{0\}$.*

In contrast to Corollary 3, it is important to mention that in this work we will show a particularity of this kind of family: The existence of a periodic period 3 orbit prevents the existence of transitivity.

Corollary 4. *The map $B : \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ defined by $B(x) = \dfrac{|x| - 1}{x}$ is a transitive SACAH. Curiously, $\cup_{n\geq 0} B^{-n}(0) = \mathbb{Q}$.*

It is important to mention that the geometric model of the Boole transformation, called expansive increasing alternating systems, forms a set of transitive maps with non-empty interior (see [11]); the alternating systems with asymptotes at the diagonal line $y = x$ and those with horizontal asymptotes (like the models shown in this paper) are border elements of the set of transitive maps of $\mathbb{R}$ (maps with a discontinuity). Another important point is that the type of maps shown in this article and those studied in [11] appear naturally as projections along invariant foliations (or leaves) in the study of transitive diffeomorphisms of the plane $\mathbb{R}^2$ minus a curve of discontinuities, and there is a relation between the plane dynamics and the projected one (see [10] and recently [8]). These two latest works are inspired by the work of Devaney [4], who shows an example of a transitive diffeomorphism of the plane $\mathbb{R}^2$ minus a line of discontinuity. Devaney's example was introduced by Hénon in [6] as a system associated with the movement of the three restricted bodies of classical mechanics.

The article is organized in the following manner: In the first section, of notations and basic results, we grouped the points of the pre-images from zero to generate a partition, which will be very useful in proving our Main Theorem. In addition, we will show some properties with respect to the generated partition. Then, in the next section we will concentrate on the proof of the Main Theorem, together with the two corollaries enunciated above.

2. Notations and basic results

In this section f denotes a SACAH. Let $x_0 < 0$ and $x_1 > 0$ as in Definition 1. On the other hand, $\mathbb{R}\setminus\{0\}$ is the union of two connected components which we will denote by $\mathbb{R}_0 = (-\infty, 0)$ and $\mathbb{R}_1 = (0, +\infty)$. Denote by $f_0 = f|_{\mathbb{R}_0}$ and $f_1 = f|_{\mathbb{R}_1}$, where $f_0(x) = f(x)$ for $x \in \mathbb{R}_0$ and $f_1(x) = f(x)$ for $x \in \mathbb{R}_1$. Since $f^{-1}(0) \neq \emptyset$, consider the following notation:

$$A_n = \bigcup_{j=1}^{n} f^{-j}(0) \cup \{0\}, \quad \text{for all } n \geq 1 \qquad \text{and} \qquad A_f = \bigcup_{n\geq 1} f^{-n}(0) \cup \{0\}.$$

From Definition 1 we obtain:

Remark 1.

(a) $f(0, x_1) = \mathbb{R}_0$ and $f(x_0, 0) = \mathbb{R}_1$;
(b) $f(\mathbb{R}_0) = (x_0, +\infty)$ and $f(\mathbb{R}_1) = (-\infty, x_1)$;
(c) $f^2(x_1, +\infty) = \mathbb{R}_0$ and $f^2(-\infty, x_0) = \mathbb{R}_1$.

This remark tells us that for every point $x \in \mathbb{R}\setminus A_f$, the orbit of x with respect to f visits each connected component $\mathbb{R}_0$ and $\mathbb{R}_1$ infinitely many times. The following definition is of great importance in the characterization of transitivity.

Definition 5. *Let $f : \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ be a SACAH. f is expansive if for each $x, y \in \mathbb{R}\setminus A_f$, there exists $N > 1$ such that $f^N(x).f^N(y) < 0$.*

Next we show that the existence of a periodic orbit of period 3, for this type of transformations, prevents the existence of transitivity and expansiveness.

Lemma 6. *Let $f : \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ be a SACAH. If there exists $\overline{x}$ a periodic point of period 3, then f is neither expansive nor transitive.*

Proof. Let $\overline{x}$ be a periodic point of period 3. Denote the orbit of $\overline{x}$ by $\overline{x} = \overline{x}_0$, $f(\overline{x}) = \overline{x}_1$ and $f^2(\overline{x}) = \overline{x}_2$. From Remark 1 $\overline{x}_i \in (-\infty, x_0)$ or $\overline{x}_i \in (x_1, +\infty)$, for some $i \in \{0, 1, 2\}$. If $\overline{x}_i \in (-\infty, x_0)$, then $B = (-\infty, \overline{x}_i] \cup (x_0, f(\overline{x}_i)] \cup (0, f^2(\overline{x}_i)]$ is f-invariant, that is, $f(B) \subset B$. On the other hand, if $\overline{x}_i \in (x_1, +\infty)$, then we have that $B = [\overline{x}_i, +\infty) \cup [f(\overline{x}_i), x_1) \cup [f^2(\overline{x}_i, 0)$ is f-invariant. In any case f is neither expansive nor transitive. □

In the proof of the Main Theorem we will need to make use of the symbolic dynamics. For it, consider the following space of sequences:

$$\Sigma_2 = \{a = (a_0, a_1, \dots) : a_j \in \{0,1\} \text{ and } j \geq 0\},$$

with the usual topology induced by the metric $d_2(a,b) = \sum_{n=0}^{+\infty} \frac{|a_n - b_n|}{2^n}$, for $a, b \in \Sigma_2$. Let us consider the subset

$$\Sigma(2,2) = \{a \in \Sigma_2 : (1,1,1) \notin a \text{ and } (0,0,0) \notin a\},$$

where $(1,1,1) \notin a$ and $(0,0,0) \notin a$ mean that $(1,1,1) \neq (a_j, a_{j+1}, a_{j+2})$ and $(0,0,0) \neq (a_j, a_{j+1}, a_{j+2})$, for all $j \geq 0$, respectively. It is well known that Σ_2 and $\Sigma(2,2)$ are compact and the shift $\sigma : \Sigma_2 \to \Sigma_2$ defined by $\sigma(a_0, a_1, a_2, \ldots) = (a_1, a_2, a_3, \ldots)$ is continuous and transitive. Also, $\Sigma(2,2)$ is invariant with respect to σ and σ is transitive restricted to $\Sigma(2,2)$. Let us denote by $\mathrm{Per}(\sigma, 3)$ the set of all periodic orbits of period 3 belonging to σ. With this notation in mind consider the set

$$\Sigma^*(2,2) = \{a \in \Sigma(2,2) : \sigma^j(a) \notin \mathrm{Per}(\sigma,3), \text{ for all } j \geq 0\}.$$

Observe that $\Sigma^*(2,2)$ is not compact and $\Sigma^*(2,2)$ is invariant with respect to σ. Also, σ restricted to $\Sigma^*(2,2)$ is a transitive function; denote this restriction by σ^*.

Definition 7. *Let K be a finite set of $\mathbb{R}$. $\mathbb{P}_K$ shall denote a partition of $\mathbb{R}$ generated by K, if its elements are intervals of $\mathbb{R}$ satisfying:*

(a) *$\partial I \subset K$ and $I \cap K = \emptyset$ for all $I \in \mathbb{P}_K$.*

(b) *$\bigcup_{I \in \mathbb{P}_K} I = \mathbb{R} \setminus K$;*

(c) *If $I, J \in \mathbb{P}_K$ with $I \neq J$, then $I \cap J = \emptyset$.*

$\mathbb{P}_K$ is the set of all the connected components of $\mathbb{R} \setminus K$. Consider the following lemma of set theory.

Lemma 8. *Let $\{K_n\}$, $n \geq 1$, be a sequence of finite sets of $\mathbb{R}$ such that $K_n \subset K_{n+1}$ and $\bigcup_{n\geq 1} K_n$ is dense in $\mathbb{R}$. If for each sequence I_n of bounded atoms of $\mathbb{P}_{K_n}$, $I_n \subset \mathbb{P}_{K_n}$, $I_{n+1} \subset I_n$, then* $\mathrm{diam}(I_n) \to 0$.

Observe that if $A \subset B$, then for each $J \in \mathbb{P}_B$ there is a single $I \in \mathbb{P}_A$ such that $J \subset I$.

Lemma 9. *Let $n > 1$ and $\mathbb{P}_{A_n}$ be the partition generated by A_n. If I is an atom of $\mathbb{P}_{A_n}$, then $f^n(I) \in \{\mathbb{R}_0, (x_0, 0), (0, x_1), \mathbb{R}_1\}$.*

Proof. By induction. The case $n = 1$ is clear from Remark 1 and the fact that

$$\mathbb{P}_{A_1} = \{\mathbb{R}_0, (x_0, 0), (0, x_1), \mathbb{R}_1\}.$$

Let us suppose that the lemma is true for $n \geq 1$. Let $J \in \mathbb{P}_{A_{n+1}}$; since $A_n \subset A_{n+1}$, there exists $I \in \mathbb{P}_{A_n}$ such that $J \subset I$. By the inductive hypothesis we have four options. The first: If $f^n(I) = \mathbb{R}_0$, then from Remark 1, $f^{n+1}(I) = (x_0, +\infty)$; this means that there exists $y \in I$ such that $f^{n+1}(y) = 0$, that is, from Definition 7, there exist J_1 and J_2 in $\mathbb{P}_{A_{n+1}}$ such that $J_1 \cup J_2 \cup \{y\} = I$, $\{y\} = \partial J_1 \cap \partial J_2$, $f^{n+1}(J_1) = (x_0, 0)$ and $f^{n+1}(J_2) = \mathbb{R}_1$. From this together with Definition 7 it follows that $J = I_1$ or $J = J_2$; in any case we have that $f^{n+1}(J) \in \{\mathbb{R}_0, (x_0, 0), (0, x_1), \mathbb{R}_1\}$. The second option: If $f^n(I) = (x_0, 0)$, then $f^{n+1}(I) = \mathbb{R}_1$, therefore $I \cap A_{n+1} = \emptyset$, so by Definition 7, $I \in \mathbb{P}_{A_{n+1}}$ and consequently $J = I$. This shows that $f^{n+1}(J) = (x_0, 0)$. The other two options are similar. So $f^{n+1}(J) \in \{\mathbb{R}_0, (x_0, 0), (0, x_1), \mathbb{R}_1\}$. □

Lemma 10. *For each $n > 0$, we have that* $f^{-n}(\mathbb{R}_0) \cup f^{-n}(\mathbb{R}_1) = \bigcup_{I \in \mathbb{P}_{A_n}} I.$

Proof. By induction. For $n = 1$ the equality is clear from Remark 1. Let us suppose that the lemma is true for $n \geq 1$. First, using the inductive hypothesis, note that

$$f^{-n-1}(\mathbb{R}_0) \cup f^{-n-1}(\mathbb{R}_1) = f^{-n-1}(\mathbb{R} \setminus \{0\}) = \bigcup_{I \in \mathbb{P}_{A_n}} f^{-1}(I).$$

Let $x \in f^{-n-1}(\mathbb{R}_0) \cup f^{-n-1}(\mathbb{R}_1)$; then $x \in \bigcup_{I \in \mathbb{P}_{A_n}} f^{-1}(I)$, that is, there exists $I \in \mathbb{P}_{A_n}$ such that $x \in f^{-1}(I)$; since $I \cap A_n = \emptyset$, it follows that $x \in \mathbb{R} \setminus A_{n+1}$; therefore, $x \in \bigcup_{I \in \mathbb{P}_{A_{n+1}}} I$ and consequently

$$f^{-n-1}(\mathbb{R}_0) \cup f^{-n-1}(\mathbb{R}_1) \subset \bigcup_{I \in \mathbb{P}_{A_{n+1}}} I.$$

Let $y \in \bigcup_{I \in \mathbb{P}_{A_{n+1}}} I$, then there exists $J \in \mathbb{P}_{A_{n+1}}$ such that $y \in J$. So, $f(y) \cap A_n = \emptyset$, and therefore there exists $I \in \mathbb{P}_{A_a}$ such that $f(y) \in I$. Consequently $y \in \bigcup_{I \in \mathbb{P}_{A_n}} f^{-1}(I)$, which completes the proof. □

Lemma 11. *Let f be a SACAH. f is expansive if and only if* $A_f = \bigcup_{n=1}^{\infty} A_n \cup \{0\}$ *is dense in* $\mathbb{R} \setminus \{0\}$.

Proof. ($\Rightarrow$) Suppose that f is expansive, and that $A_f = \bigcup_{n=1}^{\infty} A_n \cup \{0\}$ is not dense in $\mathbb{R} \setminus \{0\}$. Then, there is an interval J such that $J \cup f^{-n}(0) = \emptyset$ for all $n \geq 0$; thus, $f^n(J) \subset \mathbb{R}_0$ or $f^n(J) \subset \mathbb{R}_1$ for all $n \geq 0$. Let $x \neq y$ be in J, then since f is expansive there exists $N > 0$ such that $f^N(x) \cdot f^N(y) < 0$; from this, $f^N(J) \cap \mathbb{R}_0 \neq \emptyset$ and $f^N(J) \cap \mathbb{R}_1 \neq \emptyset$. This is a contradiction.

($\Leftarrow$) Suppose that $A_f = \bigcup_{n=1}^{\infty} A_n \cup \{0\}$ is dense in $\mathbb{R} \setminus \{0\}$. Let $a \neq b$ be in $\mathbb{R}_f$ and consider, without loss of generality, that $a < b$. Since A_f is dense there exists $x \in A_f$ such that $x \in (a, b)$. Let $N \leq 1$ be the minimum such that $f^{-N}(0) \cap (a, b) \neq \emptyset$. Then, $f^N(a) \cdot f^N(b) < 0$. □

3. Proof of the main results

Main Theorem. ($\Rightarrow$) Suppose that f is expansive. Then, from Lemma 11, $\mathbb{R}_f$ is totally disconnected and $f(\mathbb{R}_f) = \mathbb{R}_f$. Consider the function $h : \mathbb{R}_f \longrightarrow \Sigma^*(2, 2)$

defined by $h(x) = a$, where

$$a_n = \begin{cases} 0 & \text{if } f^n(x) < 0, \\ 1 & \text{if } f^n(x) > 0. \end{cases}$$

Note that from Remark 1, h is well defined. Our objective is to show that h is a homeomorphism and that $\sigma^* \circ h = h \circ f$, that is, h is a topological conjugation between f and σ^*. That means f is transitive since σ^* is, so the proof follows.

h is one to one. Indeed, let $x \neq y$ in $\mathbb{R}_f$ and let us denote $a = h(x)$ and $b = h(y)$. Since f is expansive, there exists $N > 1$ such that $f^N(x) \cdot f^N(y) < 0$, therefore $a_N \neq b_N$ and so $a \neq b$.

h is continuous. Indeed, let $x \in \mathbb{R}_f$ and let us denote $a = h(x)$. Let $\epsilon > 0$, then there exists $M > 1$ such that, if $b \in \Sigma^*(2,2)$ with $a_j = b_j$ for all $0 \leq j \leq M$, then $d_2(a,b) < \epsilon$. On the other hand, there exists $I \in \mathbb{P}_{A_{M+1}}$ such that $x \in I$. From this follows that $f^j(I) \cap \{0\} = \emptyset$ for all $0 \leq j \leq M+1$, then from definition of h, $f^j(I) \subset \mathbb{R}_{a_j}$ for all $0 \leq j \leq M$. Now, let $\delta = d(x, \partial I)$; then for $y \in (x-\delta, x+\delta) \cap \mathbb{R}_f \subset I$ it follows that $f^j(y) \in \mathbb{R}_{a_j}$ for all $0 \leq j \leq M$. Then, calling $b = h(y)$ we conclude that $d_2(h(x), h(y)) < \epsilon$.

h is onto. Indeed, let $a = (a_0, a_1, a_2, \dots) \in \Sigma^*(2,2)$ and remember that $\mathbb{R}_0 = (-\infty, 0)$, $\mathbb{R}_1 = (0, +\infty)$. Consider the sets

$$\begin{aligned} H_0 &= \mathbb{R}_{a_0}; \\ H_1 &= \mathbb{R}_{a_0} \cap f^{-1}(\mathbb{R}_{a_1}); \\ &\;\vdots \\ H_n &= \mathbb{R}_{a_0} \cap f^{-1}(\mathbb{R}_{a_1}) \cap \cdots \cap f^{-n}(\mathbb{R}_{a_n}), \quad \text{for all } n \geq 1. \end{aligned}$$

Observe that if $x \in H_n$, then $f^j(x) \in \mathbb{R}_{a_j}$ for all $0 \leq j \leq n$. To continue with the proof, we need to prove the following claims.

Claim 1: $H_n \neq \emptyset$ and $H_n \in \mathbb{P}_{A_n}$, for all $n \geq 1$.

Proof. By induction. Let us show that for $n = 1$ the result is true. Note that $H_0 = \mathbb{R}_0$ or $\mathbb{R}_1$ and $f^{-1}(0) = \{x_0, x_1\}$. Also, $f^{-1}(\mathbb{R}_0) = (-\infty, x_0) \cup (0, x_1)$ and $f^{-1}(\mathbb{R}_1) = (x_0, 0) \cup (x_1, +\infty)$. Then, intersecting it follows that $H_1 \neq \emptyset$ and $H_1 \in \mathbb{P}_{A_1}$.

Suppose that $H_n \neq \emptyset$ and $H_n \in \mathbb{P}_{A_n}$ for $n \geq 1$. Suppose that $H_{n+1} = \emptyset$, that is, $H_n \cap f^{-n-1}(\mathbb{R}_0) = \emptyset$, and suppose also that $a_{n+1} = 0$. From the inductive hypothesis and Lemma 9 it follows that $f^n(H_n) \in \{\mathbb{R}_0, (x_0, 0), (0, x_1), \mathbb{R}_1\}$ and since $H_{n+1} = \emptyset$ then $f^n(H_n) = (x_0, 0)$. So, $a_n = 0$. Now, since $H_{n-1} \neq \emptyset$ and from Lemma 9, $f^{n-1}(H_{n-1}) \in \{\mathbb{R}_0, (x_0, 0), (0, x_1), \mathbb{R}_1\}$. From Remark 1 it follows that $f^{n-1}(H_{n-1}) = \mathbb{R}_0$ or $f^{n-1}(H_{n-1}) = (x_0, 0)$. This proves that $a_{n-1} = 0$. Thus, $(a_{n-1}, a_n, a_{n+1}) = (0,0,0)$ but this is a contradiction with the fact $a \in \Sigma^*(2,2)$. In the case $a_{n+1} = 1$ the proof is similar obtaining $(a_{n-1}, a_n, a_{n+1}) = (1,1,1)$, which is a contradiction for the same reason as before. Consequently, $H_{n+1} \neq \emptyset$.

It remains to prove that $H_{n+1} \in \mathbb{P}_{A_{n+1}}$, indeed if $f^{-n-1}(0) \cap H_n = \emptyset$, then $H_n \in \mathbb{P}_{A_{n+1}}$. From Lemma 10 and the fact that $H_{n+1} \neq \emptyset$ there exists $I \in \mathbb{P}_{A_{n+1}}$

such that $H_{n+1} = H_n \cap I \neq \emptyset$. Then, from Definition 7, $H_n = I$, from where it follows that $H_{n+1} \in \mathbb{P}_{A_{n+1}}$. Now, if $f^{-n-1}(0) \cap H_n \neq \emptyset$ from Lemma 9 there is a single $y \in H_n$ such that $f^{n+1}(y) = 0$. Then, from Definition 7 and the fact that $A_n \subset A_{n+1}$, there exist I_1 and I_2 in $\mathbb{P}_{A_{n+1}}$ disjoint with $I_1 \cup I_2 = H_n \setminus \{y\}$. From this and Lemma 9, if $f^{n+1}(I_1) \subset \mathbb{R}_1$ then $f^{n+1}(I_2) \subset \mathbb{R}_0$ or conversely. It means that I_1 or I_2 is contained in $f^{-n-1}(\mathbb{R}_{a_{n+1}})$. So from Lemma 10 $H_{n+1} = H_n \cap I_1$ or $H_{n+1} = H_n \cap I_2$; in any case, $H_{n+1} \in \mathbb{P}_{A_{n+1}}$. This completes the proof of Claim 1. □

Claim 2: There exists $N \geq 1$ such that H_N is a bounded atom of $\mathbb{P}_{A_N}$.

Proof. Suppose that H_N is not bounded for all $N \geq 1$.

Case 1. If $a_0 = 0$, then $a_{3k} = 0$, $a_{3k+1} = 0$ and $a_{3k+2} = 1$, for all $k \geq 0$.

By induction. Since $H_0 = \mathbb{R}_0$ and H_1 is not bounded it follows that $H_1 = (-\infty, x_0)$. From Remark 1 $f(H_1) = (x_0, 0)$ and $f^2(H_1) = \mathbb{R}_1$, then $a_1 = 0$ and $a_2 = 1$. So, this proves the case $k = 0$.

Suppose that $a_{3k} = 0$, $a_{3k+1} = 0$ and $a_{3k+2} = 1$. Then, from this and Lemma 9, it follows that $f^{3k}(H_{3k}) = \mathbb{R}_0$ and $f^{3k+1}(H_{3k+1}) = (x_0, 0)$; therefore, $f^{3k+2}(H_{3k+2}) = \mathbb{R}_1$. Since $f^{3k+3}(H_{3k+2}) = (-\infty, x_1)$, there exists $y \in H_{3k+2}$ such that $f^{3k+3}(y) = 0$. Since H_{3k+3} is not bounded, we have that $H_{3k+3} = (-\infty, y)$ and $f^{3k+3}(H_{3k+3}) = \mathbb{R}_0$, and consequently $a_{3k+3} = 0$. Then, $f^{3(k+1)+1}(H_{3k+3}) = (x_0, +\infty)$. So there exists $y_1 \in H_{3k+3}$ such that $f^{3(k+1)+1}(y_1) = 0$. From Lemma 9 and the fact that $H_{3(k+1)+1} \subset H_{3k+3}$ is not bounded, $H_{3(k+1)+1} = (-\infty, y_1)$. Therefore $f^{3(k+1)+1}(H_{3(k+1)+1}) = (x_0, 0)$ and $f^{3(k+1)+2}(H_{3(k+1)+1} = \mathbb{R}_1$, so this shows that $a_{3(k+1)+1} = 0$ and $a_{3(k+1)+2} = 1$.

Case 2. If $a_0 = 1$, then $a_{3k} = 1$, $a_{3k+1} = 1$ and $a_{3k+2} = 0$, for all $k \geq 0$.

The proof follows similarly.

Note that in both cases 1 and 2 it is shown that $a \in \Sigma^*(2, 2)$ is a periodic orbit of period 3 with respect to the shift σ, but this is a contradiction from the supposition that H_N is not bounded for all $N \geq 1$. □

Claim 3: If $x \in \bigcap_{n \geq 1} \overline{H_n}$, then $x \in \mathbb{R}_f$.

Proof. Suppose that $x \in A_f$. From Claim 2, there exists $N > 1$ such that H_n is bounded for $n \geq N$ and from Claim 1 there exists $y \in A_N$ such that $H = (x, y)$ or $H = (y, x)$. We have the following possible cases.

(A1) Suppose that $H_N = (x, y)$ and $a_N = 1$. Then

$$a_{N+3k} = 1, \quad a_{N+3k+1} = 0, \quad \text{and} \quad a_{N+3k+2} = 0, \qquad \text{for all } k \geq 0.$$

We will prove the claim (A1) by induction on k. For $k = 0$ the claim is valid. Indeed, from our hypothesis and Lemma 9, $f^{N+1}(H_{N+1}) = \mathbb{R}_0$; consequently, $a_{N+1} = 0$. Then, $f^{N+2}(H_{N+1}) = (x_0, +\infty)$. From this there exists $y_1 \in H_{N+1}$ such that $f^{N+2}(y_1) = 0$. So, from Claim 1 and the fact that $x \in \overline{H_{N+2}}$ and $H_{N+2} \subset (x, y)$, it follows that $f^{N+2}(H_{N+2}) = (x_0, 0)$. Therefore $a_{N+2} = 0$.

Suppose that the claim is valid for some $k \geq 0$. Since

$$x \in \overline{H_{N+3k+2}} \subset [x,y] \quad \text{and} \quad H_{N+3k+2} \subset H_{N+3k+1} \subset (x,y),$$

it follows that $f^{N+3k+2}(H_{N+3k+2}) = \mathbb{R}_1$. Denote by $m = N+3k+2$. Then, applying f, $f^{m+1}(H_m) = (-\infty, x_1)$ and we obtain that there exists $y_1 \in H_m$ such that $f^{m+1}(y_1) = 0$. Then, from Claim 1 and the fact that $x \in \overline{H_{m+1}}$, it follows that $H_{m+1} = (x, y_1)$ and $f^{m+1}(H_{m+1}) = \mathbb{R}_0$; consequently, $a_{m+1} = 0$. Applying f we have that $f^{m+2}(H_{m+1}) = (x_0, +\infty)$. Since $x \in \overline{H_{m+2}}$ and using the same last argument, $f^{m+2}(H_{m+2}) = (x_0, 0)$ and $f^{m+3}(H_{m+2}) = \mathbb{R}_1$. So $a_{m+2} = 0$ and $a_{m+3} = 1$. Replacing m we have that

$$a_{N+3(k+1)} = 0, \quad a_{N+3(k+1)+1} = 0, \quad \text{and} \quad a_{N+3(k+1)+2} = 1,$$

and that is what we wanted to prove.

The demonstrations of the following claims follow similarly as in the case (A1). In each case we indicate the components of the sequence a.

(A2) Suppose that $H_N = (x,y)$ and $a_N = 0$. Then,

2.1) if $f^N(H_N) = (x_0, 0)$, then $a_{N+3k} = 0$, $a_{N+3k+1} = 1$ and $a_{N+3k+2} = 0$ for all $k \geq 0$;

2.2) if $f^N(H_N) = \mathbb{R}_0$, then $a_{N+3k} = 0$, $a_{N+3k+1} = 0$ and $a_{N+3k+2} = 1$ for all $k \geq 0$.

(A3) Suppose that $H_N = (y,x)$ and $a_N = 0$. Then

$$a_{N+3k} = 0, \quad a_{N+3k+1} = 1, \quad \text{and} \quad a_{N+3k+2} = 1 \quad \text{for all} \quad k \geq 0.$$

(A4) Suppose that $H_N = (y,x)$ and $a_N = 1$. Then,

4.1) if $f^N(H_N) = (0, x_1)$, then $a_{N+3k} = 1$, $a_{N+3k+1} = 0$, and $a_{N+3k+2} = 1$ for all $k \geq 0$;

4.2) if $f^N(H_N) = \mathbb{R}_1$, then $a_{N+3k} = 1$, $a_{N+3k+1} = 1$, and $a_{N+3k+2} = 0$ for all $k \geq 0$.

In all cases we proved that the point $(\sigma*)^N(a)$ is periodic of period 3 and this is a contradiction. Consequently, $x \in R_f$. $\square$

Let $a \in \Sigma^*(2,2)$. From Claim 2, $\bigcap_{n\geq 1} \overline{H_n} \neq \emptyset$; from the injectivity of h and Claim 2, there exists a single $x \in \bigcap_{n\geq 1} \overline{H_n}$, and from Claim 3, $x \in \mathbb{R}_f$. So, from definitions of h and the sets H_n it follows that $h(x) = a$. This proves that h is onto $\Sigma^*(2,2)$.

h^{-1} **is continuous.** Let $a \in \Sigma^*(2,2)$ and $\epsilon > 0$. Let us denote $x = h^{-1}(a)$. In the same way of constructing the H_n's in the proof that h is onto, we have that $x \in H_n$ for all $n \geq 1$. Then, from Lemma 8, $\operatorname{diam}(H_n) \to 0$ when $n \to +\infty$. Therefore, there exists $N > 1$ such that $H_N \subset (x-\epsilon, x+\epsilon)$. On the other hand, there exists $\delta > 0$ such that if $b \in \Sigma^*(2,2)$ with $d_2(a,b) < \delta$ then $a_j = b_j$ for all $0 \leq j \leq N$. Then, $y = h^{-1}(b) \in H_j$ for $0 \leq j \leq N$, that is, $y = h^{-1}(b) \in H_N \subset (x-\epsilon, x+\epsilon)$. This proves the continuity of h^{-1}.

Finally, it remains to prove that $h \circ f = \sigma^* \circ h$. Indeed, given $x \in \mathbb{R}_f$ and denoting $a = h(f(x))$ we have that $a_n = 0$ if $f^{n+1}(x) < 0$, or $a_n = 1$ if $f^{n+1}(x) > 0$ for all $n \geq 0$. On the other hand, if $b = h(x)$ we have that $\sigma^*(h(x)) = (b_1, b_2, b_3, \ldots)$.

Note that $(b_1, b_2, b_3, \ldots,) = (a_0, a_1, a_2, \ldots)$. This shows that $h \circ f(x) = \sigma^* \circ h(x)$, for all $x \in \mathbb{R}_f$.

($\Leftarrow$) Suppose that f is transitive and suppose (by contradiction) that f is not expansive. Then, there exists $x \neq y$ such that $f^j(x).f^j(y) > 0$, for all $j \geq 0$. Suppose, without loss of generality, that $x < y$. That means $f^j([x,y]) \cap A_f = \emptyset$ for all $j \geq 0$; from this we conclude that $f^j([x,y])$ does not have points of the set $\bigcup_{n\geq 0} f^{-n}(0)$. Denote by $I = [x,y]$. Since f is transitive there exists $k \geq 1$ such that $f^k(I) \cap I \neq \emptyset$. Since f is continuous, increasing in each connected component of $\mathbb{R}\backslash\{0\}$ and $f^j(I) \subset \mathbb{R}_{a_j}, \forall\, j \geq 0$, it follows that $f^k(I) \cup I$ is an interval. Note that $f^{2k}(I) \cap (f^k(I) \cup I) \neq \emptyset$, so $f^{2k}(I) \cup f^k(I)$ or I is an interval and also contained in $\mathbb{R}_0$ or $\mathbb{R}_1$. Given $J = \bigcup_{n\geq 1} f^{nk}(I)$, then J is invariant by f^k, but this is a contradiction since f is transitive. □

Proof of Corollary 3. In the proof of the Main Theorem, it is shown that f is topologically conjugated with $\sigma^* : \Sigma^*(2,2) \longrightarrow \Sigma^*(2,2)$ and the set of periodic points of σ^* is dense in $\Sigma^*(2,2)$; from this the result follows. □

Corollary 12. *All maps $f : \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$ which are SACAH and transitive are topologically conjugated with each other.*

Proof. Let f and g be SACAH maps. Then, from the Main Theorem there exist homeomorphisms $h_1 : \mathbb{R}_f \to \Sigma^*(2,2)$ and $h_2 : \mathbb{R}_g \to \Sigma^*(2,2)$ such that $h_1 \circ f = \sigma^* \circ h_1$ and $h_2 \circ g = \sigma^* \circ h_2$. Taking $h = h_2^{-1} \circ h_1 : \mathbb{R}_f \to R_g$ we have that $h \circ f = g \circ h$. □

Let us now consider an interesting example of this kind of maps. Let $B : \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$ be defined by

$$B(x) = \frac{|x| - 1}{x} = \begin{cases} 1 - \dfrac{1}{x}, & x > 0, \\[2ex] -1 - \dfrac{1}{x}, & x < 0. \end{cases}$$

Lemma 13. $\displaystyle\bigcup_{n\geq 0} B^{-n}(0) = \mathbb{Q}$.

Proof of Corollary 4. The proof follows from Lemma 13 and the Main Theorem. □

Proof of Lemma 13. Observe that if $a \in \mathbb{Q}$, then $B^{-1}(a) \in \mathbb{Q}$. From this we conclude that $\bigcup_{n\geq 0} B^{-n}(0) \subset \mathbb{Q}$. Now, we will show the other inclusion, that is, $\mathbb{Q} \subset \bigcup_{n\geq 0} B^{-n}(0)$. First observe that:

$$\text{for all } |x| > 1, \text{ it follows that } |B(x)| < 1. \tag{1}$$

Also, note that

$$\text{if } B^k(x) \in A_B, \text{ then } B^j(x) \in A_B, \text{ for } 0 \leq j \leq k. \tag{2}$$

Also note that $B(-x) = -B(x)$, for all $x \neq 0$. In general,

$$\text{for all } n \geq 1 \text{ and } x \in \mathbb{R}_B \text{ it follows that } B^n(-x) = -B^n(x), \tag{3}$$

where $\mathbb{R}_B = \mathbb{R} \setminus A_B$ and $A_B = \cup_{n\geq 1} B^{-n}(0) \cup \{0\}$.

Case 1. For all $n \in \mathbb{Z}$, we have that $n \in A_B$.

From (2) it is sufficient to show that $n \in A_B$ for all $n \geq 1$. The proof is by induction. For $n = 1$ it is clear since $B^{-1}(0) = \{-1, 1\}$. Suppose that $j \in A_B$ for $1 \leq j \leq n$. Then, $B(n+1) = \frac{n-1}{n}$ and applying B we have $B^2(n+1) = -\frac{1}{n}$. Applying B again we obtain $B^3(n+1) = n - 1$. Therefore, by our inductive hypothesis $n-1 \in A_B$; from this and (2) it follows that $(n+1) \in A_B$. This proves Case 1.

From the proof of Case 1, from (1) and (2), we have

$$\frac{1}{n} \in A_B \quad \text{for all } n \in \mathbb{Z} \setminus \{0\}. \tag{4}$$

Case 2. Consider $k > 0$, $n > k$ such that $n = q \cdot k + r$, where $r \in \{0, 1, \ldots, k-1\}$ with $q \geq 3$. Then, for $t \geq 1$ and $2t + 1 \leq q$ we have

$$\begin{aligned} B^{3t-1}\left(-\frac{k}{n}\right) &= \frac{n - 2tk}{n - (2t-1)k} \\ B^{3t}\left(-\frac{k}{n}\right) &= \frac{-k}{n - 2tk} \\ B^{3t+1}\left(-\frac{k}{n}\right) &= \frac{n - (2t+1)k}{k}. \end{aligned}$$

We will do the proof of Case 2 by finite induction. Let us see that the equations are valid for $t = 1$:

$$B^2\left(-\frac{k}{n}\right) = 1 - \frac{k}{n-k} = \frac{n-2k}{n-k}$$

since $q \geq 2$, $n - 2k > 0$, it follows that $\dfrac{n-2k}{n-k} > 0$. Therefore,

$$B^3\left(-\frac{k}{n}\right) = \frac{-k}{n-2k} < 0 \quad \text{and} \quad B^4\left(-\frac{k}{n}\right) = \frac{n-3k}{k}.$$

Suppose that the equations are valid for t and suppose that $2(t+1) + 1 \leq q$. Let us see that the equations are valid for $t + 1$. Note that $3t + 2 = 3(t+1) - 1$ and $3t+3 = 3(t+1)$. On the other hand, since $2(t+1)+1 \leq q$, then $\dfrac{n-(2t+1)k}{k} > 0$. So applying B to $B^{3t+1}\left(-\dfrac{k}{n}\right)$ we obtain

$$B^{3t+2}\left(-\frac{k}{n}\right) = 1 - \frac{k}{n-(2t+1)k} = \frac{n-(2t+2)k}{n-(2t+1)k},$$

where $n - (2t+2)k > 0$ because $2(t+1) + 1 \leq q$. Applying B again,

$$\begin{aligned} B^{3t+3}\left(-\frac{k}{n}\right) &= \frac{-k}{n-(2t+2)k} < 0 \quad \text{and} \\ B^{3(t+1)+1}\left(\frac{n-k}{k}\right) &= \frac{n-(2(t+1)+1)k}{k}, \end{aligned}$$

which is what we wanted to prove.

Case 3. For each $n \geq 1$, $\dfrac{n}{m} \in A_B$, $\forall\, m \geq n$.

Let us show this by induction on n. For $n = 1$ it is valid from (4). Suppose that it is valid for n, that is, $\dfrac{j}{m} \in A_B$, for all $m \geq j$ and for $1 \leq j \leq n$. Denote $k = n+1$. If $m = k$, then the proof follows since $1 \in A_B$. Suppose that $m > k$, then there exists $q \geq 1$ such that $m = q.k + r$, where $r \in \{0, 1, \dots, n\}$. Observe that

$$B\left(-\frac{k}{m}\right) = -1 + \frac{m}{k} = \frac{m-k}{k}.$$

If $q = 1$, we have that $m = k + r$. Then,

$$B\left(-\frac{k}{m}\right) = \frac{k+r-k}{k} = \frac{r}{k}.$$

Since $0 \leq r \leq k-1$, by the inductive hypothesis we have $\dfrac{r}{k} \in A_B$. Then, from (2) and (3) it follows that $\dfrac{k}{m} \in A_B$, $\forall\, m \in \{k, \dots, 2k-1\}$. Now, suppose that $q = 2$, then $m = 2k + r$. Therefore $\dfrac{m-k}{k} > 0$ and

$$B^2\left(-\frac{k}{m}\right) = \frac{m-2k}{m-k} = \frac{r}{k+r}.$$

By the inductive hypothesis, $\dfrac{r}{k+r} \in A_B$. Then, from (2) and (3), $\dfrac{k}{m} \in A_B$ for all $m \in \{2k, 2k+1, \dots, 3k-1\}$. Finally, suppose that $q \geq 3$, then from Case 2, for $t \geq 1$, $2t + 1 \leq q$ and $m = q.k + r$,

$$\begin{aligned}
B^{3t-1}\left(-\frac{k}{m}\right) &= \frac{m-2tk}{m-(2t-1)k} \\
B^{3t}\left(-\frac{k}{m}\right) &= -\frac{k}{m-2tk} \\
B^{3t+1}\left(-\frac{k}{m}\right) &= -\frac{m-(2t+1)k}{k}.
\end{aligned}$$

Take $t_0 \geq 1$ such that $2t_0 + 1 = q - 1$ or $2t_0 + 1 = q$. If $2t_0 + 1 = q - 1$, then $m - (2t+1)k = q.k + r - (q-1)k = k + r$.

$$B^{3t_0+1}\left(-\frac{k}{m}\right) = \frac{k+r}{k} \quad \text{and remember that} \quad k = n+1,$$

$$B^{3t_0+2}\left(-\frac{n+1}{m}\right) = B^{3t_0+2}\left(-\frac{k}{m}\right) = 1 - \frac{k}{k+r} = \frac{r}{k} = \frac{r}{n+1} \in A_B,$$

by our inductive hypothesis since $r \in \{0, 1, \dots, n\}$. From (2) and (3) it follows that $\dfrac{n+1}{m} \in A_B$, for all $m \geq 3(n+1)$. Now, if $2t_0 + 1 = q$,

$$B^{3t_0+1}\left(-\frac{n+1}{m}\right) = \frac{q.k + r - q.k}{n+1} = \frac{r}{n+1} \in A_B$$

by the inductive hypothesis since $r \in \{0, 1, \ldots, n\}$. Therefore, from (2) and (3) it follows that $\dfrac{n+1}{m} \in A_B$, for all $m \geq 3(n+1)$. The proof of Case 3 is complete.

To finish the proof of Lemma 13, let $p \in \mathbb{Q} \setminus \{0\}$. If $0 < p \leq 1$, from Case 3 we have $p \in A_B$. If $p > 1$, then from (1) $B(p) \in (0,1)$ and $B(p) \in \mathbb{Q}$; consequently, there exist integers $n \geq 1$ and $m > n$ such that $B(p) = \frac{n}{m}$. From Case 3 and (2), $p \in A_B$. Finally, if $p < 0$, by (1) and (2) we have that $p \in A_B$. This completes the proof of Lemma 13. $\square$

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