Revista de la Unión Matemática Argentina Accepted article · Early view version

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FROBENIUS PROPERTY FOR FUSION CATEGORIES OF DIMENSION 120

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Abstract. In this paper we prove that fusion categories of Frobenius-Perron dimensions 120 are of Frobenius type. Combining this with known results in the literature, we get that all weakly integral fusion categories of Frobenius-Perron dimension less than 126 are of Frobenius type.

1. Introduction

There is a classical result in group theory that the dimension of a simple module of a finite group divides the order of the group. This result was first proved by Frobenius. In honor of Frobenius for his work, we call a fusion category $\mathcal{C}$ is of Frobenius type if, for every simple object X of $\mathcal{C}$, the Frobenius-Perron dimension of X divides the Frobenius-Perron dimension of $\mathcal{C}$, i.e., the ratio $\frac{\mathrm{FPdim}(\mathcal{C})}{\mathrm{FPdim}(X)}$ is an algebraic integer. In [7, Appendix 2], Kaplansky conjectured that the representation category of a finite dimensional semisimple Hopf algebra is of Frobenius type. Although some results on the conjecture have been obtained, the conjecture is still open.

In [5], the authors introduced the notion of a weakly group-theoretical fusion category and proved that this class of fusion categories have the Frobenius property, see [5, Theorem 1.5].

In [1], Dong, Natale and Vendramin proved that fusion categories of dimension 84 and 90 are of Frobenius type. Combining the results in [5] they obtained that every weakly integral fusion category of Frobenius-Perron dimension less than 120 is of Frobenius type.

In the present paper, we prove that a fusion category of dimension 120 is also of Frobenius type. Together with the results in the literature, we obtain that every weakly integral fusion category of Frobenius-Perron dimension less than 126 is of Frobenius type.

The paper is organized as follows. In Section 2, we recall some basic definitions and results on fusion categories. Some of them have appeared in the context of category of representations of a semisimple Hopf algebra. We also get some useful lemmas in this section. In section 3, we prove our main result on fusion categories of dimension 120.

2020 *Mathematics Subject Classification.* 18M20, 16T05.

Key words and phrases. Fusion category; Frobenius property; Frobenius-Perron dimension.

Submitted: August 16, 2023
Accepted: May 14, 2024
Published (early view): September 3, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version

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Throughout this paper, we shall work over an algebraically closed field k of characteristic 0. We refer to [3] for the main notions about fusion categories.

2. Preliminaries

Let $\mathcal{C}$ be a fusion category over k and let $\mathrm{Irr}(\mathcal{C})$ be the set of isomorphism classes of simple objects of $\mathcal{C}$. Then the set $\mathrm{Irr}(\mathcal{C})$ is a basis of the Grothendieck ring $K_0(\mathcal{C})$ of $\mathcal{C}$.

The Frobenius-Perron dimension $\mathrm{FPdim}(X)$ of $X \in \mathrm{Irr}(\mathcal{C})$ is the largest eigenvalue of the matrix of left multiplication by X in $K_0(\mathcal{C})$ with respect to the basis $\mathrm{Irr}(\mathcal{C})$. The Frobenius-Perron dimension of $\mathcal{C}$ is defined as the number

$$\mathrm{FPdim}(\mathcal{C}) = \sum_{X \in \mathrm{Irr}(\mathcal{C})} \mathrm{FPdim}(X)^2.$$

A fusion category is weakly integral if $\mathrm{FPdim}(\mathcal{C})$ is an integer. If $\mathrm{FPdim}(X)$ is an integer for every $X \in \mathrm{Irr}(\mathcal{C})$ then $\mathcal{C}$ is integral.

A fusion subcategory of $\mathcal{C}$ is a full tensor subcategory $\mathcal{D}$ such that if X is an object of $\mathcal{C}$ isomorphic to a direct summand of an object Y of $\mathcal{D}$, then X is in $\mathcal{D}$. By [4, Proposition 8.15], if $\mathcal{D}$ is a fusion subcategory of $\mathcal{C}$ then $\mathrm{FPdim}(\mathcal{D})$ divides $\mathrm{FPdim}(\mathcal{C})$, i.e. $\frac{\mathrm{FPdim}(\mathcal{C})}{\mathrm{FPdim}(\mathcal{D})}$ is an algebraic integer.

A simple object X of $\mathcal{C}$ is invertible if $\mathrm{FPdim}(X) = 1$. We use $G(\mathcal{C})$ to denote the group of isomorphism classes of invertible simple objects of a fusion category $\mathcal{C}$. All invertible simple objects of $\mathcal{C}$ generate a fusion subcategory $\mathcal{C}_{pt}$ of $\mathcal{C}$. It is the largest pointed fusion subcategory of $\mathcal{C}$. A fusion category is pointed if all simple objects are invertible.

Let $1 = d_0 < d_1 < \cdots < d_s$ be positive real numbers and $n_0, n_1, \cdots, n_s$ be positive integers. We say $\mathcal{C}$ is of type $(d_0, n_0; d_1, n_1; \cdots, d_s, n_s)$ if n_i is the number of the non-isomorphism simple objects of Frobenius-Perron dimension d_i, for all i.

Lemma 2.1 and Lemma 2.2 below are proved by Nichols and Richmond in the setting of semisimple Hopf algebra. Their proofs also work in the fusion category setting because their proofs only make use of the properties of the Grothendieck ring.

Lemma 2.1. [8, Theorems 9,10] *Let $\mathcal{C}$ be a fusion category and $X \in \mathrm{Irr}(\mathcal{C})$. Let $m(X, Y) = \dim \mathrm{Hom}_{\mathcal{C}}(X, Y)$ denote the multiplicity of X in an object Y. Then*

(1) $m(X, Y \otimes Z) = m(Y^*, Z \otimes X^*) = m(Y, X \otimes Z^*)$ and $m(X, Y) = m(X^*, Y^*)$.

(2) Assume $Y \in \mathrm{Irr}(\mathcal{C})$ and $g \in G(\mathcal{C})$. Then $m(g, X \otimes Y) = 1$ if $Y = X^* \otimes g$, otherwise $m(g, X \otimes Y) = 0$.

In particular, part (2) implies that $m(g, X \otimes Y) = 0$ if $\mathrm{FPdim}(X) \neq \mathrm{FPdim}(Y)$. Moreover, $m(g, X \otimes X^*) > 0$ if and only if $m(g, X \otimes X^*) = 1$ if and only if $g \otimes X = X$.

The set of isomorphism classes of invertible simple objects in the decomposition of $X \otimes X^*$ will be denoted by $G[X]$. It is a subgroup of $G(\mathcal{C})$ whose order divides $\mathrm{FPdim}(X)^2$, see [1, Lemma 2.2].

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Accepted: May 14, 2024
Published (early view): September 3, 2024

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Lemma 2.2. [8, Theorem 11] *Assume that the integral fusion category $\mathcal{C}$ contains a 2-dimensional simple object X. Then one of the following holds:*

(1) $G[X] \neq \{1\}$.

(2) $\mathcal{C}$ has a fusion subcategory $\mathcal{D}$ of type $(1,2;2,1;3,2)$ such that $X \notin \mathrm{Irr}(\mathcal{D})$. Moreover, $\mathcal{D}$ has an invertible simple object g of order 2 such that $g \otimes X \neq X$.

(3) $\mathcal{C}$ has a fusion subcategory of type $(1,3;3,1)$ or $(1,1;3,2;4,1;5,1)$.

In particular, if $G[X] = \{1\}$ then $\mathcal{C}$ has a fusion subcategory of dimension $12, 24$ or 60.

A fusion category $\mathcal{C}$ is called a G-extension of a fusion category $\mathcal{D}$ if it has a faithful grading $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$ such that the tensor product of $\mathcal{C}$ maps $\mathcal{C}_g \times \mathcal{C}_h$ to $\mathcal{C}_{gh}$, $(\mathcal{C}_g)^* = \mathcal{C}_{g^{-1}}$ and the trivial component $\mathcal{C}_e$ is equivalent to $\mathcal{D}$.

It is known that any fusion category $\mathcal{C}$ has a canonical faithful grading $\mathcal{C} = \oplus_{g\in\mathcal{U}(\mathcal{C})}\mathcal{C}_g$ whose trivial component $\mathcal{C}_e$ is the adjoint subcategory $\mathcal{C}_{ad}$ which is generated by all simple objects in $X \otimes X^*$, $X \in \mathrm{Irr}(\mathcal{C})$. The group $\mathcal{U}(\mathcal{C})$ is called the universal grading group of $\mathcal{C}$, see [6].

Let $\mathcal{C}$ be a fusion category and let $\mathcal{Z}(\mathcal{C})$ be its Drinfeld center. Consider the group homomorphism $F_0 : G(\mathcal{Z}(\mathcal{C})) \to G(\mathcal{C})$ induced by the forgetful tensor functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$. Let N be the kernel of F_0. By [1, Lemma 2.1], $\mathcal{C}$ is faithful graded by the group $\hat{N}$. Moreover, if $\mathcal{U}(\mathcal{C})$ is trivial then the group homomorphism F_0 is injective. These results implies the lemma below.

Lemma 2.3. *Let $I : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ be the right adjoint functor of the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$. If $I(\mathbf{1})$ contains a nontrivial invertible simple object g then $\mathcal{C}$ has a nontrivial faithful grading. Moreover, $\mathcal{Z}(\mathcal{C})$ contains a nontrivial Tannakian subcategory.*

Proof. Since I is the right adjoint functor of F, we have $0 \neq \mathrm{Hom}_{\mathcal{C}}(F(g), \mathbf{1}) = \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(g, I(\mathbf{1}))$. Hence $F(g) = \mathbf{1}$. Thus the kernel of $F_0 : G(\mathcal{Z}(\mathcal{C})) \to G(\mathcal{C})$ is not trivial. By [1, Lemma 2.1], $\mathcal{C}$ is faithfully graded by some finite group. Then $\mathcal{Z}(\mathcal{C})$ contains a nontrivial Tannakian subcategory by [5, Proposition 2 9(ii)]. □

For the the existence and the structure of the right adjoint to the forgetful functor, the reader is directed to [5, Section 3]. The following lemma will be frequently used in our proof. It is contained in the proof of [5, Lemma 9.17].

Lemma 2.4. *Let $\mathcal{C}$ be an integral fusion category. If $\mathcal{C}$ has a fusion subcategory $\mathcal{D}$ then $I(\mathbf{1})$ has a subalgebra $\mathcal{B}$ corresponding to $\mathcal{D}$ such that $\mathrm{FPdim}(\mathcal{B}) = \mathrm{FPdim}(\mathcal{C}) / \mathrm{FPdim}(\mathcal{D})$.*

Lemma 2.5. *Let $\mathcal{C}$ be a fusion category. If the Drinfeld center $\mathcal{Z}(\mathcal{C})$ has a nontrivial symmetric category $\mathcal{E}$ and the order of $G(\mathcal{C})$ is odd, then $\mathcal{Z}(\mathcal{C})$ has a nontrivial Tannakian subcategory*

Proof. The proof is by considering the universal grading of $\mathcal{C}$. If $\mathcal{C}$ has a nontrivial universal grading then [5, Proposition 2.9(ii)] shows that $\mathcal{Z}(\mathcal{C})$ contains a nontrivial Tannakian subcategory.

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If $\mathcal{C}$ has a trivial universal grading then the group homomorphism $F_0 : G(\mathcal{Z}(\mathcal{C})) \to G(\mathcal{C})$ is injective by [1, Lemma 2.1]. Hence the order of $G(\mathcal{Z}(\mathcal{C}))$ is odd since the order of $G(\mathcal{C})$ is odd. It follow that $\mathcal{Z}(\mathcal{C})$ can not have fusion subcategory of dimension 2. This implies that the dimension of $\mathcal{E}$ is bigger than 2. Thus $\mathcal{E}$ contains a nontrivial Tannakian subcategory by [2, Corollary 2.50]. $\square$

Lemma 2.6. *Assume that $\mathcal{C}$ has type $(1, n_1; d_2, n_2; \cdots, d_s, n_s)$. Then $\mathcal{C}$ has a fusion subcategory of type $(1, n_1; d_2, n_2; \cdots, d_k, n_k)$ if one of the following holds:*
(1) $d_k^2 < d_{k+1}$;
(2) $d_k^2 = d_{k+1}$ and $G[X] \cap G[Y] \neq \{\mathbf{1}\}$ for all simple object X and Y of dimension d_k.

Proof. (1) The assumption $d_k^2 < d_{k+1}$ means that the tensor product of two simple objects of dimension $\leq d_k$ is a sum of simple objects of dimension $\leq d_k$. Hence all simple objects of dimension $\leq d_k$ generates a fusion subcategory.

(2) By [1, Lemma 2.5], $G[X] \cap G[Y] \neq \{\mathbf{1}\}$ means that the tensor product of two simple objects of dimension d_k can not be a simple object, and hence it is a sum of simple objects of dimension $\leq d_k$. Hence, all simple objects of dimension $\leq d_k$ generate a fusion subcategory. $\square$

3. Fusion categories of dimension 120

Lemma 3.1. *Let $\mathcal{C}$ be an integral fusion category of dimension 120. If the Drinfeld center $\mathcal{Z}(\mathcal{C})$ has a nontrivial Tannakian subcategory $\mathrm{Rep}(G)$ then $\mathcal{C}$ is weakly group-theoretical. In particular, $\mathcal{C}$ has the Frobenius Property.*

Proof. If the dimension of $\mathrm{Rep}(G)$ is a power of 2 then G is a solvable group. Hence $\mathrm{Rep}(G)$ has a subcategory $\mathrm{Rep}(H)$ of dimension 2. Under the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$, the image of $\mathrm{Rep}(H)$ is either Vec or $\mathrm{Rep}(H)$, where Vec is the trivial category. Then $\mathcal{C}$ is an H-extension of a fusion category of dimension 60 or $\mathcal{C}$ is an H-equivariantization of a fusion category of dimension 60, see [5, Propositions 2.9, 2.10]. By [5, Theorem 9.16], a fusion category of dimension 60 is weakly group-theoretical. Hence $\mathcal{C}$ is weakly group-theoretical by [5, Proposition 4.1].

If the dimension of $\mathrm{Rep}(G)$ has prime factor 3 or 5 then we consider the de-equivariantization $\mathcal{Z}(\mathcal{C})_G$ of $\mathcal{Z}(\mathcal{C})$ by $\mathrm{Rep}(G)$. Set $\mathcal{D} = \mathcal{Z}(\mathcal{C})_G$. Then $\mathcal{D} = \oplus_{g \in G} \mathcal{D}_g$ is faithfully graded by G. The dimension of the trivial component $\mathcal{D}_e$ is $\frac{120}{|G|^2}$, see [2, Proposition 4.56]. Under our assumption, $\mathrm{FPdim}(\mathcal{D}_e)$ has at most 2 prime factors. Hence $\mathcal{D}_e$ is solvable by [5, Theorem 1.6]. It follows that $\mathcal{D}_e$ is also weakly group-theoretical. Thus $\mathcal{C}$ is weakly group-theoretical by [5, Proposition 4.1]. $\square$

Lemma 3.2. *Let $\mathcal{C}$ be an integral fusion category of dimension 120. Assume that $\mathcal{C}$ has a fusion subcategory $\mathcal{D}$ of dimension ≥ 4. Then $\mathcal{Z}(\mathcal{C})$ has a nontrivial symmetric subcategory.*

Proof. By Lemma 2.4, $I(\mathbf{1})$ contains a subalgebra B corresponding to the fusion subcategory $\mathcal{D}$ such that $\mathrm{FPdim}(B) = \frac{\mathrm{FPdim}(\mathcal{C})}{\mathrm{FPdim}(\mathcal{D})} \leq 30$. By Lemma 2.3, we may assume that B contains no nontrivial invertible simple objects.

Submitted: August 16, 2023
Accepted: May 14, 2024
Published (early view): September 3, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version

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In view of [5, Theorem 2.11], the Frobenius-Perron dimensions of simple objects of $\mathcal{Z}(\mathcal{C})$ divide 120. Hence the possible decomposition of B as an object of $\mathcal{Z}(\mathcal{C})$ shows that B must contains simple objects with prime power dimension. Then $\mathcal{Z}(\mathcal{C})$ contains a nontrivial symmetric subcategory by [2, Corollary 7.2]. □

Theorem 3.3. *Let $\mathcal{C}$ be an integral fusion category of dimension 120. Then $\mathcal{C}$ has the Frobenius property.*

Proof. Assume that $\mathcal{C}$ does not have the Frobenius property. Then $\mathcal{C}$ has simple objects of dimension 7 or 9. The result then follows from the Lemma 3.4 and Lemma 3.5 below. □

Lemma 3.4. *Let $\mathcal{C}$ be an integral fusion category of dimension 120. Then $\mathcal{C}$ can not have simple objects of dimension 7.*

Proof. Assume on the contrary that $\mathcal{C}$ has simple objects of dimension 7. Then $\mathcal{C}$ has one of following types:
(1,1;3,1;5,1;6,1;7,1),(1,1;3,2;4,1;6,1;7,1), (1,1;3,5;5,1;7,1),(1,1;3,6;4,1;7,1),
(1,2;2,1;4,1;7,2),(1,1;2,m;$\cdots$;7,1).

Type $(1,1;3,1;5,1;6,1;7,1)$: Let X_i be simple object of dimension i, where $i = 3,5,6,7$. Then $X_3 \otimes X_3 = \mathbf{1} \oplus X_3 \oplus X_5$. From $m(X_5, X_3 \otimes X_3) = m(X_3, X_5 \otimes X_3) = 1$, we can write

$$\text{(i)}\ \ X_5 \otimes X_3 = X_3 \oplus 2X_6 \ \text{ or }\ \text{(ii)}\ \ X_5 \otimes X_3 = X_3 \oplus X_5 \oplus X_7.$$

Case (i): From $(X_6, X_5 \otimes X_3) = m(X_5, X_6 \otimes X_3) = 2$, we can write $X_6 \otimes X_3 = 2X_5 \oplus W$, where W does not contain simple objects of dimension 5. In other words, W is a direct sum of simple objects of dimension $3,6$ or 7. It is impossible since $\mathrm{FPdim}(W) = 8$.

Case (ii): From $(X_7, X_5 \otimes X_3) = m(X_5, X_7 \otimes X_3) = 1$, we can write $X_7 \otimes X_3 = X_5 \oplus W$, where W does not contain simple objects of dimension 5. In addition, W does not contain simple objects of dimension 3. In fact, if $m(X_3, X_7 \otimes X_3) \geq 1$ then $m(X_7, X_3 \otimes X_3) \geq 1$. This contradicts the decomposition of $X_3 \otimes X_3$. Hence W is a direct sum of simple objects of dimension 6 or 7. It is also impossible since $\mathrm{FPdim}(W) = 16$.

Type $(1,1;3,2;4,1;6,1;7,1)$: Let X_3 be simple object of dimension 3. Then $X_3 \otimes X_3^* = \mathbf{1} \oplus 2X_4$, where X_4 is the unique simple object of dimension 4. From $m(X_4, X_3 \otimes X_3^*) = m(X_3, X_4 \otimes X_3) = 2$, we get $X_4 \otimes X_3 = 2X_3 + X_6$, where X_6 is the unique simple object of dimension 6. From $m(X_6, X_4 \otimes X_3) = m(X_4, X_6 \otimes X_3^*) = 1$, we get $X_6 \otimes X_3^* = 2X_7 + X_4$, where X_7 is the unique simple object of dimension 7. From $m(X_7, X_6 \otimes X_3^*) = m(X_6, X_7 \otimes X_3) = 2$, we can write $X_7 \otimes X_3 = 2X_6 + W$, where $\mathrm{FPdim}(W) = 9$. The possible decomposition of W is $W = aX_3 + bX_3'$, where $a+b = 3$ and X_3' is another simple object of dimension 3. But $a = m(X_3, X_7 \otimes X_3) = m(X_7, X_3 \otimes X_3^*) = 0$ and $b = m(X_3', X_7 \otimes X_3) = m(X_7, X_3' \otimes X_3^*) \leq 1$, which contradicts $a + b = 3$.

Submitted: August 16, 2023
Accepted: May 14, 2024
Published (early view): September 3, 2024

Revista de la Unión Matemática Argentina Accepted article · Early view version
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Type $(1,1;3,5;5,1;7,1)$: Let X_3 be a simple object of dimension 3. Then $X_3 \otimes X_3^*$ has the unique possible decomposition;

$$X_3 \otimes X_3^* = \mathbf{1} \oplus X_3' \oplus X_5.$$

where X_3' is a simple object of dimension 3 and X_5 is the unique simple object of dimension 5.

From $m(X_5, X_3 \otimes X_3^*) = (X_3, X_5 \otimes X_3) = 1$, we have $X_5 \otimes X_3 = X_3 \oplus X$, where FPdim($X$)=12. If X contain a simple object X_3'' of dimension 3 then $m(X_3'', X_5 \otimes X_3) = (X_5, X_3'' \otimes X_3^*) = 1$. This shows that $X_3'' \otimes X_3^* = X_5 \oplus Y$, where FPdim($Y$)=4, which implies that Y must contain the unique invertible simple object $\mathbf{1}$. This is impossible since $X_3'' \neq X_3$. Hence we have

$$X_5 \otimes X_3 = X_3 \oplus X_5 \oplus X_7,$$

where X_7 is the unique simple object of dimension 7. Without loss of generality, we get $X_5 \otimes Z = Z \oplus X_5 \oplus X_7$ for any simple object Z of dimension 3. From $m(X_5, X_5 \otimes Z) = m(X_5, Z^* \otimes X_5) = m(Z^*, X_5 \otimes X_5^*) = m(Z, X_5 \otimes X_5^*) = 1$, we know

$$X_5 \otimes X_5^* = \mathbf{1} \oplus X_3^1 \oplus X_3^2 \oplus X_3^3 \oplus X_3^4 \oplus X_3^5 \oplus U$$

where FPdim(U)=9 and X_3^1,X_3^2,X_3^3,X_3^4,X_3^5 are all simple object of dimension 3. This is impossible since U is a direct sum of simple objects of dimension 5 or 7.

Type $(1,1;3,6;4,1;7,1)$: Let X_3 be a simple object of dimension 3. Then

$$X_3 \otimes X_3^* = \mathbf{1} \oplus 2X_4,$$

where X_4 is the unique simple object of dimension 4.

From $m(X_4, X_3 \otimes X_3^*) = m(X_3, X_4 \otimes X_3) = 2$, we have$X_4 \otimes X_3 = 2X_3 \oplus X$, where FPdim($X$)=6 and X is a direct sum of simple objects of dimension 3. Let X_3' be a summand of X. Then $m(X_3', X_4 \otimes X_3) = m(X_4, X_3' \otimes X_3^*) \leq 2$, which shows that $X_3' \otimes X_3^* = aX_4 \oplus Y$ where $a = 1$ or 2, FPdim(Y)=9-4a. This is impossible since $X_3' \otimes X_3^*$ can not contain $\mathbf{1}$.

Type $(1,2;2,1;4,1;7,2)$: Let X_2 and X_4 be simple objects of dimension 2 and 4, respectively. Then $X_2 \otimes X_4 = 2X_4$ which means that $X_4 \otimes X_4^* = \mathbf{1} \oplus g \oplus 2X_2 \oplus X$, where $G(\mathcal{C}) = \{1, g\}$, FPdim($X$)=10. This is impossible since X is a direct sum of simple objects of dimension of 4 or 7.

Type $(1,1;2,m;\cdots;7,1)$: By Theorem 2.2 $\mathcal{C}$ has a fusion subcategory of dimension 6 or 12 or 60. It follows from Lemma 3.2 that $\mathcal{Z}(\mathcal{C})$ contains a nontrivial symmetric subcategory. By Lemma 2.5, $\mathcal{Z}(\mathcal{C})$ has a Tannakian subcategory. Then $\mathcal{C}$ is weakly group-theoretical by Lemma 3.1. By [5, Theorem 1.2], $\mathcal{C}$ has the Frobenius property. This is a contradiction. ☐

Lemma 3.5. *Let $\mathcal{C}$ be an integral fusion category of dimension 120. Then $\mathcal{C}$ can not have simple objects of dimension 9.*

Proof. Assume on the contrary that $\mathcal{C}$ has a simple objects of dimension 9. Then $\mathcal{C}$ has one of following types:

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$(1,1;2,1;3,1;5,1;9,1)$,$(1,1;2,1;3,2;4,1;9,1)$, $(1,1;2,5;3,2;9,1)$, $(1,3;3,4;9,1)$,$(1,3;2,9;9,1)$, $(1,3;6,1;9,1)$.

Types $(1,1;2,1;3,1;5,1;9,1),(1,1;2,1;3,2;4,1;9,1),(1,1;2,5;3,2;9,1)$: $\mathcal{C}$ has a fusion subcategory of dimension $6, 12$ or 60, by Theorem 2.2. Then $\mathcal{Z}(\mathcal{C})$ has nontrivial symmetric subcategory by Lemma 3.2. Since the order of $G(\mathcal{C})$ is odd, $\mathcal{Z}(\mathcal{C})$ has a nontrivial Tannakian subcategory by Lemma 2.5. Hence $\mathcal{C}$ is weakly group-theoretical by Lemma 3.1. By [5, Theorem 1.2], $\mathcal{C}$ has the Frobenius property, a contradiction.

Types $(1,3;3,4;9,1),(1,3;2,9;9,1)$: By Lemma 2.6, $\mathcal{C}$ has fusion subcategory of dimension 39 which does not divide 120. This is also impossible.

Type $(1,3;6,1;9,1)$: Let X_6 and X_9 be simple objects of dimension 6 and 9, respectively. Consider the action of $G(\mathcal{C})$ on $\mathrm{Irr}(\mathcal{C})$, we get $G[X_6]=G(\mathcal{C})$ since X_6 is the unique simple objects of dimension 6. Similarly, we have $G[X_9]=G(\mathcal{C})$. Then $X_6\otimes X_6=\mathbf{1}\oplus g_1\oplus g_2\oplus X_6\oplus 3X_9$, where $\{\mathbf{1},g_1,g_2\}=G(\mathcal{C})$. From $m(X_9,X_6\otimes X_6)=m(X_6,X_9\otimes X_6)=3$, we get $X_9\otimes X_6=3X_6+4X_9$. From $m(X_9,X_9\otimes X_6)=m(X_9,X_6\otimes X_9)=m(X_6,X_9\otimes X_9)=4$, we get $X_9\otimes X_9=\mathbf{1}\oplus g_1\oplus g_2\oplus 4X_6+6X_9$. Hence the matrices of left tensor product by g_1,g_2,X_6,X_9 are

$$M_{g_1}=\begin{pmatrix}0&0&1&0&0\\1&0&0&0&0\\0&1&0&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},M_{g_2}=\begin{pmatrix}0&1&0&0&0\\0&0&1&0&0\\1&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},$$

$$M_{X_6}=\begin{pmatrix}0&0&0&1&0\\0&0&0&1&0\\0&0&0&1&0\\1&1&1&1&3\\0&0&0&3&4\end{pmatrix},M_{X_9}=\begin{pmatrix}0&0&0&1&0\\0&0&0&1&0\\0&0&0&1&0\\1&1&1&1&4\\0&0&0&4&6\end{pmatrix}.$$

Let $A=I_5+M_{g_1}M_{g_2}+M_{g_2}M_{g_1}+M_{X_6}^2+M_{X_9}^2$. The eigenvalues of A are $120,8,5,3,3$. These eigenvalues are called the formal codegrees of $\mathcal{C}$ in [9].

It is known that $I(\mathbf{1})$ is an algebra in $\mathcal{Z}(\mathcal{C})$, and hence $\mathbf{1}$ is a summand of $I(\mathbf{1})$. Since $K(\mathcal{C})$ is commutative and semisimple, it either has five 1-dimensional irreducible representations, or has one trivial representation and one 2-dimensional irreducible representation. If the later case holds true then $I(\mathbf{1})$ contains 2 simple objects: one with multiplicity 1 and another one with multiplicity 2 by [9, Theorem 2.13]. But 3 different formal codegrees implies that $I(\mathbf{1})$ contains at least 3 simple objects with distinct dimensions, also by [9, Theorem 2.13]. Hence only former case holds true. It follows from [9, Theorem 2.13] that the object $I(\mathbf{1})$ is a sum of 5 simple objects and every object has multiplicity 1. So we can write

$$I(\mathbf{1})=\mathbf{1}\oplus A\oplus B\oplus C\oplus \mathcal{D}.$$

Again by [9, Theorem 2.13], we have

$$\mathrm{FPdim}(A)=15,\mathrm{FPdim}(B)=24,\mathrm{FPdim}(C)=\mathrm{FPdim}(D)=40.$$

Submitted: August 16, 2023
Accepted: May 14, 2024
Published (early view): September 3, 2024

Revista de la Unión Matemática Argentina **Accepted article · Early view version**

This peer-reviewed unedited article has been accepted for publication. The final copyedited version may differ in some details. Volume, issue, and page numbers will be assigned at a later stage. Cite using this DOI, which will not change in the final version: https://doi.org/10.33044/revuma.4390.

By [4, Proposition 5.4], we have

$$\begin{aligned} F(I(\mathbf{1})) &= F(\mathbf{1}) \oplus F(A) \oplus F(B) \oplus F(C) \oplus F(D) \\ &= \bigoplus_{T \in \mathrm{Irr}(\mathcal{C})} T \otimes \mathbf{1} \otimes T^* \\ &= 5 \cdot \mathbf{1} \oplus 2g_1 \oplus 2g_2 \oplus 5X_6 \oplus 9X_9. \end{aligned}$$

Assume $F(A) = 1 + a_1g_1 \oplus a_2g_2 \oplus a_3X_6 \oplus a_4X_9$, $F(B) = 1 + b_1g_1 \oplus b_2g_2 \oplus b_3X_6 \oplus b_4X_9$, $F(C) = 1 + c_1g_1 \oplus c_2g_2 \oplus c_3X_6 \oplus c_4X_9$, $F(D) = 1 + d_1g_1 \oplus d_2g_2 \oplus d_3X_6 \oplus d_4X_9$. Applying FPdim on both sides, we have a system of equations:

$$\begin{aligned} 1 + a_1 + a_2 + 6a_3 + 9a_4 &= 15, & a_1 + b_1 + c_1 + d_1 &= 2, \\ 1 + b_1 + b_2 + 6b_3 + 9b_4 &= 24, & a_2 + b_2 + c_2 + d_2 &= 2, \\ 1 + c_1 + c_2 + 6c_3 + 9c_4 &= 40, & a_3 + b_3 + c_3 + d_3 &= 5, \\ 1 + d_1 + d_2 + 6d_3 + 9d_4 &= 40, & a_4 + b_4 + c_4 + d_4 &= 9. \end{aligned}$$

It is easy to check that this system of equations does not have solutions. This completes the proof. □

Theorem 3.6. *Let $\mathcal{C}$ be a weakly integral fusion category of dimension less than* 126. *Then $\mathcal{C}$ has the Frobenius property.*

Proof. If $\mathrm{FPdim}(\mathcal{C}) = p^a q^b$ then $\mathcal{C}$ is solvable by [5, Theorem 1.6], where p, q are prime numbers, $a, b \geq 0$. If $\mathrm{FPdim}(\mathcal{C}) = pqr$, then either $\mathcal{C}$ is integral and thus group-theoretical [5, Theorem 9.2], or $\mathcal{C}$ is a $\mathbb{Z}_2$-extension of a fusion subcategory $\mathcal{D}$ by [6, Theorem 3.10]. We may assume that $p = 2$ and $\mathrm{FPdim}(\mathcal{D}) = qr$. Then $\mathcal{D}$ is solvable by [5, Theorem 1.6]. Hence $\mathcal{C}$ is solvable by [5, Proposition 4.5]. In all cases, $\mathcal{C}$ is weakly group-theoretical and hence has the Frobenius property.

By the main result of [1], every weakly integral fusion category of dimension less than 120 has the Frobenius property. It remains to consider the cases when $\mathrm{FPdim}(\mathcal{C}) = 120$. If $\mathcal{C}$ is integral then the result follows in this case from Theorems 3.3. If $\mathcal{C}$ is not integral then $\mathcal{C}$ is a G-extension of a fusion subcategory $\mathcal{D}$ by [6, Theorem 3.10], where G is an elementary abelian 2-group. Then $\mathrm{FPdim}(\mathcal{D}) = 60$, 30, or 15. Hence $\mathcal{C}$ is weakly group-theoretical by [5, Proposition 4.1]. Thus $\mathcal{C}$ has the Frobenius property by [5, Theorem 1.5]. □

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Submitted: August 16, 2023
Accepted: May 14, 2024
Published (early view): September 3, 2024

Revista de la Unión Matemática Argentina **Accepted article · Early view version**

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Submitted: August 16, 2023
Accepted: May 14, 2024
Published (early view): September 3, 2024