Revista de la Unión Matemática Argentina Accepted article · Early view version

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RECURRENCE FOR WEIGHTED PSEUDO-SHIFT OPERATORS

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Abstract. In this work, we provide a characterization of multiply recurrent operators that act on a Fréchet space. As an application, we extend the weighted shift's results established by G. Costakis et al. in [9]. We achieve this by characterizing topologically multiply recurrent pseudo-shifts acting on an F-sequence space indexed by an arbitrary countable infinite set. This characterization is in terms of the weights, the OP-basis and the shift mapping. Additionally, we establish that the recurrence and the hypercyclicity of pseudo-shifts are equivalent.

1. Introduction and preliminary results

Let $\mathbb{Z}$ or $\mathbb{N}$ denote the set of all integers and positive integers, respectively. Throughout this paper, unless stated otherwise, X refers to an F-sequence space indexed by an arbitrary countable infinite set I, that is, a subspace of the space $\mathbb{K}^I$ of all scalar families $(x_i)_{i\in I}$ which is endowed with its natural product topology.

The bilateral weighted backward (respectively, forward) shift $B_{\mathbf{w}}$ on a bilateral sequence space is defined as follows

$$B_{\mathbf{w}}(x_k)_{k\in\mathbb{Z}} := (w_{k+1}x_{k+1})_{k\in\mathbb{Z}},$$

respectively,

$$B_{\mathbf{w}}(x_k)_{k\in\mathbb{Z}} := (w_{k-1}x_{k-1})_{k\in\mathbb{Z}},$$

where the weight sequence $\mathbf{w} = (w_k)_{k\in\mathbb{Z}}$ of the bilateral weighted shift is assumed to be a bounded sequence of positive real numbers. The unilateral weighted backward and forward shifts on a sequence space indexed by $\mathbb{N}$ are defined similarly with $x_0 = w_0 = 0$.

As an extension of weighted shifts, K.-G. Grosse Erdemann introduced in [12] the concept of a weighted pseudo-shift T that acts on X, defined by the existence of a sequence of non-zero scalars $(w_i)_{i\in I}$, called the weight sequence, and an injective mapping $f : I \to I$ such that $T(x_i)_{i\in I} = \big(w_i x_{f(i)}\big)_{i\in I}$. Since X is an F-space, the continuity of T follows from the closed graph theorem.

Recall that an operator T acting on a Fréchet space Y is hypercyclic if there is some x such that the orbit of x under T, that is, the set $\mathrm{Orb}(T, x) := \{T^n x; n \geq 0\}$, is dense in Y. Such a vector x is called a hypercyclic vector. The set of all hypercyclic vectors for T is denoted by $\mathrm{HC}(T)$. Verifying the hypercyclicity of

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2020 *Mathematics Subject Classification.* Primary 47B37, 37B20; Secondary 47A16.

Key words and phrases. Sequence spaces, Weighted shifts, Weighted pseudo-shifts, Recurrence, Hypercyclicity.

Submitted: April 19, 2023
Accepted: February 27, 2024
Published (early view): September 2, 2024

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an operator often involves demonstrating through an equivalent formulation that the orbit of any non-empty open set under the operator is dense in Y. For more background and examples about hypercyclicity, we refer the books by F. Bayart et al. [3] and by G.-K. Grosse-Erdmann et al. [13].

The concept of hypercyclicity was first introduced by S. Rolewicz in 1969, who showed that the multiple of a backward shift in a Banach space is hypercyclic if and only if the multiplication factor is greater than one. Since then, the study of weighted shifts has provided a wealth of examples and counterexamples in linear dynamics, as they offer a means to investigate the dynamic characteristics of continuous linear operators. In this context, H.N. Salas [18] established a characterization of hypercyclic weighted backward shifts. Later on, K.-G. Grosse-Erdmann [12] expanded upon Salas' results by characterizing hypercyclic weighted pseudo-shifts acting on an arbitrary F-sequence space. In recent years, the study of pseudo-shifts has garnered considerable attention, resulting in a surge of research and numerous studies into the topic. For example, Y. Wang et al. [19] investigated disjoint hypercyclicity and disjoint supercyclicity of tuples of distinct powers of weighted pseudo-shifts. On the other hand, Ç. Nurhan et al. [17] characterized disjoint hypercyclicity of weighted pseudo-shifts that are raised to the same power. In addition, M. Özgür et al. [16] have studied the so-called disjoint frequently hypercyclic pseudo-shifts.

Recall that an operator T acting on a Fréchet space Y is hypercyclic if there is some x such that the orbit of x under T, that is, the set $\mathrm{Orb}(T,x) := \{T^n x; n \geq 0\}$, is dense in Y. Such a vector x is called a hypercyclic vector. The set of all hypercyclic vectors for T is denoted by $\mathrm{HC}(T)$. Verifying the hypercyclicity of an operator often involves demonstrating through an equivalent formulation that the orbit of any non-empty open set under the operator is dense in Y. For more background and examples about hypercyclicity, we refer the books by F. Bayart et al. [3] and by G.-K. Grosse-Erdmann et al. [13].

Recurrence is one of the fundamental concepts in the theory of dynamical systems. Its study was first initiated in 1890 by H. Poincaré with the Poincaré recurrence theorem [15] and later generalized by H. Furstenberg in 1976 with the multiple recurrence theorem [11]. Recently, these notions were studied systematically in linear dynamics in two fundamental papers by G. Costakis et al. in [8, 9].

According to these works, an operator T is said to be recurrent if for each non-empty open set U, there exists a positive integer n such that $T^n(U) \cap U \neq \emptyset$. A vector x is called a recurrent vector for T if there exists a strictly increasing sequence of positive integers $(n_k)_k$ such that $T^{n_k}x \longrightarrow x$ as $k \longrightarrow \infty$. The set of all recurrent vectors for T is denoted by $\mathrm{Rec}(T)$. Notice that T is recurrent if and only if $\mathrm{Rec}(T)$ is dense. An operator T is said to be topologically multiply recurrent or simply multiply recurrent if for every positive integer m and every non-empty open set U, there exists $n \in \mathbb{N}$ such that $\bigcap_{l=0}^{m} T^{-ln}U \neq \emptyset$. If $m = 1$, then T satisfies $U \cap T^{-n}U \neq \emptyset$, thus T is recurrent. For a deeper understanding of recurrence, see [1, 2, 4, 5, 6, 7, 8, 9, 14].

Submitted: April 19, 2023
Accepted: February 27, 2024
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To proceed further, it is necessary to collect several definitions of some key notions that will be utilized for our main result.

Definition 1.1 (OP-basis)**.** Let $(e_i)_{i\in I}$ denote the canonical unit vectors in a topological sequence space X over an arbitrary countable infinite set I. We say that $(e_i)_{i\in I}$ is an OP-basis or (Ovsepian Petczynski basis) if span$\{e_i : i \in I\}$ is a dense subspace of X and the family of coordinate projections $x \mapsto x_i e_i (i \in I)$ on X is equicontinuous.

Remarks 1.2.
(1) Every separable Banach space is isomorphic to a Banach sequence space in which $(e_n)_{n\in\mathbb{N}}$ is an OP-basis.
(2) Note that in a Banach sequence space over I, the family of coordinate projections is equicontinuous if and only if $\sup_{i\in I} \|e_i x_i\| < \infty$.

Definition 1.3 (Weighted pseudo-shift)**.** [12] Let X and Y be topological sequence spaces over countable infinite sets I and J, respectively. Then a linear operator $T : X \to Y$ is called a weighted pseudo-shift if there is a sequence $(w_j)_{j\in J}$ of non-zero scalars and an injective mapping $f : J \to I$ such that

$$T\,(x_i)_{i\in I} = \left(w_j x_{f(j)}\right)_{j\in J}$$

for $(x_i)_i \in X$. We then write $T = T_{w,f}$. $(w_j)_{j\in J}$ is called the weight sequence.

Remarks 1.4.
(1) We note that if X and Y are F-sequence spaces, then the continuity of T is a straightforward conclusion through the application of the closed graph theorem.
(2) Every unilateral or bilateral weighted backward shift is a weighted pseudo-shift with w_{n+1} as weight, and $f(n) = n + 1$. Likewise, every bilateral weighted forward shift is a weighted pseudo-shift with the weight w_{n-1} and $f(n) = n - 1$. In contrast, unilateral weighted forward shifts cannot be considered pseudo-shifts due to their definition in the first component.
(3) If $X = Y$, then the identity operator on X defines a pseudo-shift.
(4) If $T = T_{w,f} : X \to X$ is a weighted pseudo-shift, then for each $n \geq 1$, T^n is also a weighted pseudo-shift. Additionally,

$$T^n\,(x_i)_{i\in I} = \left(w_{n,i} x_{f^n(i)}\right)_{i\in I}$$

where

$$f^n(i) = (f \circ f \circ \cdots \circ f)(i) \quad (n\text{-fold})$$

$$w_{n,i} = w_i w_{f(i)} \cdots w_{f^{n-1}(i)} = \prod_{v=0}^{n-1} w_{f^v(i)}.$$

(5) We consider the inverse $f^{-1} : f(I) \to I$ of the mapping f. We also set

$$w_{f^{-1}(i)} = 0 \quad \text{and} \quad e_{f^{-1}(i)} = 0 \quad \text{if } i \in I \backslash f(I).$$

Then for all $i \in I$,

$$T_{w,f^{-1}} e_i = w_{f^{-1}(i)} e_{f^{-1}(i)}.$$

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In this paper, motivated by Grosse-Erdmann's work [12] and in order to generalize the weighted shift's characterizations given in [9] we study the topological multiple recurrence and the recurrence of weighted pseudo-shifts.

The rest of the paper is structured as follows: Section 2 provides equivalent conditions for topologically multiply recurrent operators that act on Fréchet space. In Section 3, we characterize topologically multiply recurrent pseudo-shifts that act on an F-sequence space in terms of the weights, the OP-basis and the shift mapping. Furthermore, we establish that recurrence and hypercyclicity of pseudo-shifts are equivalent.

2. Topologically multiply recurrent operators

This section is devoted to the study of topologically multiply recurrent operators that act on Fréchet space X. In the following, we recall their definition from [9].

Definition 2.1. An operator T that acts on X is called topologically multiply recurrent operator if for every non-empty open subset U of X and every positive integer m there exists a positive integer n such that

$$U \cap T^{-n}U \cap \cdots \cap T^{-mn}U \neq \emptyset.$$

Remark 2.2. In the provided definition, when $m = 1$, we obtain the recurrence of the operator T.

The subsequent example demonstrates that the converse implication does not hold in full generality.

Example 2.3. Let $T : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$ be the bilateral backward shift with weights $(w_n)_{n\in\mathbb{Z}}$ defined as follows

$$w_n = \begin{cases} 2, & \text{if } n \leq 0, \\ \frac{1}{1+n}, & \text{if } n > 0. \end{cases}$$

T is recurrent but not topologically multiply recurrent. Indeed, since $w_n > 0$, we have

$$\lim_{k\to+\infty} \prod_{j=0}^{k} \frac{1}{w_{-j}} = \lim_{k\to+\infty} \prod_{j=0}^{k} \frac{1}{2} = \lim_{k\to+\infty} \left(\frac{1}{2}\right)^k = 0$$

and

$$\lim_{k\to+\infty} \prod_{j=1}^{k} w_j = \lim_{k\to+\infty} \prod_{j=1}^{k} \frac{1}{1+j} = \lim_{k\to+\infty} \frac{1}{(1+k)!} = 0.$$

Then, according to [10, Theorem 3.2] and [9, Proposition 5.1], T is recurrent. However, for all strictly increasing sequences of positive integers $(n_k)_k$, we have

$$\lim_{k\to+\infty} \prod_{j=1}^{n_k} w_j = \lim_{k\to+\infty} \prod_{j=0}^{n_k} \frac{1}{w_{-j}} < \infty.$$

Then, by [9, Proposition 5.3], T is not topologically multiply recurrent.

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The following theorem characterizes sequentially the topological multiple recurrence of an operator.

Theorem 2.4. *Let T be an operator that acts on a Fréchet space X. Then, T is topologically multiply recurrent if and only if for every $m \in \mathbb{N}$ and every $x \in X$, there exist a sequence of vectors $(x_k)_k$ and a strictly increasing sequence of positive integers $(n_k)_k$ such that for every $l \in \{1, 2, \ldots, m\}$*

$$x_k \longrightarrow x \quad \textit{and} \quad T^{ln_k}(x_k) \longrightarrow x \quad \textit{as} \quad k \longrightarrow \infty.$$

Proof. Let $m \in \mathbb{N}$ and $x \in X$. Consider the open ball $B\left(x, \frac{1}{2}\right)$. Since T is topologically multiply recurrent, there exists a positive integer n_1 such that

$$B\left(x, \frac{1}{2}\right) \cap T^{-n_1} B\left(x, \frac{1}{2}\right) \cap \cdots \cap T^{-mn_1} B\left(x, \frac{1}{2}\right) \neq \emptyset.$$

Thus, there exists a vector x_1 in X such that for every $l \in \{1, 2, \ldots, m\}$

$$\|x - x_1\| < \frac{1}{2} \quad \text{and} \quad \|x - T^{ln_1} x_1\| \leq \frac{1}{2}.$$

Continuing inductively, we construct $(x_k)_k$ in X and $(n_k)_k$ in $\mathbb{N}$ such that for every $l \in \{1, 2, \ldots, m\}$

$$\|x - x_k\| < \frac{1}{2^k} \quad \text{and} \quad \|x - T^{ln_k} x_k\| < \frac{1}{2^k}.$$

As k goes to infinity, we obtain for every $l \in \{1, 2, \ldots, m\}$

$$x_k \longrightarrow x \quad \text{and} \quad T^{ln_k}(x_k) \longrightarrow x.$$

Conversely, let U be a non-empty open subset of X and $m \in \mathbb{N}$. Then there exist $x \in U$ and $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$. Since there are $(x_k)_k$ in X and $(n_k)_k$ in $\mathbb{N}$ such that

$$x_k \longrightarrow x \quad \text{and} \quad T^{ln_k}(x_k) \longrightarrow x \quad \text{as} \quad k \longrightarrow \infty, \quad \text{for each } l \in \{1, 2, \ldots, m\}.$$

Thus, there exists $k \in \mathbb{N}$ such that

$$x_k \in B(x, \varepsilon) \subset U \quad \text{and} \quad T^{ln_k} x_k \in B(x, \varepsilon) \subset U, \quad \text{for each } l \in \{1, 2, \ldots, m\}.$$

Hence, $x \in \bigcap_{l=0}^{m} T^{-ln_k} U$. Finally, T is a topologically multiply recurrent operator. □

3. Weighted pseudo-shifts and recurrence

Throughout this section, X is an F-sequence space over an arbitrary set I in which $(e_i)_{i \in I}$ forms an OP-basis. Before providing the main result, we need to discuss the following concept.

Definition 3.1 (Run-away sequence)**.** Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of maps acting on I. $(f_n)_{n \in \mathbb{N}}$ is called a run-away sequence if for each pair of finite subsets $I_0 \subset I$ and $J_0 \subset I$, there exists an $n_0 \in \mathbb{N}$ such that $f_n(J_0) \cap I_0 = \emptyset$ for every $n \geq n_0$.

The following illustrations represent maps for which the run-away property holds for their sequences of iterates.

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Examples 3.2.
(1) Let f be the translation map defined on the set of positive integers $\mathbb{N}$ by $f(j)=j+1$. Then, $(f^n)_{n\in\mathbb{N}}$ is a run-away sequence. Let I_0 and J_0 be two finite subsets of $\mathbb{N}$. Notice that $f^n(J_0)=\{n+j \mid j\in J_0\}$. By setting $n_0=\max(I_0)+1$, we obtain that $f^n(J_0)\cap I_0=\emptyset$ for every $n\geq n_0$. As a result, the sequence $(f^n)_{n\in\mathbb{N}}$ is indeed run-away.
(2) Let $f(j)=2j$ be a map defined on the set of strictly positive integers $\mathbb{N}\setminus\{0\}$. Let I_0 and J_0 be two finite subsets of $\mathbb{N}\setminus\{0\}$. Observe that $f^n(J_0)=\{2^n j \mid j\in J_0\}$. Then, $f^n(J_0)\cap I_0=\emptyset$ for every $n\geq\max(I_0)$. Hence, $(f^n)_{n\in\mathbb{N}}$ is a run-away sequence.

The next examples illustrate that the run-away property may not hold for f^n.

Examples 3.3.
(1) Consider the identity map $\mathrm{Id}_{\mathbb{Z}}$ defined on the set of integers $\mathbb{Z}$. Then $(\mathrm{Id}_{\mathbb{Z}}^n)_{n\in\mathbb{N}}$ is not a run-away sequence. This is because there exists a nonempty finite subset $I_0\subset\mathbb{Z}$ with $\mathrm{Id}_{\mathbb{Z}}^n(I_0)\cap I_0=I_0\neq\emptyset$ for every $n\in\mathbb{N}$.
(2) Let us define $f(j)=j^2$ on the set of positive integers $\mathbb{N}$. Then $(f^n)_{n\in\mathbb{N}}$ is not a run-away sequence. Indeed, by setting $I_0=\{1\}$, we observe that $f^n(I_0)\cap I_0=I_0\neq\emptyset$ for all $n\in\mathbb{N}$. Hence, the sequence $(f^n)_{n\in\mathbb{N}}$ is not run-away.

In the upcoming proposition, we will present a characterization of the run-away property.

Proposition 3.4. *Let f be a map acting on I. f has no periodic points if and only if $(f^n)_{n\in\mathbb{N}}$ is a run-away sequence.*

Proof. Suppose that f has no periodic points. Let I_0 be a finite subset of I. Then for any $i\in I$ there exists a positive integer $n_0\in\mathbb{N}$ with $f^n(i)\notin I_0$ for $n\geq n_0$. This shows that $(f^n)_{n\in\mathbb{N}}$ is a run-away sequence.

Conversely, for the sake of contradiction, let's assume that f admits a periodic point. This implies the existence of $x,k\in I\times\mathbb{N}$ such that $f^k(x)=x$.
Set $I_0=\{x\}$. Since $(f^n)_{n\in\mathbb{N}}$ is run-away, there exists a positive integer n_0 such that $f^nI_0\cap I_0=\emptyset$, for every $n\geq n_0$.
If $k\geq n_0$, then $f^kI_0\cap I_0=\{x\}\neq\emptyset$, which contradicts the run-away assumption.
If $k<n_0$, there exists a positive integer p such that $pk\geq n_0$. However, $f^{pk}(x)=x$, and thus $f^{pk}I_0\cap I_0=\{x\}\neq\emptyset$. □

The following theorem represents our main result.

Theorem 3.5. *Let T be a weighted pseudo-shift that acts on X with weights $(w_i)_{i\in I}$. If the mapping $f: I\to I$ has no periodic points, then the following assertions are equivalent:*
(1) *T is topologically multiply recurrent;*
(2) *For every $\varepsilon>0$ and every strictly positive integers m and N_0, there exists an integer $n>N_0$ such that for every $i\in I$ and every $l\in\{1,2,\ldots,m\}$*

$$\|w_{ln,i}^{-1}e_{f^{ln}(i)}\|<\varepsilon \quad \textit{and} \quad \left\|\prod_{v=1}^{ln}w_{f^{-v}(i)}e_{f^{-ln}(i)}\right\|<\varepsilon.$$

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Proof. Let's first prove that (1) implies (2). Fix a positive integer m, $\varepsilon > 0$, and also choose a positive integer N_0. Given a finite subset $I_0 \subset I$, by the equicontinuity of the coordinate projections in X, there exists some $\mu > 0$ such that for every $x \in X$ and every $i \in I_0$

$$\|x_i e_i\| < \frac{\varepsilon}{2}, \quad \text{if } \|x\| < \mu. \tag{3.1}$$

Since the mapping $f : I \to I$ has no periodic points, by Proposition 3.4, there exists an integer $N \in \mathbb{N}$ such that for every $l \in \{1, 2, \dots, m\}$ and every $n > N$

$$f^{ln}(I_0) \cap I_0 = \emptyset. \tag{3.2}$$

It follows that

$$f^{-ln}\left(I_0 \cap f^{ln}(I)\right) \cap I_0 = \emptyset. \tag{3.3}$$

Consider the open ball $B\left(\sum_{i\in I_0} e_i, \mu\right)$. Since T is topologically multiply recurrent, there exists a positive integer $n > \max(N, N_0)$ such that

$$B\left(\sum_{i\in I_0} e_i, \mu\right) \cap T^{-n}\left(B\left(\sum_{i\in I_0} e_i, \mu\right)\right) \cap \dots \cap T^{-mn}\left(B\left(\sum_{i\in I_0} e_i, \mu\right)\right) \neq \emptyset.$$

Thus, by virtue of Theorem 2.4, there exist $x \in X$ and $n > \max(N, N_0)$, such that for every $l \in \{1, 2, \dots, m\}$

$$\left\|x - \sum_{i\in I_0} e_i\right\| < \mu \quad \text{and} \quad \left\|T^{ln}x - \sum_{i\in I_0} e_i\right\| < \mu. \tag{3.4}$$

By (3.1), (3.2), and the first inequality in (3.4), it follows that

$$\|x_i e_i\| < \frac{\varepsilon}{2}, \quad \text{if } i \notin I_0.$$

Then, by (3.2) we have for every $i \in I_0$ and every $l \in \{1, 2, \dots, m\}$,

$$\|x_{f^{ln}(i)} e_{f^{ln}(i)}\| < \frac{\varepsilon}{2}. \tag{3.5}$$

By continuous inclusion of X into $\mathbb{K}^I$ we have

$$\sup_{i\in I_0} |x_i - 1| \le \frac{1}{2} \quad \text{and} \quad \sup_{i\in I_0} \left|y_i^{(l)} - 1\right| \le \frac{1}{2}. \tag{3.6}$$

where $y^{(l)} := T^{ln}x = (w_{ln,i} x_{f^{ln}(i)})_{i\in I}$.
Consequently, by the second inequality we have for every $i \in I_0$ and every $l \in \{1, 2, \dots, m\}$

$$x_{f^{ln}(i)} \neq 0 \quad \text{and} \quad \left|w_{ln,i} x_{f^{ln}(i)} - 1\right| \le \frac{1}{2}. \tag{3.7}$$

(3.5) and (3.7) imply that, for every $i \in I_0$ and every $l \in \{1, 2, \dots, m\}$

$$\begin{aligned} \left\|w_{ln,i}^{-1} e_{f^{ln}(i)}\right\| &= \left\|\frac{1}{w_{ln,i} x_{f^{ln}(i)}} x_{f^{ln}(i)} e_{f^{ln}(i)}\right\| \\ &\le \left\|x_{f^{ln}(i)} e_{f^{ln}(i)}\right\| + \left\|\left(\frac{1}{w_{ln,i} x_{f^{ln}(i)}} - 1\right) x_{f^{ln}(i)} e_{f^{ln}(i)}\right\| < \varepsilon. \end{aligned}$$

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Hence the first inequality of (2) holds.
Now, by (3.1) and the second inequality in (3.4) we obtain that for every $i \notin I_0$ and every $l \in \{1, 2, \dots, m\}$

$$\| \prod_{v=0}^{ln-1} w_{f^v(i)} x_{f^{ln}(i)}.e_i \| < \frac{\varepsilon}{2}. \tag{3.8}$$

Notice that

$$e_{f^{-ln}(i)} = 0, \quad \text{if } i \in I \backslash f^{ln}(I).$$

Then, by (3.3) and (3.8) we conclude that for every $i \in I_0$ and every $l \in \{1, 2, \dots, m\}$

$$\left\| \prod_{v=0}^{ln-1} w_{f^v(f^{-ln(i)})} x_{f^{ln}(f^{-ln}(i))} e_{f^{-ln}(i)} \right\| = \left\| \prod_{v=1}^{ln} w_{f^{-v}(i)} x_i e_{f^{-ln}(i)} \right\| < \frac{\varepsilon}{2}. \tag{3.9}$$

From the first inequality in (3.6) we have

$$2\,|x_i| \geq 1, \quad \text{for each } i \in I_0. \tag{3.10}$$

Now, (3.9) and (3.10) imply that for every $i \in I_0$ and every $l \in \{1, 2, \dots, m\}$

$$\begin{aligned} \left\| \left(\prod_{v=1}^{ln} w_{f^{-v}(i)} \right) e_{f^{-ln}(i)} \right\| &= \left\| \frac{1}{2x_i} 2 \left(\prod_{v=1}^{ln} w_{f^{-v}(i)} \right) x_i e_{f^{-ln}(i)} \right\| \\ &\leq \left\| 2 \left(\prod_{v=1}^{ln} w_{f^{-v}(i)} \right) x_i e_{f^{-ln}(i)} \right\| < \varepsilon. \end{aligned} \tag{3.11}$$

This completes the proof of the implication (1)$\Longrightarrow$(2).
Conversely, fix $i \in I$, $m \in \mathbb{N}$ and suppose that there is an increasing sequence of positive integers $(n_k)_k$ satisfying the both inequalities of (2). We consider the linear mapping $S_n : X \to X$ given by

$$S_n(e_i) = \left(\prod_{v=0}^{n-1} w_{f^v(i)} \right)^{-1} e_{f^n(i)} \quad (i \in I).$$

According to the first inequality of (2) we have

$$y_k := \sum_{l=1}^{m} S_{ln_k}(e_i) = \sum_{l=1}^{m} \left(\prod_{v=0}^{ln_k-1} w_{f^v(i)} \right)^{-1} e_{f^{ln_k}(i)} \underset{k\to+\infty}{\longrightarrow} 0.$$

Set $x_k := y_k + e_i$. We have

$$x_k - e_i = y_k \underset{k\to+\infty}{\longrightarrow} 0. \tag{3.12}$$

On the other hand, by virtue of the second inequality of (2) we have for any $l \in \{1, 2, \dots, m\}$

$$T^{ln_k} e_i = \left(\prod_{v=1}^{ln_k} w_{f^{-v}(i)} \right) e_{f^{-ln_k}(i)} \underset{k\to+\infty}{\longrightarrow} 0.$$

Submitted: April 19, 2023
Accepted: February 27, 2024
Published (early view): September 2, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4120.

Notice that by hypothesis we obtain that for any $1 \leq s < l \leq m$,

$$T^{ln_k} S_{sn_k} e_i = \left(\prod_{v=0}^{sn_k-1} w_{f^v(i)}\right)^{-1} \left(\prod_{v=1}^{ln_k} w_{f^{-v}(f^{sn_k}(i))}\right) e_{f^{(s-l)n_k}(i)} \underset{k\to+\infty}{\longrightarrow} 0$$

and

$$T^{sn_k} S_{ln_k} e_i = \left(\prod_{v=0}^{ln_k-1} w_{f^v(i)}\right)^{-1} \left(\prod_{v=1}^{sn_k} w_{f^{-v}(f^{ln_k}(i))}\right) e_{f^{(l-s)n_k}(i)} \underset{k\to+\infty}{\longrightarrow} 0.$$

Since $T^{ln_k} S_{ln_k} e_i = e_i$, we conclude that for any $l = 1, \ldots, m$

$$T^{ln_k} y_k = T^{ln_k} (\sum_{j=1}^{m} S_{jn_k}(e_i)) \underset{k\to+\infty}{\longrightarrow} e_i.$$

Finally

$$T^{ln_k} x_k - e_i = T^{ln_k} e_i + T^{ln_k} y_k - e_i \underset{k\to+\infty}{\longrightarrow} 0. \tag{3.13}$$

According to (3.12), (3.13) and Theorem 2.4, it follows that T is topologically multiply recurrent. □

The next example shows that the equivalence stated in the previous theorem fails without the assumption of no periodic points for f nor without the run-away assumption for the sequence of maps f^n.

Example 3.6. Let $T_{w,f} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the weighted pseudo-shift with weights $(w_i)_{i\in\mathbb{Z}}$, where $w_i = \lambda$ for each $i \in \mathbb{Z}$ such that $|\lambda| = 1$ and the shift mapping $f := Id_{\mathbb{Z}}$. According to Examples 3.3, $(f^n)_{n\in\mathbb{N}}$ is not run-away. Furthermore, $T_{w,f}$ is topologically multiply recurrent, yet it does not fulfill the inequalities stated in (2) of the previous theorem. Indeed, let $m \in \mathbb{N}$ and $\varepsilon > 0$ and let U be a non-empty open subset of the complex space $\ell^2(\mathbb{Z})$ and $x \in U$. Since $|\lambda^l| = 1$ for any $l \in \{1, 2, \ldots, m\}$, the set $\{n \geq 0 : \forall l \in \{1, 2, \ldots, m\} |\lambda^{ln} - 1| < \varepsilon\}$ has bounded gaps. Then, there exists a strictly increasing sequence of positive integers $(n_k)_k$ such that $\lambda^{ln_k} \longrightarrow 1$ as $k \longrightarrow \infty$ for any $l \in \{1, 2, \ldots, m\}$. Hence, $T^{ln_k}_{w,f} x \longrightarrow x$ as $k \longrightarrow \infty$. Which means that $\bigcap_{l=1}^m T^{ln_k}_{w,f} U \cap U \neq \emptyset$. Then, $T_{w,f}$ is topologically multiply recurrent.
However, for every $n \in \mathbb{N}$ we have

$$\begin{cases} \left\| \left(\prod_{v=0}^{n-1} w_{f^v(i)}\right)^{-1} e_{f^n(i)} \right\| = |\left(\prod_{i=0}^{n-1} w_i\right)^{-1}| = 1, \\ \left\| \left(\prod_{v=1}^{n} w_{f^{-v}(i)}\right) e_{f^{-n}(i)} \right\| = |\prod_{i=1}^{n} w_i| = 1. \end{cases} \tag{3.14}$$

In the following, we consider the special case of Theorem 3.5 where $m = 1$. This leads us to a characterization of recurrent or hypercyclic pseudo-shifts in terms of conditions on the weights which extends [9, Proposition 5.1] and [12, Theorem 5].

Submitted: April 19, 2023
Accepted: February 27, 2024
Published (early view): September 2, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version

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Corollary 3.7. *Let T be weighted pseudo-shift that acts on X with weights $(w_i)_{i\in I}$. If the mapping $f : I \to I$ has no periodic points. Then the following assertions are equivalent:*

1. *T is recurrent;*
2. *T is hypercyclic;*
3. *For every $\varepsilon > 0$ and every strictly positive integer N_0 there exists an integer $n > N_0$ such that for every $i \in I$*

$$\|w_{n,i}^{-1} e_{f^n(i)}\| < \varepsilon \quad \textit{and} \quad \|w_{n,f^{-n}(i)} e_{f^{-n}(i)}\| < \varepsilon.$$

Proof. Suppose that T is recurrent. Arguing as in the proof of Theorem 3.5 with $m = 1$, we obtain that for every $\varepsilon > 0$ and every N_0 there is $n > N_0$ such that for any $i \in I$

$$\begin{cases} \|w_{n,i}^{-1} e_{f^n(i)}\| < \varepsilon, \\ \|w_{n,f^{-n}(i)} e_{f^{-n}(i)}\| < \varepsilon. \end{cases}$$

Hence we have the proof of (1) ⇒ (3). By [12, Theorem 5] we have that (3) implies (2). Finally the implication (2) ⇒ (1) holds trivially and this completes the proof of the equivalence of statements (1)−(3) of the theorem. □

A bilateral weighted backward shift is an example of pseudo-shift with the strictly increasing map $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ given by $f(n) = n + 1$. Utilizing this observation, the characterization of topologically multiply recurrent backward shifts in terms of the weights established by G. Costakis et al. [9] is now a consequence of Theorem 3.5.

Proposition 3.8. [9, Proposition 5.3.] *Let $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be a bilateral weighted shift with weight sequence $(w_n)_{n\in\mathbb{Z}}$. The following are equivalent:*

1. *T is topologically multiply recurrent;*
2. *For every $m \in \mathbb{N}$ the operator $T \oplus T^2 \oplus \cdots \oplus T^m$ is hypercyclic on X^m;*
3. *For every $m \in \mathbb{N}$ the operators $T, T^2, \ldots, T^m$ are densely d-hypercyclic;*
4. *For every $m, q \in \mathbb{N}$ and every $\varepsilon > 0$ there exists a positive integer $n := n(m, q, \varepsilon)$ such that for every integer j with $|j| \le q$ and every $l = 1, \ldots, m$*

$$\prod_{i=0}^{\ln -1} w_{j+i} > \frac{1}{\varepsilon} \quad \textit{and} \quad \prod_{i=1}^{\ln} w_{j-i} < \varepsilon.$$

Proof. Let m and q be positive integers and let $\varepsilon > 0$. Suppose that T is topologically multiply recurrent. Then there exists a positive ineteger n such that for every integer j with $|j| \le q$ and every $l = 1, \ldots, m$

$$\prod_{i=0}^{\ln -1} w_{j+i} > \frac{1}{\varepsilon} \quad \text{and} \quad \prod_{i=1}^{\ln} w_{j-i} < \varepsilon.$$

Submitted: April 19, 2023
Accepted: February 27, 2024
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Notice that

$$\begin{cases} \left\| \left(\prod_{v=0}^{ln-1} w_{f^v(i)} \right)^{-1} e_{f^{ln}(i)} \right\| = \left(\prod_{i=0}^{ln-1} w_{j+i} \right)^{-1} < \varepsilon, \\ \left\| \left(\prod_{v=1}^{ln} w_{f^{-v}(i)} \right) e_{f^{-ln}(i)} \right\| = \prod_{i=1}^{ln} w_{j-i} < \varepsilon. \end{cases} \tag{3.15}$$

□

If each $i \in I$ lies outside $f^n(I)$ for all sufficiently large n, then the following statement can be considered as a special case of Theorem 3.5.

Theorem 3.9. *Let X be an F-sequence space over I, in which $(e_i)_{i\in I}$ is an OP-basis. Let $T = T_{w,f} : X \to X$ be a weighted pseudo-shift with weight sequence $(w_i)_{i\in I}$, so that each $i \in I$ lies outside $f^n(I)$ for all sufficiently large n. If the mapping $f : I \to I$ has no periodic points. Then the following assertions are equivalent:*

(1) *T is topologically multiply recurrent;*
(2) *For every positive integer m and every $\varepsilon > 0$, there exists an increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that for every $i \in I$ and every $l \in \{1, \dots, m\}$*

$$\left\| \left(\prod_{v=0}^{ln_k-1} w_{f^v(i)} \right)^{-1} e_{f^{ln_k}(i)} \right\| < \varepsilon.$$

Proof. Suppose that T is topologically multiply recurrent. Since each $i \in I$ lies outside $f^n(I)$ for all sufficiently large n, it follows that for every $i \in I$, there exists a positive integer N_i such that $e_{f^{-n}(i)} = 0$ when $n > N_i$. Then the inequality $\left\| \prod_{v=1}^{ln} w_{f^{-v}(i)} e_{f^{-ln}(i)} \right\| < \varepsilon$ is always satisfied. Hence by virtue of Theorem 3.5 the result follows. □

Remark 3.10. Note that every unilateral weighted shift is topologically multiply recurrent.

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Submitted: April 19, 2023
Accepted: February 27, 2024
Published (early view): September 2, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version

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Submitted: April 19, 2023
Accepted: February 27, 2024
Published (early view): September 2, 2024