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A HIGH-ACCURACY COMPACT FINITE DIFFERENCE SCHEME FOR TIME-FRACTIONAL DIFFUSION EQUATIONS

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Abstract. In this paper, we propose a compact finite difference (CFD) scheme for the solution of time-fractional diffusion equation (TFDE) with the Caputo-Fabrizio derivative. The Caputo-Fabrizio derivative is discussed in time direction and is discretized by a special discrete scheme. The compact difference operator is introduced in space direction. We prove the unconditionally stability and convergence of proposed scheme. We show that the convergence order is $O(\tau^3 + h^4)$, where τ and h are the temporal stepsize and spatial stepsize, respectively. Our main purpose is to show that Caputo-Fabrizio derivative without singular term can improve the accuracy of the discrete scheme. Numerical examples are given to show the efficiency of the proposed scheme. It is illustrated that the numerical results are in good agreement with theoretical ones.

1. Introduction

In recent decades, fractional calculus has become more and more widely used in different engineering fields [1–4], so that the discussion of related fractional differential equations (FDEs) has become a hot spot for many scholars. There are various forms of fractional derivative; commonly used ones include Caputo, Riemann-Liouville (R-L) and Grünwald-Letnikov (G-L). For more details refer to [5–11]. We know that analytical solutions for fractional differential equations are difficult to obtain, however, there are still many scholars want to find these solutions [12–14]. Compared to analytical methods, numerical methods are more important in practical applications, the results of fractional differential equations using numerical methods can refer to these works [15–18].

The diffusion equation is a class of partial differential equation used to describe changes in the density of matter in diffusion phenomena. It is also commonly used to spread similar phenomena, such as the spread of alleles in population genetics. If we replace the integer order derivative in time direction with a fractional derivative of order α $(0 < \alpha < 1)$, then diffusion equation with time-fractional

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derivative can be obtained and which can more accurately and appropriately describe some anomalous diffusion phenomena. TFDEs are derived by considering continuous time random walk problems, which are in general non-Markovian processes. The physical interpretation of the fractional derivative is that it represents a degree of memory in the diffusing material [19, 20]. There are many results about the numerical solutions of TFDE. For example, in [21], Gorenflo et al. considered the numerical solution of TFDE in the fractional Sobolev spaces. Authors of [22] introduced some initial-boundary-value problems for the TFDE in open bounded one-dimensional domains. Authors of [23] proposed a new compact alternating direction implicit method for solving two dimensional time fractional diffusion equation with Caputo-Fabrizio (C-F) derivative.

C-F derivative was first presented by Caputo and Fabrizio [24]. This operator is important and interesting for describing the behavior of some complex physical materials. Another interesting aspect for C-F derivative is that it can provide a new perspectives for some areas of mechanical phenomena. This derivative is a promising differentiation operator and it has been widely applied to several models arising in many fields, such as biology, physics, control systems, material science, fluid dynamics and real-world problems [25, 26]. Up to now, there are many results about C-F derivative. In [27], a new discretization of Caputo-Fabrizio derivative was discussed. Authors of [28] introduced a numerical method for TFDEs based on C-F operator, in which a finite difference method and a spectral method were used. For the latest results about C-F derivative, one can refer to [29–35], among others.

CFD operator has good applications for solving FDEs. It has been studied and applied by many scholars. In [36], CFD method was used to obtain a fully discrete implicit scheme for the fractional diffusion equation. In [37], the compact difference scheme for distributed-order time-fractional diffusion-wave equation on bounded domains was considered by Ye et al.. Authors of [38] presented a class of new compact difference schemes, which were used for solving the fourth-order time fractional sub-diffusion equation. In [39], Wang and Vong studied CFD schemes for two types of fractional partial differential equations. In [40], a high-order accurate scheme by using compact difference operator was proposed for time-fractional advection-diffusion equations. Among the numerous numerical methods for solving FDEs, CFD proves to be an effective method for constructing high-order schemes.

Therefore, in this paper, we want to construct high-order numerical schemes for solving TFDE by CFD. The TFDE is as follows:

$$\begin{cases} {}_0^{CF}D_t^{\alpha} u(x,t) = a\dfrac{\partial^2 u}{\partial x^2} + f(x,t), & (x,t) \in (0,L)\times(0,T), \\ u(x,t)|_{t=0} = \varphi(x), & x \in \Omega = [0,L], \\ u(0,t) = u(L,t) = 0, & t \in [0,T], \end{cases} \tag{1.1}$$

where $a > 0$ represents the diffusion coefficient, L is the length of the space, T is the termination time. $f(x,t)$ and $\varphi(x)$ are all given and sufficiently smooth functions.

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We use ${}_{0}^{CF}D_t^{\alpha}$ to denote the C-F derivative [24], which is defined by

$$ {}_{0}^{CF}D_t^{\alpha}u(t) = \frac{1}{1-\alpha}\int_0^t u'(s)e^{-\frac{\alpha}{1-\alpha}(t-s)}\mathrm{d}s = \frac{1}{1-\alpha}\int_0^t u'(s)e^{-\varrho(t-s)}\mathrm{d}s, $$

where $\varrho = \frac{\alpha}{1-\alpha}$ and $0 < \alpha < 1$.

The rest of the paper is organized as follows. In Section 2, some notations will be given and a discrete scheme will be proposed for diffusion equation Eq. (1.1). The stability and error estimation of proposed discrete scheme will be discussed in Section 3. In Section 4, in order to confirm the efficiency and useful of the proposed discrete scheme, some numerical experiments will be considered. Finally, some conclusions are given in Section 5.

2. Construction of the numerical scheme for TFDE

In the present section, the construction of discrete scheme for Eq. (1.1) will be considered. For development of our discrete scheme, we provide some notations which will be used in this section and others. For given positive integers M and N, let $x_j = jh$, $j = 0, 1, \cdots, M$, where $h = L/M$ is space stepsize, and let $t_n = n\tau$, $n = 0, 1, \cdots, N$, where $\tau = T/N$ is time stepsize. We use $[N]$ to denote the set of $\{1, 2, \cdots, N\}$ and $[M]$ to denote the set of $\{1, 2, \cdots, M\}$. We define $u_j^n = u(x_j, t_n)$ and $f_j^n = f(x_j, t_n)$. $V_h = \{V|V = (V_0, V_1, \cdots, V_M), V_0 = V_M = 0\}$ denotes the grid function space. For any function $V, W \in V_h$, let

$$ \delta_x V_{j-\frac{1}{2}} = \frac{V_j - V_{j-1}}{h}, \; \delta_x V_{j+\frac{1}{2}} = \frac{V_{j+1} - V_j}{h}, $$

$$ \delta_x^2 V_j = \frac{\delta_x V_{j+\frac{1}{2}} - \delta_x V_{j-\frac{1}{2}}}{h} = \frac{V_{j+1} - 2V_j + V_{j-1}}{h^2}, $$

define the inner products and Sobolev norms (or seminorms)

$$ (V, W) = h\sum_{j=1}^{M-1} V_j\overline{W_j}, \qquad \|V\| = \sqrt{(V, V)}, $$

$$ \langle \delta_x V_j, \delta_x W_j\rangle = h\sum_{j=1}^{M} \delta_x V_{j-\frac{1}{2}}\overline{\delta_x W_{j-\frac{1}{2}}}, \qquad (\delta_x^2 V_j, \delta_x^2 W_j) = h\sum_{j=1}^{M-1} \delta_x^2 V_j\overline{\delta_x^2 W_j}, $$

$$ (V, W)_A = \langle \delta_x V_j, \delta_x W_j\rangle - \frac{h^2}{12}(\delta_x^2 V_j, \delta_x^2 W_j), \qquad \|V\|_A = \sqrt{(V, V)_A}. $$

We use $\mathcal{H}$ to represent a compact operator, which has the following form,

$$ \mathcal{H}u_j^n = \begin{cases} \frac{1}{12}u_{j-1}^n + \frac{10}{12}u_j^n + \frac{1}{12}u_{j+1}^n = (1 + \frac{h^2}{12}\delta_x^2)u_j^n, & j \in [M-1], \\ u_j^n, & j = 0 \text{ or } M. \end{cases} \tag{2.1} $$

For convenience, we allow C to represent different values at different positions. Next, we introduce some lemmas that are helpful for understanding the construction of our discrete scheme.

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Lemma 2.1. *[40] Let $\mathcal{H}$ be the compact operator defined in (2.1). Suppose $u(x) \in C^6(\Omega)$, then one has*

$$\mathcal{H}u_{xx}(x_j, t_n) = \delta_x^2 u_j^n + O(h^4),$$

where h is space stepsize and $j \in [M]$.

Lemma 2.2. *[41] Let $\varrho = \frac{\alpha}{1-\alpha}$ with $0 < \alpha < 1$. Let $u(t)$ be sufficiently smooth function for $t > 0$. Then*

$${}_0^{CF}D_t^\alpha u^n = \lambda \sum_{k=1}^{n} e^{-\varrho(n-k)\tau}(u^k - u^{k-1}) + \beta \sum_{k=2}^{n} e^{-\varrho(n-k)\tau}(u^k - 2u^{k-1} + u^{k-2}) + O(\tau^3),$$

where $\lambda = \frac{1-e^{-\varrho\tau}}{\alpha\tau}$ and $\beta = \frac{2(\alpha-1)(1-e^{-\varrho\tau})+\alpha\tau(1+e^{-\varrho\tau})}{2\alpha^2\tau^2}$.

By Lemma 2.2, we have

$$\begin{aligned}
{}_0^{CF}D_t^\alpha u^n =& -(\lambda e^{-\varrho\tau} - \beta)e^{-\varrho(n-2)\tau}u^0 + \big(\lambda e^{-2\varrho\tau} - ((\lambda + 2\beta)e^{-\varrho\tau} - \beta)\big)e^{-\varrho(n-3)\tau}u^1 \\
& - \sum_{k=2}^{n-2}(1 - e^{-\varrho\tau})((\lambda + \beta)e^{-\varrho\tau} - \beta)e^{-\varrho(n-k-2)\tau}u^k \\
& - ((\lambda + \beta)(1 - e^{-\varrho\tau}) + \beta)u^{n-1} + (\lambda + \beta)u^n + O(\tau^3).
\end{aligned}$$

Let $A_1 = (\lambda + 2\beta)e^{-\varrho\tau} - \beta$, $A_2 = (1 - e^{-\varrho\tau})((\lambda + \beta)e^{-\varrho\tau} - \beta)$ and $A_3 = (\lambda + \beta)(1 - e^{-\varrho\tau}) + \beta$. By the definition of compact operator $\mathcal{H}$, Lemma 2.1 and Lemma 2.2, one has

$$\begin{cases}
(\lambda + \beta)\mathcal{H}u_j^n - a\delta_x^2 u_j^n = (\lambda e^{-\varrho\tau} - \beta)e^{-\varrho(n-2)\tau}\mathcal{H}u_j^0 - (\lambda e^{-2\varrho\tau} - A_1)e^{-\varrho(n-3)\tau}\mathcal{H}u_j^1 \\
+ A_2 \sum_{k=2}^{n-2} e^{-\varrho(n-k-2)\tau}\mathcal{H}u_j^k + A_3\mathcal{H}u_j^{n-1} + \mathcal{H}f_j^n + R_j^n, \quad j \in [M-1],\ n \in [N-1], \\
u_j^0 = \varphi(x_j), \quad 0 \le j \le M, \\
u_0^n = u_M^n = 0, \quad n \in [N],
\end{cases} \tag{2.2}$$

where $\|R_j^n\| \le C(\tau^3 + h^4)$.

We use U_j^n to represent the numerical approximation of $u(x,t)$ at the mesh point (x_j, t_n). If we ignore the truncation error term R_j^n in Eq. (2.2), then we can get the CFD scheme for Eq. (1.1) as follows,

$$\begin{cases}
(\lambda + \beta)\mathcal{H}U_j^n - a\delta_x^2 U_j^n = (\lambda e^{-\varrho\tau} - \beta)e^{-\varrho(n-2)\tau}\mathcal{H}U_j^0 - (\lambda e^{-2\varrho\tau} - A_1)e^{-\varrho(n-3)\tau}\mathcal{H}U_j^1 \\
+ A_2 \sum_{k=2}^{n-2} e^{-\varrho(n-k-2)\tau}\mathcal{H}U_j^k + A_3\mathcal{H}U_j^{n-1} + \mathcal{H}f_j^n, \quad j \in [M-1],\ n \in [N-1], \\
U_j^0 = \varphi(x_j), \quad 0 \le j \le M, \\
U_0^n = U_M^n = 0, \quad n \in [N].
\end{cases} \tag{2.3}$$

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3. Stability analysis and error estimates

The main purpose of this section is to provide and prove the stability and error estimation for discrete scheme Eq. (2.3). Next, we introduce some lemmas that are helpful for understanding the proof of our main results. We will ignore the subscript j in the following discussion.

Lemma 3.1. *[42, 43] If $V, W \in V_h$, then $(\delta_x^2 V, W) = -\langle \delta_x V, \delta_x W\rangle$.*

Lemma 3.2. *For any grid function $V, W \in V_h$, then it holds that*

$$-(\delta_x^2 V, \mathcal{H}W) = (V, W)_A.$$

Proof. Applying the definition of operator $\mathcal{H}$, we can get

$$-(\delta_x^2 V, \mathcal{H}W) = -(\delta_x^2 V, (1 + \frac{h^2}{12}\delta_x^2)W) = -(\delta_x^2 V, W) - \frac{h^2}{12}(\delta_x^2 V, \delta_x^2 W).$$

By Lemma 3.1, $(\delta_x^2 V, W) = -\langle \delta_x V, \delta_x W\rangle$, it follows that

$$-(\delta_x^2 V, \mathcal{H}W) = \langle \delta_x V, \delta_x W\rangle - \frac{h^2}{12}(\delta_x^2 V, \delta_x^2 W) = (V, W)_A.$$

Then the proof is complete. □

Lemma 3.3. *[44] For any grid function $U \in V_h$, it holds that $\frac{1}{3}\|U\|^2 \le \|\mathcal{H}U\|^2 \le \|U\|^2$.*

The proof of the following lemma is simple, and we will omit its process here.

Lemma 3.4. *Let $\varrho = \frac{\alpha}{1-\alpha}$ with $0 < \alpha < 1$, it holds that*

$$\sum_{k=1}^{n-1} e^{-\varrho(n-1-k)\tau}(1 - e^{-\varrho\tau}) < 1.$$

Lemma 3.5. *[41] Let $\varrho = \frac{\alpha}{1-\alpha}$ with $0 < \alpha < 1$, and let*

$$T_k^n = \big(2(\alpha - 1)(1 - e^{-\varrho\tau}) + \alpha\tau(1 + e^{-\varrho\tau})\big)e^{-\varrho(n-k)\tau}, 2 \le k \le n.$$

It holds that

$$T_n^n > T_{n-1}^n > \cdots > T_k^n > T_{k-1}^n > \cdots > T_2^n > 0.$$

Lemma 3.6. *Let $0 < \alpha < 1$ with $\varrho = \frac{\alpha}{1-\alpha}$. It holds that*

$$0 < \frac{A_2}{\lambda + \beta}\sum_{k=2}^{n-2} e^{-\varrho(n-k-2)\tau} < 1,$$

where $\lambda = \frac{1-e^{-\varrho\tau}}{\alpha\tau}$, $\beta = \frac{2(\alpha-1)(1-e^{-\varrho\tau})+\alpha\tau(1+e^{-\varrho\tau})}{2\alpha^2\tau^2}$ and $A_2 = (1 - e^{-\varrho\tau})((\lambda + \beta)e^{-\varrho\tau} - \beta)$.

Proof. Since $A_2 = (1 - e^{-\varrho\tau})((\lambda + \beta)e^{-\varrho\tau} - \beta)$, we can get that

$$\frac{A_2}{\lambda + \beta}\sum_{k=2}^{n-2} e^{-\varrho(n-k-2)\tau} = \frac{(\lambda + \beta)e^{-\varrho\tau} - \beta}{\lambda + \beta}\sum_{k=2}^{n-2}(1 - e^{-\varrho\tau})e^{-\varrho(n-k-2)\tau}.$$

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Firstly, we will show that $0 < \frac{(\lambda+\beta)e^{-\varrho\tau}-\beta}{\lambda+\beta} < 1$. For this purpose, we consider $(\lambda+\beta)e^{-\varrho\tau}-\beta$ to be expanded into the following form,

$$\begin{aligned}
&(\lambda+\beta)e^{-\varrho\tau}-\beta\\
&=\lambda e^{-\varrho\tau}+\beta(e^{-\varrho\tau}-1)\\
&=\frac{e^{-\varrho\tau}-e^{-2\varrho\tau}}{\alpha\tau}+\frac{(2\alpha-2)(2e^{-\varrho\tau}-e^{-2\varrho\tau}-1)+\alpha\tau(e^{-2\varrho\tau}-1)}{2\alpha^2\tau^2}\\
&=\frac{(2-2\alpha-\alpha\tau)(e^{-2\varrho\tau}-2e^{-\varrho\tau}+1)}{2\alpha^2\tau^2}.
\end{aligned}$$

For $0<\alpha<1$, we know that $\lim_{\tau\to 0}(2-2\alpha-\alpha\tau)>0$ and $\lim_{\tau\to 0}(e^{-2\varrho\tau}-2e^{-\varrho\tau}+1)>0$, then we can obtain that $(\lambda+\beta)e^{-\varrho\tau}-\beta>0$. By Lemma 3.5, we know that $\beta>0$. If $\tau\to 0$, we get that

$$\lambda+\beta-((\lambda+\beta)e^{-\varrho\tau}-\beta)=\lambda(1-e^{-\varrho\tau})+\beta(2-e^{-\varrho\tau})>0.$$

This means that $\lambda+\beta>0$, and we get that $0<\frac{(\lambda+\beta)e^{-\varrho\tau}-\beta}{\lambda+\beta}<1$.

By Lemma 3.4, we know that $0<\sum_{k=2}^{n-2}(1-e^{-\varrho\tau})e^{-\varrho(n-k-2)\tau}<1$. Thus,

$$0<\frac{A_2}{\lambda+\beta}\sum_{k=2}^{n-2}e^{-\varrho(n-k-2)\tau}<1.$$

Thus, the proof is finished. □

By the proof of Lemma 3.6, we know that $(\lambda+\beta)e^{-\varrho\tau}-\beta>0$. Then it is easy to reach the following remark.

Remark 3.7. Let $A_1=(\lambda+2\beta)e^{-\varrho\tau}-\beta$, $A_2=(1-e^{-\varrho\tau})((\lambda+\beta)e^{-\varrho\tau}-\beta)$ and $A_3=(\lambda+\beta)(1-e^{-\varrho\tau})+\beta$. Then each of the following holds.
(i) $A_1>0$, $A_2>0$ and $A_3>0$.
(ii) $\lambda e^{-2\varrho\tau}-A_1<0$.

First we will give the proof of stability for the discrete scheme Eq. (2.3).

Theorem 3.8. *Let U^n be the solution of Eq. (2.3) with respect to the initial and boundary conditions, then*

$$\|U^n\|\le C(\|U^0\|+\max_{1\le s\le n}\|f^s\|),$$

where C is a positive constant and $n\in[N]$.

Proof. By Eq. (2.3), we get that

$$\begin{aligned}
&(\lambda+\beta)\mathcal{H}U^n-a\delta_x^2U^n\\
&=(\lambda e^{-\varrho\tau}-\beta)e^{-\varrho(n-2)\tau}\mathcal{H}U^0-(\lambda e^{-2\varrho\tau}-A_1)e^{-\varrho(n-3)\tau}\mathcal{H}U^1\\
&+A_2\sum_{k=2}^{n-2}e^{-\varrho(n-k-2)\tau}\mathcal{H}U^k+A_3\mathcal{H}U^{n-1}+\mathcal{H}f^n,
\end{aligned}\tag{3.1}$$

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Multiplying both sides of Eq. (3.1) by $\mathcal{H}U^n$ and integrating on Ω, then it yields,

$$\begin{aligned}&(\lambda+\beta)(\mathcal{H}U^n,\mathcal{H}U^n)-a(\delta_x^2U^n,\mathcal{H}U^n)\\&=(\lambda e^{-\varrho\tau}-\beta)e^{-\varrho(n-2)\tau}(\mathcal{H}U^0,\mathcal{H}U^n)-(\lambda e^{-2\varrho\tau}-A_1)e^{-\varrho(n-3)\tau}(\mathcal{H}U^1,\mathcal{H}U^n)\\&+A_2(\sum_{k=2}^{n-2}e^{-\varrho(n-k-2)\tau}\mathcal{H}U^k,\mathcal{H}U^n)+A_3(\mathcal{H}U^{n-1},\mathcal{H}U^n)+(\mathcal{H}f^n,\mathcal{H}U^n),\end{aligned}$$

By Lemma 3.2, $-a(\delta_x^2U^n,\mathcal{H}U^n)=a(U^n,U^n)_A\geq 0$, then we have

$$\begin{aligned}&(\lambda+\beta)(\mathcal{H}U^n,\mathcal{H}U^n)\\&\leq(\lambda e^{-\varrho\tau}-\beta)e^{-\varrho(n-2)\tau}(\mathcal{H}U^0,\mathcal{H}U^n)-(\lambda e^{-2\varrho\tau}-A_1)e^{-\varrho(n-3)\tau}(\mathcal{H}U^1,\mathcal{H}U^n)\\&+A_2(\sum_{k=2}^{n-2}e^{-\varrho(n-k-2)\tau}\mathcal{H}U^k,\mathcal{H}U^n)+A_3(\mathcal{H}U^{n-1},\mathcal{H}U^n)+(\mathcal{H}f^n,\mathcal{H}U^n).\end{aligned}\tag{3.2}$$

By Remark 3.7, we know that $A_2>0$, $A_3>0$ and $-(\lambda e^{-2\varrho\tau}-A_1)>0$. In Eq. (3.2), if $\lambda e^{-\varrho\tau}-\beta>0$, we can get

$$\begin{aligned}&(\lambda+\beta)\|\mathcal{H}U^n\|\\&\leq(\lambda e^{-\varrho\tau}-\beta)e^{-\varrho(n-2)\tau}\|\mathcal{H}U^0\|-(\lambda e^{-2\varrho\tau}-A_1)e^{-\varrho(n-3)\tau}\|\mathcal{H}U^1\|\\&+A_2\sum_{k=2}^{n-2}e^{-\varrho(n-k-2)\tau}\|\mathcal{H}U^k\|+A_3\|\mathcal{H}U^{n-1}\|+\|\mathcal{H}f^n\|,\end{aligned}\tag{3.3}$$

and if $\lambda e^{-\varrho\tau}-\beta\leq 0$, we have

$$\begin{aligned}&(\lambda+\beta)\|\mathcal{H}U^n\|\\&\leq-(\lambda e^{-2\varrho\tau}-A_1)e^{-\varrho(n-3)\tau}\|\mathcal{H}U^1\|\\&+A_2\sum_{k=2}^{n-2}e^{-\varrho(n-k-2)\tau}\|\mathcal{H}U^k\|+A_3\|\mathcal{H}U^{n-1}\|+\|\mathcal{H}f^n\|.\end{aligned}\tag{3.4}$$

Next, we want to prove the following inequality by mathematical induction,

$$\|\mathcal{H}U^n\|\leq C(\|\mathcal{H}U^0\|+\max_{1\leq s\leq n}\|\mathcal{H}f^s\|),$$

where $n\in[N]$.

From the proof process of Lemma 3.6 , it can be seen that $\lambda+\beta>0$. If $n=1$ and $\lambda e^{-\varrho\tau}-\beta>0$, by Eq. (3.3), we have

$$(\lambda+\beta)\|\mathcal{H}U^1\|\leq(\lambda e^{-\varrho\tau}-\beta)e^{\varrho\tau}\|\mathcal{H}U^0\|-(\lambda e^{-2\varrho\tau}-A_1)e^{2\varrho\tau}\|\mathcal{H}U^1\|+A_3\|\mathcal{H}U^0\|+\|\mathcal{H}f^1\|.$$

i.e.,

$$(2\lambda+\beta-A_1e^{2\varrho\tau})\|\mathcal{H}U^1\|\leq(\lambda-\beta e^{\varrho\tau}+A_3)\|\mathcal{H}U^0\|+\|\mathcal{H}f^1\|.\tag{3.5}$$

For the above formula, we need to verify that

$$2\lambda+\beta-A_1e^{2\varrho\tau}\geq c>0.\tag{3.6}$$

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By Eq. (3.6), we get that

$$2\lambda+\beta-A_1e^{2\varrho\tau}=\lambda(2-e^{\varrho\tau})+\beta(1+e^{2\varrho\tau}-2e^{\varrho\tau}).$$

From Lemma 3.5, $\beta>0$, therefore, $\beta(1+e^{2\varrho\tau}-2e^{\varrho\tau})=\beta(e^{\varrho\tau}-1)^2>0$. For the term of $\lambda(2-e^{\varrho\tau})$, we can get that

$$\lambda(2-e^{\varrho\tau})=\frac{(1-e^{-\varrho\tau})(2-e^{\varrho\tau})}{\alpha\tau}. \tag{3.7}$$

In Eq. (3.7), as $\tau\to 0$, we can get that $e^{-\varrho\tau}\to 1^-$ and $e^{\varrho\tau}\to 1^+$, which means that

$$\lim_{\tau\to 0}(1-e^{-\varrho\tau})>0 \text{ and } \lim_{\tau\to 0}(2-e^{\varrho\tau})>0,$$

i.e., $(1-e^{-\varrho\tau})(2-e^{\varrho\tau})>0$. Thus, as the parameter is given, $2\lambda+\beta-A_1e^{2\varrho\tau}\geq c>0$ holds. Combining $2\lambda+\beta-A_1e^{2\varrho\tau}\geq c>0$ and $\lambda-\beta e^{\varrho\tau}+A_3>0$, we can arrive at

$$0<\frac{\lambda-\beta e^{\varrho\tau}+A_3}{2\lambda+\beta-A_1e^{2\varrho\tau}}<C.$$

In summary, Eq. (3.5) can be written as

$$\|\mathcal{H}U^1\|\leq C(\|\mathcal{H}U^0\|+\|\mathcal{H}f^1\|). \tag{3.8}$$

Similarly, if $n=1$ and $\lambda e^{-\varrho\tau}-\beta\leq 0$, by Eq. (3.4), we also can get the same result as (3.8).

Assuming that

$$\|\mathcal{H}U^k\|\leq C_{k-1}(\|\mathcal{H}U^0\|+\max_{1\leq s\leq k}\|\mathcal{H}f^s\|) \tag{3.9}$$

holds as $k=2,3,\cdots,n-1$, where C_{k-1} is a positive constant and independent of n. Then, for $k=n$, we want to prove the following inequality

$$\|\mathcal{H}U^n\|\leq C(\|\mathcal{H}U^0\|+\max_{1\leq s\leq n}\|\mathcal{H}f^s\|).$$

For $k=n$ and $\lambda e^{-\varrho\tau}-\beta>0$, by Eq. (3.3) and Cauchy-Schwarz inequality, one has

$$\begin{aligned}
&\|\mathcal{H}U^n\|\\
&\leq\frac{(\lambda e^{-\varrho\tau}-\beta)e^{-\varrho(n-2)\tau}}{\lambda+\beta}\|\mathcal{H}U^0\|-\frac{(\lambda e^{-2\varrho\tau}-A_1)e^{-\varrho(n-3)\tau}}{\lambda+\beta}\|\mathcal{H}U^1\|\\
&+\frac{A_2}{\lambda+\beta}\sum_{k=2}^{n-2}e^{-\varrho(n-k-2)\tau}\|\mathcal{H}U^k\|+\frac{A_3}{\lambda+\beta}\|\mathcal{H}U^{n-1}\|+\frac{1}{\lambda+\beta}\|\mathcal{H}f^n\|.
\end{aligned} \tag{3.10}$$

Denoting

$$B_k=\frac{A_2}{\lambda+\beta}e^{-\varrho(n-k-2)\tau} \text{ with } 2\leq k\leq n-2, \tag{3.11}$$

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we have

$$\begin{aligned}&\sum_{k=2}^{n-2} B_k\|\mathcal{H}U^k\|\\ &\le\ B_2C_1(\|\mathcal{H}U^0\|+\max_{1\le s\le 2}\|\mathcal{H}f^s\|)+B_3C_2(\|\mathcal{H}U^0\|+\max_{1\le s\le 3}\|\mathcal{H}f^s\|)+\cdots\\ &+B_{n-2}C_{n-3}(\|\mathcal{H}U^0\|+\max_{1\le s\le n-2}\|\mathcal{H}f^s\|).\end{aligned}$$

By Lemma 3.6, we know that $0<\sum_{k=2}^{n-2}B_k<1$. Let $C=\max\{C_1,C_2,\dots,C_{n-3}\}$ which is independent of n. For the third term to right hand of Eq. (3.10), it can be further obtained that

$$\begin{aligned}\sum_{k=2}^{n-2} B_k\|\mathcal{H}U^k\| &\le (B_2+B_3+\cdots+B_{n-2})C(\|\mathcal{H}U^0\|+\max_{1\le s\le n-2}\|\mathcal{H}f^s\|)\\ &\le \sum_{k=2}^{n-2}B_kC(\|\mathcal{H}U^0\|+\max_{1\le s\le n-2}\|\mathcal{H}f^s\|)\\ &\le C(\|\mathcal{H}U^0\|+\max_{1\le s\le n-2}\|\mathcal{H}f^s\|).\end{aligned}$$

For the rest terms to right hand of Eq. (3.10), we just need to make sure that the coefficients of these terms are greater than zero and bounded. In fact, $\lambda e^{-\varrho\tau}-\beta>0$ and Remark 3.7 imply that the coefficients of the rest terms are greater than zero and bounded. In this way, we get

$$\|\mathcal{H}U^n\|\le C(\|\mathcal{H}U^0\|+\max_{1\le s\le n}\|\mathcal{H}f^s\|)\text{ with }n\in[N]. \tag{3.12}$$

Similarly, if $k=n$ and $\lambda e^{-\varrho\tau}-\beta\le 0$, using Eq. (3.4), by a similar proof, we also can get the same result as (3.12).

Therefore, it holds that $\|\mathcal{H}U^n\|\le C(\|\mathcal{H}U^0\|+\max_{1\le s\le n}\|\mathcal{H}f^s\|)$. Applying Lemma 3.3, we obtain

$$\|U^n\|\le C(\|U^0\|+\max_{1\le s\le n}\|f^s\|).$$

The proof of this theorem is finished. □

We can derive the convergence of the discrete scheme Eq. (2.3) as similarly as proving Theorem 3.8.

Theorem 3.9. *Let u^n be the solution of the given equation (1.1) and U^n be the solution of the CFD scheme for Eq. (2.3). Let $\varepsilon^n=u^n-U^n$, then $\varepsilon^0=0$. Then the following result holds,*

$$\|\varepsilon^n\|\le C(\tau^3+h^4),$$

where C is a positive constant and $n\in[N]$.

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Proof. Subtracting Eq. (2.3) from Eq. (2.2), we obtain

$$\begin{aligned}&(\lambda+\beta)\mathcal{H}\varepsilon^n - a\delta_x^2\varepsilon^n\\&=(\lambda e^{-\varrho\tau}-\beta)e^{-\varrho(n-2)\tau}\mathcal{H}\varepsilon^0-(\lambda e^{-2\varrho\tau}-A_1)e^{-\varrho(n-3)\tau}\mathcal{H}\varepsilon^1\\&+A_2\sum_{k=2}^{n-2}e^{-\varrho(n-k-2)\tau}\mathcal{H}\varepsilon^k+A_3\mathcal{H}\varepsilon^{n-1}+R^n.\end{aligned}\tag{3.13}$$

Multiplying both sides of Eq. (3.13) by $\mathcal{H}\varepsilon^n$ and integrating on Ω, we get that

$$\begin{aligned}&((\lambda+\beta)\mathcal{H}\varepsilon^n,\mathcal{H}\varepsilon^n)-a(\delta_x^2\varepsilon^n,\mathcal{H}\varepsilon^n)\\&=(\lambda e^{-\varrho\tau}-\beta)e^{-\varrho(n-2)\tau}(\mathcal{H}\varepsilon^0,\mathcal{H}\varepsilon^n)-(\lambda e^{-2\varrho\tau}-A_1)e^{-\varrho(n-3)\tau}(\mathcal{H}\varepsilon^1,\mathcal{H}\varepsilon^n)\\&+A_2(\sum_{k=2}^{n-2}e^{-\varrho(n-k-2)\tau}\mathcal{H}\varepsilon^k,\mathcal{H}\varepsilon^n)+A_3(\mathcal{H}\varepsilon^{n-1},\mathcal{H}\varepsilon^n)+(R^n,\mathcal{H}\varepsilon^n).\end{aligned}\tag{3.14}$$

In Eq. (3.14), since $-a(\delta_x^2\varepsilon^n,\mathcal{H}\varepsilon^n)=a(\varepsilon^n,\varepsilon^n)_A\geq 0$ and $\varepsilon^0=0$, by Cauchy-Schwarz inequality and the result of Lemma 3.1, it yields

$$\begin{aligned}&(\lambda+\beta)\|\mathcal{H}\varepsilon^n\|^2\\&\leq-(\lambda e^{-2\varrho\tau}-A_1)e^{-\varrho(n-3)\tau}\|\mathcal{H}\varepsilon^1\|\|\mathcal{H}\varepsilon^n\|\\&+A_2\sum_{k=2}^{n-2}e^{-\varrho(n-k-2)\tau}\|\mathcal{H}\varepsilon^k\|\|\mathcal{H}\varepsilon^n\|+A_3\|\mathcal{H}\varepsilon^{n-1}\|\|\mathcal{H}\varepsilon^n\|+\|R^n\|\|\mathcal{H}\varepsilon^n\|,\end{aligned}$$

i.e.,

$$\begin{aligned}&(\lambda+\beta)\|\mathcal{H}\varepsilon^n\|\\&\leq-(\lambda e^{-2\varrho\tau}-A_1)e^{-\varrho(n-3)\tau}\|\mathcal{H}\varepsilon^1\|\\&+A_2\sum_{k=2}^{n-2}e^{-\varrho(n-k-2)\tau}\|\mathcal{H}\varepsilon^k\|+A_3\|\mathcal{H}\varepsilon^{n-1}\|+\|R^n\|.\end{aligned}\tag{3.15}$$

Next, we use mathematical induction to prove $\|\mathcal{H}\varepsilon^n\|\leq C(\tau^3+h^4)$ with $n\in[N]$. For $n=1$, by Eq. (3.15), we have

$$(2\lambda+\beta-A_1e^{2\varrho\tau})\|\mathcal{H}\varepsilon^1\|\leq A_3\|\mathcal{H}\varepsilon^0\|+\|R^1\|=\|R^1\|.$$

Since $\|R^n\|\leq C(\tau^3+h^4)$, then

$$\|\mathcal{H}\varepsilon^1\|\leq\frac{1}{2\lambda+\beta-A_1e^{2\varrho\tau}}\|R^1\|\leq C(\tau^3+h^4).$$

Assuming that

$$\|\mathcal{H}\varepsilon^k\|\leq C(\tau^3+h^4)\tag{3.16}$$

holds as $k=2,\cdots,n-1$. Then we will show that $\|\mathcal{H}\varepsilon^n\|\leq C(\tau^3+h^4)$. By Eqs. (3.11) and (3.16), the second term to right hand of Eq. (3.15) can be obtained as

$$\sum_{k=2}^{n-2}B_k\|\mathcal{H}\varepsilon^k\|\leq B_2C_1(\tau^3+h^4)+B_3C_2(\tau^3+h^4)+\cdots+B_{n-2}C_{n-3}(\tau^3+h^4).$$

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Let $C = \max\{C_1, C_2, \ldots, C_{n-2}\}$ be a positive constant and independent of n, by Eqs. (3.15)-(3.16) and Lemma 3.6, we obtain

$$\|\mathcal{H}\varepsilon^n\| \le C(\tau^3 + h^4)\sum_{k=2}^{n-2} B_k + \|R^n\| \le C(\tau^3 + h^4) + \|R^n\|.$$

Therefore, it holds that $\|\mathcal{H}\varepsilon^n\| \le C(\tau^3 + h^4)$. Applying Lemma 3.3, we arrive at $\frac{1}{3}\|\varepsilon^n\|^2 \le \|\mathcal{H}\varepsilon^n\|^2 \le \|\varepsilon^n\|^2$, so that $\|\varepsilon^n\| \le \sqrt{3}\|\mathcal{H}\varepsilon^n\|$, then

$$\|\varepsilon^n\| \le C(\tau^3 + h^4).$$

The proof of this theorem is thus finished. □

The error estimation of our method with Caputo-Fabrizio derivative and the same one in [20] with Caputo derivative imply that we can get the following remark, which shows that Caputo-Fabrizio derivative without singular term can improve the accuracy of the discrete scheme.

Remark 3.10. The temporal convergence rate (TCR) of our method is τ^3, and the TCR of the method in [20] is $\tau^{2-\alpha}$.

4. Numerical examples

In order to support our theoretical analysis in Section 3, we present some numerical examples in this section. Let $\tau = T/N$ be the time stepsize and $h = L/M$ the space stepsize with N and M are positive integers. In the process of implementing numerical experiments, we use the Matlab 2020a with PC of AMD-Ryzen-5-3500U and 8 GB memory. We use the following error norm

$$E_n(\tau, h) = \max_{1\le n\le N} \|u^n - U^n\|.$$

Denote temporal convergence rate (TCR) as

$$TCR = \log(E_n(\tau_1, h)/E_n(\tau_2, h))/\log(\tau_1/\tau_2),$$

where $E_n(\tau_1, h)$ and $E_n(\tau_2, h)$ are errors for the errors of mesh sizes τ_1 and τ_2, respectively. And denote spatial convergence rate (SCR) as

$$SCR = \log(E_n(\tau, h_1)/E_n(\tau, h_2))/\log(h_1/h_2),$$

where $E_n(\tau, h_1)$ and $E_n(\tau, h_2)$ are errors for the errors of mesh sizes h_1 and h_2, respectively. Also, in order to investigate the SCR, we can use $SCR = \log(E_n(\tau_1, h_1)/E_n(\tau_2, h_2))/\log(h_1/h_2)$, where $\tau_1 << h_1$ and $\tau_2 << h_2$ (we use $\tau = 1/1000$ in Table 2, and $\tau_1 = h_1^2$ and $\tau_2 = h_2^2$ in Table 4).

Example 1. In the first example, for $a = 1$ and $(x, t) \in [0, 1] \times [0, 1]$, Eq. (1.1) will be

$$\begin{cases} {}_0^{CF}D_t^{\alpha} u(x,t) = \dfrac{\partial^2 u}{\partial x^2} + f(x,t), \\ u(x,0) = 0, \\ u(0,t) = u(1,t) = 0, \end{cases}$$

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where $f(x,t)=\sin(\pi x)\frac{3}{1-\alpha}(\frac{t^2}{\varrho}-\frac{2t}{\varrho^2}+\frac{2}{\varrho^3}(1-e^{-\varrho t}))+\pi^2t^3\sin(\pi x)$ and $\varrho=\frac{\alpha}{1-\alpha}$.

In this example, the exact solution of the equation can be obtained through calculation. However, in order to verify the effectiveness of our method, we assume that the exact solution of the equation is unknown and take the solution on the finer grid (that is $M=N=2000$) as the corresponding exact solution. First, the proposed scheme will be used to test the accuracy in the direction of time. In this case, we take $M=200$. The errors, TCRs and CPU times at different α ($\alpha=0.1,0.3,0.5$ and 0.7) are shown in Table 1. The data in Table 1 show that the TCR is about 3. Furthermore, we will test the accuracy of our scheme for space. We chose $N=1000$ for $\alpha=0.2,0.4,0.6$ and 0.8 at different M ($M=10,20,40$ and 80). The errors, SCRs and CPU times are shown in Table 2. From Table 2 we can see that SCRs are $O(h^4)$. The above data indicates that the numerical experimental results are consistent with the theoretical analysis. Similar numerical example can be found in [20] (Example 1 in [20]), where the fractional derivative is the form of Caputo fractional derivative (with singular term). A comparison of the obtained results with other existing methods reveals that our method is more accurate and efficient for the time-fractional diffusion equation, where the fractional derivative is the form of Caputo-Fabrizio derivative (without singular term).

We present the solution on finer grid (that is $M=N=2000$), numerical solution ($M=200$ and $N=2000$), absolute error and contour plot of absolute error in Fig. 1, in which we use $\alpha=0.02$. The error of numerical and exact solutions is small. In Fig. 2a, we show the TCRs of Example 1 for $\alpha=0.1,0.3,0.5$ and 0.7, respectively. In Fig. 2b, we show the SCRs of Example 1 for $\alpha=0.2,0.4,0.6$ and 0.8, respectively.

Table 1 Errors, TCRs and CPU times for Example 1.

	M	N	$E_n(\tau,h)$	TCR	CPUtime(s)
$\alpha=0.1$	200	50	6.2229×10^{-8}	—	0.0603
		60	3.6062×10^{-8}	2.99	0.0807
		70	2.2686×10^{-8}	3.01	0.1039
		80	1.5145×10^{-8}	3.03	0.1240
$\alpha=0.3$	200	50	2.6388×10^{-7}	—	0.0610
		60	1.5328×10^{-7}	2.98	0.0808
		70	9.6736×10^{-8}	2.99	0.0992
		80	6.4872×10^{-8}	2.99	0.1247
$\alpha=0.5$	200	50	6.6895×10^{-7}	—	0.0627
		60	3.8869×10^{-7}	2.98	0.0777
		70	2.4543×10^{-7}	2.98	0.1026
		80	1.6471×10^{-7}	2.99	0.1259
$\alpha=0.7$	200	50	1.6354×10^{-6}	–	0.0620
		60	9.4984×10^{-7}	2.98	0.0832
		70	5.9965×10^{-7}	2.98	0.1054
		80	4.0243×10^{-7}	2.99	0.1388

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Table 2 Errors, SCRs and CPU times for Example 1.

	M	N	$E_n(\tau, h)$	SCR	CPUtime(s)
$\alpha = 0.2$	10	1000	3.6409×10^{-5}	–	0.5056
	20		2.2689×10^{-6}	4.00	0.5842
	40		1.4172×10^{-7}	4.00	0.7930
	80		8.8756×10^{-9}	4.00	1.5948
$\alpha = 0.4$	10	1000	3.5612×10^{-5}	–	0.5024
	20		2.2192×10^{-6}	4.00	0.5903
	40		1.3858×10^{-7}	4.00	0.7865
	80		8.6458×10^{-9}	4.00	1.7361
$\alpha = 0.6$	10	1000	3.4486×10^{-5}	–	0.5081
	20		2.1489×10^{-6}	4.00	0.5872
	40		1.3412×10^{-7}	4.00	0.7849
	80		8.2944×10^{-9}	4.02	1.7771
$\alpha = 0.8$	10	1000	3.2843×10^{-5}	–	0.5218
	20		2.0464×10^{-6}	4.00	0.5998
	40		1.2751×10^{-7}	4.00	0.7798
	80		7.6744×10^{-9}	4.05	1.7200

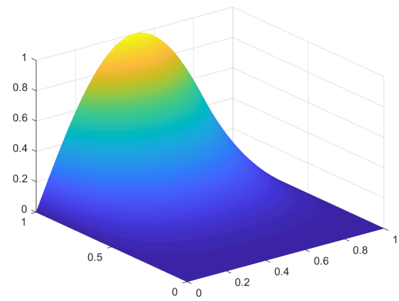

Fig. 1a. Solution on finer grid

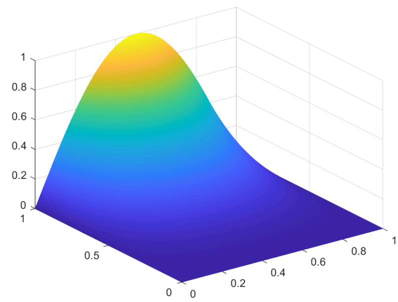

Fig. 1b. Numerical solution

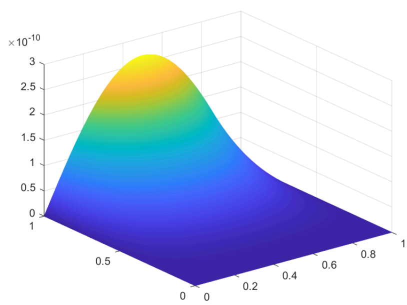

Fig. 1c. Absolute error

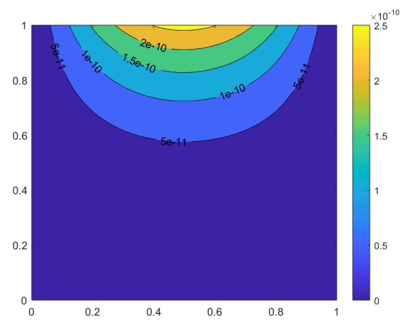

Fig. 1d. Contour plot of absolute error

Fig. 1. The results of Example 1 with $\alpha = 0.02$.

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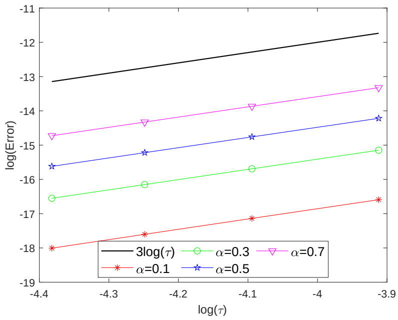

Fig. 2a. $\alpha = 0.1, 0.3, 0.5, 0.7$

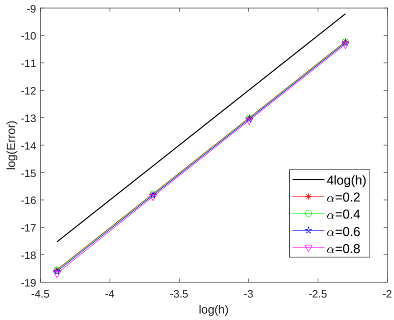

Fig. 2b. $\alpha = 0.2, 0.4, 0.6, 0.8$

Fig. 2. The TCRs and SCRs of Example 1 with given α.

Example 2. For $a = 1$, $T = 1$ and $L = 1$, we want to consider the equation Eq. (1.1) with exact solution $u(x,t) = t^3x^3(1-x)^3$. Then the source term is $f(x,t) = \frac{3}{1-\alpha}x^3(1-x)^3(\frac{t^2}{\varrho} - \frac{2t}{\varrho^2} + \frac{2}{\varrho^3}(1-e^{-\varrho t})) + 3t^3x(1-x)(2(1-x)^2 - 6x(1-x) + 2x^2)$, where $\varrho = \frac{\alpha}{1-\alpha}$. The function $\varphi(x)$ can be found by the exact solution $u(x,t)$.

In order to verify TCRs, we will fix the spatial meshes as $M = 1000$. We present the errors, TCRs and CPU times in Table 3. For different values of α, the TCRs of the CFD scheme reach the third order. In order to verify SCRs, we choose the spatial meshes M and temporal meshes N as $N = M^2$. The errors, SCRs and CPU times are shown in Table 4. For different values of α, the SCRs of the CFD scheme reach the fourth order. The data of these tables show that our method provides an approximate solution with high accuracy for Example 2.

Table 3 Errors, TCRs and CPU times for Example 2.

	M	N	L^∞-Error	L^∞-Rate	CPUtime(s)
$\alpha = 0.1$	1000	40	1.5563×10^{-9}	—	5.7114
		80	1.9781×10^{-10}	2.98	14.1234
		120	5.8767×10^{-11}	2.99	28.0687
		160	2.4749×10^{-11}	3.01	47.1467
$\alpha = 0.3$	1000	80	8.3603×10^{-10}	–	14.7304
		120	2.4876×10^{-10}	2.99	27.9238
		160	1.0492×10^{-10}	3.00	46.8027
		200	5.3733×10^{-11}	3.00	70.6237
$\alpha = 0.5$	1000	80	2.1186×10^{-9}	–	14.0054
		120	6.3084×10^{-10}	2.99	27.9049
		160	2.6649×10^{-10}	3.00	47.4357
		200	1.3652×10^{-10}	3.00	70.7949
$\alpha = 0.7$	1000	80	5.1671×10^{-9}	–	14.1410
		120	1.5381×10^{-9}	2.99	28.2142
		160	6.5040×10^{-10}	2.99	46.9748
		200	3.3332×10^{-10}	3.00	70.8377

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Table 4 Errors, SCRs and CPU times for Example 2.

	M	N	L^∞-Error	L^∞-Rate	CPUtime(s)
$\alpha = 0.2$	10	100	3.3398×10^{-5}	–	0.0314
	20	400	2.0874×10^{-6}	4.00	0.2092
	40	1600	1.3046×10^{-7}	4.00	3.9552
	80	6400	8.1538×10^{-9}	4.00	133.7926
$\alpha = 0.4$	10	100	3.2643×10^{-5}	–	0.0266
	20	400	2.0402×10^{-6}	4.00	0.2169
	40	1600	1.2752×10^{-7}	4.00	3.9932
	80	6400	7.9697×10^{-9}	4.00	124.3089
$\alpha = 0.6$	10	100	3.1578×10^{-5}	–	0.0269
	20	400	1.9737×10^{-6}	4.00	0.2063
	40	1600	1.2336×10^{-7}	4.00	3.9823
	80	6400	7.7099×10^{-9}	4.00	124.0635
$\alpha = 0.8$	10	100	3.0023×10^{-5}	–	0.0259
	20	400	1.8767×10^{-6}	4.00	0.2090
	40	1600	1.1729×10^{-7}	4.00	3.9389
	80	6400	7.3310×10^{-9}	4.00	133.4166

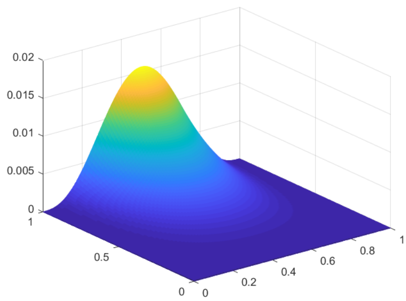

Fig. 3a. Exact solution

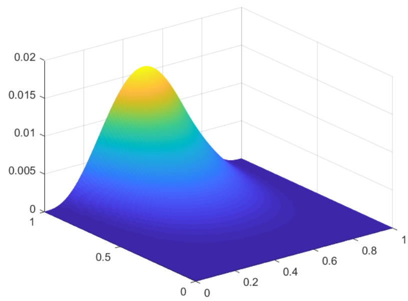

Fig. 3b. Numerical solution

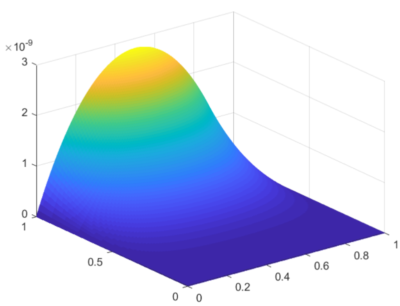

Fig. 3c. Absolute error

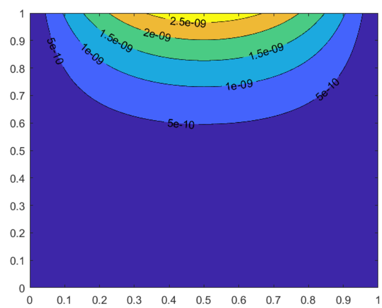

Fig. 3d. Contour plot of absolute error

Fig. 3. The results of Example 2 with $\alpha = 0.99$ at $M = 100$ and $N = 5000$.

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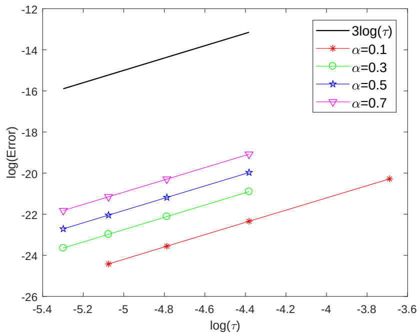

Fig. 4a. $\alpha = 0.1, 0.3, 0.5, 0.7$

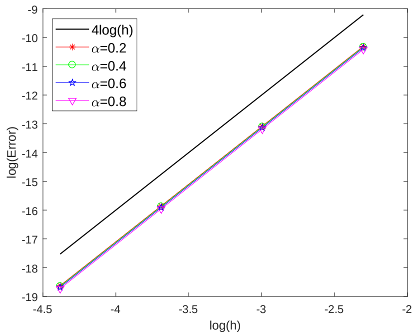

Fig. 4b. $\alpha = 0.2, 0.4, 0.6, 0.8$

Fig. 4. The TCRs and SCRs of Example 2 with given α.

In Fig. 3, we present the exact and numerical solutions, the absolute error and the contour plot of absolute error obtained from CFD scheme with $\alpha = 0.99$ at $M = 100$ and $N = 5000$. The results show that the numerical solution of the scheme has a high accuracy. In Fig. 4a, we show the TCRs of Example 2 for $\alpha = 0.1, 0.3, 0.5$ and 0.7, respectively. In Fig. 4b, we show the SCRs of Example 1 for $\alpha = 0.2, 0.4, 0.6$ and 0.8, respectively. Fig 4 is indicated that the numerical solution of our method is in excellent agreement with the exact solution.

Example 3. In the third example, we consider the 2D case for Eq. (1.1), in which $T = 1$, $L = 1$, $u(\mathbf{x}, t) = t^4 \sin(\pi x) \sin(2\pi y)$ and $a = \sin(\pi t - \pi/2)$. Then the source term is

$$f(x, y, t) = \Big(\tfrac{4}{1-\alpha}\big(\tfrac{t^3}{\sigma} - \tfrac{3t^2}{\sigma^2} + \tfrac{6t}{\sigma^3} - \tfrac{6}{\sigma^4}(1 - e^{-\sigma t})\big) + 5\pi^2 t^4 a\Big) \sin(\pi x) \sin(2\pi y),$$

where $\sigma = \frac{\alpha}{1-\alpha}$. The function $\varphi(\mathbf{x})$ can be found by substituting $u(\mathbf{x}, t)$ into Eq. (1.1).

In order to verify TCRs in this example, we will fix the spatial meshes as $M_1 = M_2 = M = 120$, where M_1 and M_2 are the numbers of meshes in x direction and y direction, respectively. We present the errors and TCRs in Table 5. From Table 5, we see that the TCRs of the CFD scheme reach the third order for different values of α. In order to verify SCRs, we choose the spatial meshes $M_1 = M_2 = M$ and temporal meshes $N = 300$. The errors and SCRs are shown in Table 6. From Table 6, we see that the SCRs of the CFD scheme reach the fourth order for different values of α. The data of these tables show that our method provides an approximate solution with high accuracy for 2D case of Eq. (1.1).

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Table 5 Errors and TCRs for Example 3.

	M	N	L^∞-Error	L^∞-Rate
$\alpha = 0.3$	120	10	4.4542×10^{-6}	–
		15	1.3860×10^{-6}	2.88
		20	5.9047×10^{-7}	2.97
		25	2.9650×10^{-7}	3.09
$\alpha = 0.6$	120	10	2.1343×10^{-5}	–
		15	6.5900×10^{-6}	2.90
		20	2.8390×10^{-6}	2.93
		25	1.4659×10^{-6}	2.96
$\alpha = 0.9$	120	10	1.7720×10^{-4}	–
		15	5.2434×10^{-5}	3.00
		20	2.2203×10^{-5}	2.99
		25	1.1405×10^{-5}	2.99

Table 6 Errors and SCRs for Example 3.

	M	N	L^∞-Error	L^∞-Rate
$\alpha = 0.2$	10	300	5.2223×10^{-4}	–
	20		3.3923×10^{-5}	3.94
	30		6.6497×10^{-6}	4.01
	40		2.1139×10^{-6}	3.98
$\alpha = 0.5$	10	300	5.2736×10^{-4}	–
	20		3.4255×10^{-5}	3.95
	30		6.7145×10^{-6}	4.01
	40		2.1342×10^{-6}	3.98
$\alpha = 0.8$	10	300	5.3864×10^{-4}	–
	20		3.4986×10^{-5}	3.95
	30		6.8559×10^{-6}	4.01
	40		2.1776×10^{-6}	3.99

5. Conclusion

In this paper, we present a new CFD scheme for TFDE. First, we use some techniques to obtain a third-order approximation for Caputo-Fabrizio derivative. For spatial discretization, we consider a fourth-order CFD scheme to discretize it. The obtained CFD scheme is led to the third-order accuracy in the temporal and the fourth-order accuracy in the spatial variables. The CFD scheme is proved to be stable. Finally, we provide numerical examples to verify the correctness of our theoretical analysis. In our future work, we will study the numerical solutions of high dimensional fractional differential equations and the problems with complex region.

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