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A NOTE ON THE SCHWARZ FRACTAL DERIVATIVE

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Abstract. We define a Schwarz fractal derivative of order n for a real-valued function $f(t)$ as the limit

$$(\mathrm{S}_n^{\alpha,\beta} f)(t_0) = \lim_{t\to t_0} \frac{\sum_{j=0}^{n} \binom{n}{j} (-1)^j f^\beta\left(t_0 + \frac{n-2j}{2}(t-t_0)\right)}{(t^\alpha - t_0^\alpha)^n},$$

where $\alpha, \beta > 0$ and $f^\beta := f|f|^{\beta-1}$. This derivative naturally generalizes the one introduced by Riemann in 1854. We study its essential properties and its relationship with other fractal derivatives recently reported in the literature. We obtain certain analogues of the mean value and Rolle theorems, together with some of their most important consequences. Finally, we propose an extension of such derivatives to the several-variable setting.

1. Introduction

The upper Schwarz derivative of a real-valued function f at $t_0 \in \mathbb{R}$ is defined by

$$(\overline{S}_n f)(t_0) = \limsup_{h\to 0} \frac{\sum_{j=0}^{n} \binom{n}{j} (-1)^j f\left(t_0 + \frac{n-2j}{2} h\right)}{h^n}.$$

The lower Schwarz derivative $(\underline{S}_n f)(t_0)$ is analogously defined by the corresponding lower limit.

When $(\overline{S}_n f)(t_0) = (\underline{S}_n f)(t_0)$, whether finite or infinite, the common value will be denoted by $(S_n f)(t_0)$ and is referred to as the n-th Schwarz derivative of f at t_0. The existence of the n-th ordinary derivative $f^{(n)}(t_0)$ implies that of $(S_n f)(t_0)$. For $n = 2$, we obtain the well-known *Riemann derivative*. Riemann was the first to realize the important role that this derivative plays in Fourier analysis [14]. On the other hand, Schwarz proved that if f is continuous and this derivative vanishes everywhere, then f must be linear. The terms "symmetric derivative" and "pseudoderivative" are often used in the literature. Good references on this topic are the works [4, 2, 5, 7, 16, 15, 12].

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Fractal calculus is very simple but extremely effective to deal with phenomena in hierarchical or porous media. Fractal theory is the theoretical basis for the fractal space-time. Many researchers have already found the intrinsic relationship between the fractional dimensions and the fractional order. The flexibility introduced by the fractal derivatives in the setting of fractal calculus allows one to look for new perspectives in several lines of research concerning approximation theory, anomalous diffusion and fractional differential equations (see, e.g., [3, 6, 8, 11]).

In this paper, the n-th *Schwarz fractal derivative* of f at t_0 is defined by the limit

$$(\mathrm{S}_n^{\alpha,\beta} f)(t_0) = \lim_{t\to t_0} \frac{\sum_{j=0}^{n} \binom{n}{j}(-1)^j f^\beta\left(t_0 + \frac{n-2j}{2}(t-t_0)\right)}{(t^\alpha - t_0^\alpha)^n},$$

where $\alpha,\beta > 0$ and $f^\beta := f|f|^{\beta-1}$. As in the classical case, such a limit exists if the upper and lower limits are equal. In the particular case when $n = 2$, we will refer to this derivative as the *Riemann fractal derivative.* First we will study basic properties of these derivatives, leading to interesting theoretical results. Subsequently, we obtain weak versions of the mean value and Rolle theorems, as well as some of their most important consequences. In Section 4 we derive a necessary condition connecting the local extremes of a continuous function with its Riemann fractal derivative. Finally, we briefly discuss a natural extension of this derivative in the context of complex analysis.

2. Preliminaries and simple facts

For $\alpha,\beta > 0$, let us define the (β,α)-*fractal derivative* of a function f at the point t_0 by

$$\frac{d^\beta f}{dt^\alpha}(t_0) = \lim_{t\to t_0} \frac{f^\beta(t) - f^\beta(t_0)}{t^\alpha - t_0^\alpha},$$

whenever the limit exists and is finite. In such a case, we say that f is (β,α)-*fractal differentiable* at t_0. The space of (β,α)-fractal differentiable functions at t_0 will be denoted by $\mathfrak{F}(t_0)$. Here, the function f^β is defined as in the introduction by $f^\beta = f|f|^{\beta-1}$. This derivative was studied in the forthcoming paper by Alfonso Santiesteban et al. [1], where it was used to find a better fitting curve for a real data set related to tuberculosis in Mexico. It is not difficult to obtain the following interesting properties of $\frac{d^\beta}{dt^\alpha}$.

Proposition 2.1. *Let f be a differentiable function at t_0. Assume also that $t_0 \neq 0$ if $\alpha > 1$, and $f(t_0) \neq 0$ if $\beta < 1$. Then f is (β,α)-fractal differentiable at t_0, and*

$$\frac{d^\beta f}{dt^\alpha}(t_0) = \frac{\beta}{\alpha}\,|t_0|^{1-\alpha}|f(t_0)|^{\beta-1} f'(t_0).$$

In particular, $d^\beta f/dt^\alpha \geq 0$ if $f' \geq 0$ and $d^\beta f/dt^\alpha \leq 0$ if $f' \leq 0$.

Proof. Since $t_0 \neq 0$ if $\alpha > 1$, and $f(t_0) \neq 0$ if $\beta < 1$, we obtain

$$\begin{aligned}\frac{d^\beta f}{dt^\alpha}(t_0) &= \lim_{t\to t_0} \frac{f^\beta(t) - f^\beta(t_0)}{t^\alpha - t_0^\alpha} = \lim_{t\to t_0} \frac{f^\beta(t) - f^\beta(t_0)}{t - t_0} \cdot \frac{t - t_0}{t^\alpha - t_0^\alpha} \\ &= \beta\, |f(t_0)|^{\beta-1} f'(t_0) \lim_{t\to t_0} \frac{1}{\alpha\, |t|^{\alpha-1}} = \frac{\beta}{\alpha}\, |t_0|^{1-\alpha} |f(t_0)|^{\beta-1} f'(t_0). \qquad \square\end{aligned}$$

Proposition 2.2. *Let f, g be (β,α)-fractal differentiable functions at t_0. Then the following statements hold:*

(1) *fg is (β,α)-fractal differentiable at t_0 and*
$$\frac{d^\beta (fg)}{dt^\alpha}(t_0) = \frac{d^\beta f}{dt^\alpha}(t_0)\, g^\beta(t_0) + f^\beta(t_0)\, \frac{d^\beta g}{dt^\alpha}(t_0).$$

(2) *If $g(t_0) \neq 0$, then $1/g$ is (β,α)-fractal differentiable at t_0 and*
$$\frac{d^\beta}{dt^\alpha}\left(\frac{1}{g}\right)(t_0) = \frac{-\frac{d^\beta g}{dt^\alpha}(t_0)}{|g(t_0)|^{2\beta}}.$$

(3) *If $g(t_0) \neq 0$, then f/g is (β,α)-fractal differentiable at t_0 and*
$$\frac{d^\beta}{dt^\alpha}\left(\frac{f}{g}\right)(t_0) = \frac{\frac{d^\beta f}{dt^\alpha}(t_0)\, g^\beta(t_0) - f^\beta(t_0)\, \frac{d^\beta g}{dt^\alpha}(t_0)}{|g(t_0)|^{2\beta}}.$$

We define the n-th Schwarz fractal derivative of a real-valued function f at a point $t_0 \in \mathbb{R}$ by

$$(\mathrm{S}_n^{\alpha,\beta} f)(t_0) = \lim_{t\to t_0} \frac{\sum_{j=0}^n \binom{n}{j} (-1)^j f^\beta\left(t_0 + \frac{n-2j}{2}(t - t_0)\right)}{(t^\alpha - t_0^\alpha)^n}, \tag{2.1}$$

where $\alpha, \beta > 0$. If the limit (2.1) exists, we say that f is n-times Schwarz fractal differentiable at t_0. This limit can exist even if the function is not continuous in t_0. The space of n-times Schwarz fractal differentiable functions at t_0 will be denoted by $\mathfrak{S}_n(t_0)$.

Remark 2.3. Using the identity

$$\sum_{j=1}^n (-1)^{j+n} \frac{j^n}{j!(n-j)!} = 1$$

and L'Hôpital's rule, in view of (2.1), we get that when the function f is n-times differentiable at t_0 then

$$(S_n^{1,1} f)(t_0) = f^{(n)}(t_0).$$

Unfortunately, for the fractal derivatives $S_n^{\alpha,\beta} f$ such a nice relationship is no longer true (see Section 4). Sometimes it is convenient to rewrite $S_n^{\alpha,\beta} f$ as

$$(S_n^{\alpha,\beta} f)(t_0) = \begin{cases} \frac{1}{\alpha^n |t_0|^{n(\alpha-1)}} \lim\limits_{h\to 0} \frac{\sum_{j=0}^n \binom{n}{j}(-1)^j f^\beta(t_0 + (n-2j)h)}{2^n h^n}, & t_0 \neq 0, \\ \lim\limits_{h\to 0} \frac{\sum_{j=0}^n \binom{n}{j}(-1)^j f^\beta((n-2j)h)}{2^{n\alpha} h^{n\alpha}}, & t_0 = 0. \end{cases}$$

This rewriting will be useful in Section 5, where the first-order Schwarz fractal derivative is studied in detail ($n = 1$).

Example 2.4. The Dirichlet function

$$f(t) = \begin{cases} 1, & t \in \mathbb{Q}, \\ 0, & t \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

has a first-order Schwarz fractal derivative equal to zero at every $t \in \mathbb{Q}$, but, in contrast, it is not first-order Schwarz fractal differentiable at any $t \in \mathbb{R} \setminus \mathbb{Q}$. On the other hand, neither f' nor $\frac{d^\beta f}{dt^\alpha}$ exist anywhere in $\mathbb{Q}$. The existence of $S_1^{\alpha,\beta} f$ does not imply the continuity of f as does the existence of f' and $\frac{d^\beta f}{dt^\alpha}$.

Example 2.5. Let

$$f(t) = \begin{cases} t^2, & t \geq 0, \\ -t^2, & t < 0. \end{cases}$$

A direct computation shows that the Riemann fractal derivative of f at 0

$$(\mathrm{S}_2^{\alpha,\beta} f)(0) = \lim_{t \to 0} \frac{f^\beta(t) + f^\beta(-t)}{t^{2\alpha}}$$

exists and is equal to zero, despite the non-existence of $f''(0)$. Assume now that $\alpha < 1$ and $\beta > 1$. We have that

$$\frac{d^\beta f}{dt^\alpha}(t_0) = \begin{cases} \frac{2\beta}{\alpha} |t_0|^{2\beta-\alpha-1} t_0, & t_0 \geq 0, \\ \frac{-2\beta}{\alpha} |t_0|^{2\beta-\alpha-1} t_0, & t_0 < 0, \end{cases}$$

whence $\frac{d^1}{dt^\alpha}\left[\frac{d^\beta f}{dt^\alpha}\right](0) = 0$. This simple example shows that ordinary differentiability is a stronger condition than Schwarz fractal differentiability. We performed some simulations in the GeoGebra environment for this example, available at https://www.geogebra.org/m/rm46bvpr.

Remark 2.6. In [10], a generalization of local fractional derivatives was studied. This generalized derivative was defined as

$$(G_T^\vartheta f)(t_0) = \lim_{h \to 0} \frac{1}{h^{\lceil \vartheta \rceil}} \sum_{k=0}^{\lceil \vartheta \rceil} (-1)^k \binom{\lceil \vartheta \rceil}{k} f(t_0 - khT(t_0, \vartheta)), \tag{2.2}$$

where $f : I \to \mathbb{R}$, $I \subset \mathbb{R}$ is an interval, $\vartheta \in \mathbb{R}^+$ and $T(t_0, \vartheta)$ is a positive continuous function on I. For $\alpha = \beta = 1$, the relation (2.1) reduces to

$$(\mathrm{S}_n^{1,1} f)(t_0) = \lim_{h \to 0} \frac{1}{h^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k f\left(t_0 - kh + \frac{n}{2}h\right). \tag{2.3}$$

Note the similarity between (2.2) and (2.3) when $\vartheta = n$ and $T(t_0, \vartheta) \equiv 1$. Therefore, using the kernel $T(t_0, \vartheta)$ we can arrive at a generalization of (2.1) by means of

$$\lim_{t \to t_0} \frac{\sum_{j=0}^{\lfloor \vartheta \rfloor} \binom{\lfloor \vartheta \rfloor}{j} (-1)^j f^\beta\left(t_0 + \frac{n-2j}{2}(t - t_0) T(t_0, \vartheta)\right)}{(t^\alpha - t_0^\alpha)^{\lfloor \vartheta \rfloor}}.$$

The following proposition is straightforward.

Proposition 2.7. *Let $\alpha, \beta > 0$, $c \in \mathbb{R}$, and let $f \in \mathfrak{S}_n(t_0)$. The following statements hold:*

(1) *We have*
$$(\mathrm{S}_n^{\alpha,\beta} f)(t_0) = (\mathrm{S}_n^{\alpha,1} f^\beta)(t_0).$$

(2) *The constant function c is Schwarz fractal differentiable of any order in the whole space, and*
$$(\mathrm{S}_k^{\alpha,\beta} c)(t_0) = 0 \quad \text{for all } t_0 \in \mathbb{R},\, k \in \mathbb{N}^*.$$
In addition,
$$(\mathrm{S}_2^{\alpha,\beta}(at+b))(t_0) = \begin{cases} \beta(\beta-1)(a/\alpha)^2 |t_0|^{2-2\alpha} |at_0+b|^{\beta-2}, & \beta \neq 1, \\ 0, & \beta = 1, \end{cases}$$
where $a, b \in \mathbb{R}$ are such that $at_0 + b \neq 0$ if $\beta \in (0,2] \setminus \{1\}$ and $t_0 \neq 0$ if $\alpha > 1$.

(3) *The function $cf \in \mathfrak{S}_n(t_0)$ and satisfies*
$$(\mathrm{S}_n^{\alpha,\beta} cf)(t_0) = c^\beta (\mathrm{S}_n^{\alpha,\beta} f)(t_0).$$

(4) *$\mathrm{S}_n^{\alpha,\beta}$ is a linear operator if and only if $\beta = 1$.*

Proof. (1) By definition, we have that
$$\begin{aligned}(\mathrm{S}_n^{\alpha,\beta} f)(t_0) &= \lim_{t \to t_0} \frac{\sum_{j=0}^{n} \binom{n}{j} (-1)^j f^\beta \left(t_0 + \frac{n-2j}{2}(t-t_0)\right)}{(t^\alpha - t_0^\alpha)^n} \\ &= (\mathrm{S}_n^{\alpha,1} f^\beta)(t_0).\end{aligned}$$

(2) Since $\sum_{j=0}^{k} \binom{k}{j}(-1)^j = 0$, $k \in \mathbb{N}^*$, it follows that
$$(\mathrm{S}_k^{\alpha,\beta} c)(t_0) = \lim_{t \to t_0} \frac{\sum_{j=0}^{k} \binom{k}{j} (-1)^j c^\beta}{(t^\alpha - t_0^\alpha)^k} = 0 \quad \text{for all } t_0 \in \mathbb{R}.$$
The function $(at+b)^\beta$, $a, b \in \mathbb{R}$, $\beta \neq 1$, satisfies
$$\frac{d((at+b)^\beta)}{dt} = a\beta |at+b|^{\beta-1}.$$
Therefore, the function $(at+b)^\beta$ is differentiable at all $t_0 \in \mathbb{R}$ except for the case where $t_0 = -\frac{b}{a}$, $a \neq 0$ and $\beta < 1$. Note that for all $t_0 \neq -\frac{b}{a}$, we have
$$\frac{d(|at+b|^{\beta-1})}{dt}(t_0) = a(\beta-1)|at_0+b|^{\beta-2}.$$
However, if $\beta \in (0,2] \setminus \{1\}$, then the function $|at+b|^{\beta-1}$ is not differentiable at $t_0 = -\frac{b}{a}$, $a \neq 0$. If $\beta > 2$ we obtain
$$\frac{d(|at+b|^{\beta-1})}{dt}\left(-\frac{b}{a}\right) = \lim_{h \to 0} \frac{|ah|^{\beta-1}}{h} = \lim_{h \to 0} |a|^{\beta-1} \frac{|h|}{h} |h|^{\beta-2} = 0.$$

Using Remark 2.3 we can rewrite $(\mathrm{S}_2^{\alpha,\beta} f)(t_0)$ as

$$(\mathrm{S}_2^{\alpha,\beta} f)(t_0) = \begin{cases} \frac{1}{\alpha^2|t_0|^{2(\alpha-1)}} \lim\limits_{h\to 0} \frac{\sum_{j=0}^{2}\binom{2}{j}(-1)^j f^\beta(t_0+(2-2j)h)}{2^2h^2}, & t_0 \neq 0, \\ \lim\limits_{h\to 0} \frac{\sum_{j=0}^{2}\binom{2}{j}(-1)^j f^\beta((2-2j)h)}{2^{2\alpha}h^{2\alpha}}, & t_0 = 0. \end{cases}$$

Since $(at+b)^\beta$ is twice continuously differentiable at $t_0 \neq 0$ if $\alpha > 1$, and such that $at_0 + b \neq 0$ if $\beta \in (0,2] \setminus \{1\}$, at this point t_0 we have

$$\frac{d^2((at+b)^\beta)}{dt^2}(t_0) = \lim_{h\to 0} \frac{\sum_{j=0}^{2}\binom{2}{j}(-1)^j\,(a(t_0+(2-2j)h)+b)^\beta}{2^2h^2},$$

hence

$$(\mathrm{S}_2^{\alpha,\beta}(at+b))(t_0) = \beta(\beta-1)(a/\alpha)^2|t_0|^{2-2\alpha}|at_0+b|^{\beta-2}.$$

When $t_0 = 0$ with $\alpha \leq 1$ and $b \neq 0$ if $\beta \in (0,2] \setminus \{1\}$, we have

$$(\mathrm{S}_2^{\alpha,\beta}(at+b))(0) = \frac{d^2((at+b)^\beta)}{dt^2}(0) \cdot \lim_{h\to 0}(2h)^{2-2\alpha} = \begin{cases} 0, & \alpha < 1, \\ a^2\beta(\beta-1)|b|^{\beta-2}, & \alpha = 1. \end{cases}$$

Analogously, if $\beta = 1$ we obtain $(\mathrm{S}_2^{\alpha,\beta}(at+b))(t_0) = 0$.

(3) We have

$$\begin{aligned} (\mathrm{S}_n^{\alpha,\beta} cf)(t_0) &= \lim_{t\to t_0} \frac{\sum_{j=0}^{n}\binom{n}{j}(-1)^j (cf)^\beta\left(t_0 + \frac{n-2j}{2}(t-t_0)\right)}{(t^\alpha - t_0^\alpha)^n} \\ &= \lim_{t\to t_0} \frac{\sum_{j=0}^{n}\binom{n}{j}(-1)^j c^\beta f^\beta\left(t_0 + \frac{n-2j}{2}(t-t_0)\right)}{(t^\alpha - t_0^\alpha)^n} \\ &= c^\beta(\mathrm{S}_n^{\alpha,\beta} f)(t_0). \end{aligned}$$

(4) Given two arbitrary real functions f and g, the result follows because the equality $(f+g)^\beta = f^\beta + g^\beta$ holds if and only if $\beta = 1$. ☐

In particular, note that if f is (β,α)-fractal differentiable at t_0 then we have

$$\begin{aligned} (\mathrm{S}_1^{\alpha,\beta} f)(t_0) &= \lim_{t\to t_0} \frac{f^\beta\left(t_0 + \frac{t-t_0}{2}\right) - f^\beta\left(t_0 - \frac{t-t_0}{2}\right)}{t^\alpha - t_0^\alpha} \\ &= \lim_{z=\frac{1}{2}t+\frac{1}{2}t_0\to t_0} \frac{f^\beta(z) - f^\beta(2t_0 - z)}{2(z^\alpha - t_0^\alpha)} \cdot \frac{2(z^\alpha - t_0^\alpha)}{(2z-t_0)^\alpha - t_0^\alpha} \\ &= \lim_{t\to t_0} \frac{f^\beta(t) - f^\beta(t_0) + f^\beta(t_0) - f^\beta(2t_0 - t)}{2(t^\alpha - t_0^\alpha)} \\ &= \frac{1}{2}\frac{d^\beta f}{dt^\alpha}(t_0) + \frac{1}{2}\lim_{t\to t_0} \frac{f^\beta(t_0) - f^\beta(2t_0 - t)}{t^\alpha - t_0^\alpha} \\ &= \frac{1}{2}\frac{d^\beta f}{dt^\alpha}(t_0) + \frac{1}{2}\lim_{2t_0 - t\to t_0} \frac{f^\beta(2t_0 - t) - f^\beta(t_0)}{(2t_0 - t)^\alpha - t_0^\alpha} \\ &= \frac{d^\beta f}{dt^\alpha}(t_0). \end{aligned}$$

Let $\epsilon > 0$ and $f, g \in \mathfrak{S}_1(t_0) \cap C(t_0 - \epsilon, t_0 + \epsilon)$. We have

$$\begin{aligned}
&(\mathrm{S}_1^{\alpha,\beta} fg)(t_0) \\
&\quad = \lim_{t \to t_0} \frac{(fg)^\beta(t) - (fg)^\beta(2t_0 - t)}{2(t^\alpha - t_0^\alpha)} \\
&\quad = \lim_{t \to t_0} \frac{(fg)^\beta(t) - f^\beta(2t_0 - t)g^\beta(t) + f^\beta(2t_0 - t)g^\beta(t) - (fg)^\beta(2t_0 - t)}{2(t^\alpha - t_0^\alpha)} \\
&\quad = (\mathrm{S}_1^{\alpha,\beta} f)(t_0) g^\beta(t_0) + f^\beta(t_0)(\mathrm{S}_1^{\alpha,\beta} g)(t_0).
\end{aligned}$$

In general, the space $\mathfrak{S}_1(t_0)$ is not a Banach algebra; however, if the continuity of the function is required in a neighborhood of t_0, then $\mathfrak{S}_1(t_0)$ becomes a Banach algebra. It will be shown in the following section that the continuity of $(\mathrm{S}_1^{\alpha,\beta} f)(t)$ at a point t_0 and the continuity of $f(t)$ in a neighborhood of t_0 imply the existence of $\frac{d^\beta f}{dt^\alpha}(t_0)$.

We shall be concerned with the following subclasses:

- $\Sigma = \{f : S_1^{\alpha,\beta} f \text{ exists everywhere}\}$,
- $m\Sigma = \{f \in \Sigma : f \text{ is measurable}\}$,
- $\sigma = \{f \in \Sigma : S_1^{\alpha,\beta} f \text{ is finite everywhere}\}$.

A function $f \in \sigma$ is symmetrically continuous at each $t \in \mathbb{R}$, that is,

$$\lim_{h \to 0} (f(t + h) - f(t - h)) = 0$$

for each $t \in \mathbb{R}$. Stein and Zygmund proved that a symmetrically continuous function is continuous almost everywhere, and therefore $\sigma \subset m\Sigma$ (see [16, Lemma 9]). A proof quite analogous to that of [12, Theorem 2.1] shows that $S_1^{\alpha,\beta} f$ belongs to the first Baire class for any $f \in \Sigma$. We refer the reader to the paper [12], many of whose results can also be applied to this fractional context.

In real analysis, two differentiable functions whose derivatives are equal throughout an interval must differ by a constant in that interval. A similar result for $S_1^{\alpha,\beta}$ is no longer true, as proved in the following example.

Example 2.8. Define the two functions as follows:

$$g(t) = \begin{cases} 0, & t = \frac{1}{n} \quad (n = \pm 1, \pm 2, \dots), \\ 3, & \text{otherwise}, \end{cases}$$

and

$$h(t) = \begin{cases} t, & t \text{ is an integer}, \\ \pi, & \text{otherwise}. \end{cases}$$

At all points, $(S_1^{\alpha,\beta} g)(t) = (S_1^{\alpha,\beta} h)(t) = 0$, but g and h obviously do not differ by a constant.

The following proposition is essential for the proof of the results to be presented below.

Proposition 2.9. *Let $f(t)$ be continuous on $a \leq t < b$ and let $\mathrm{S}_1^{\alpha,\beta}(t)$ exist on $a < t < b$. Let $f(b) > f(a)$ (resp. $f(b) < f(a)$), then there exists a point c, $a < c < b$, such that $\mathrm{S}_1^{\alpha,\beta}(c) \geq 0$ (resp. $\mathrm{S}_1^{\alpha,\beta}(c) \leq 0$).*

Proof. Since the function $x|x|^{\beta-1}$ is increasing in $\mathbb{R}$, it follows that $f^\beta(a) < f^\beta(b)$. Let d be such that $f^\beta(a) < d < f^\beta(b)$. The set $\{t : f^\beta(t) > d,\, a < t < b\}$ is bounded below by a, and applying the greatest lower bound property we obtain that it has a infimum c such that $c \neq a$ and $c \neq b$. In addition, $(\mathrm{S}_1^{\alpha,\beta} f)(c) \geq 0$, since there are points $t > c$ in every neighborhood of c such that $f^\beta(t) > f^\beta(c)$ and $f^\beta(t) \leq f^\beta(c)$ for $a \leq t \leq c$. The second part of the proof is analogous. $\square$

3. Rolle-type theorems and consequences

The classical mean value theorem does not hold for the first-order Schwarz fractal derivative. This is illustrated by the example below.

Example 3.1. Consider $f(t) = |t|^{\frac{\alpha}{\beta}}$. Then

$$\begin{aligned}
(S_1^{\alpha,\beta} f)(t_0) &= \lim_{t \to t_0} \frac{f^\beta(t_0 + \frac{1}{2}(t - t_0)) - f^\beta(t_0 - \frac{1}{2}(t - t_0))}{t^\alpha - t_0^\alpha} \\
&= \lim_{t \to t_0} \frac{|t_0 + \frac{1}{2}(t - t_0)|^\alpha - |t_0 - \frac{1}{2}(t - t_0)|^\alpha}{t^\alpha - t_0^\alpha} \\
&= \begin{cases} \frac{|t_0|}{t_0}, & t_0 \neq 0, \\ 0, & t_0 = 0. \end{cases}
\end{aligned}$$

If $a = -2$ and $b = 3$, then

$$\frac{f^\beta(b) - f^\beta(a)}{b^\alpha - a^\alpha} = \frac{3^\alpha - 2^\alpha}{3^\alpha + 2^\alpha},$$

which is not a value in the range of $S_1^{\alpha,\beta} f$.

In this section we will see a weak version of the Rolle theorem for the first-order Schwarz fractal derivative and some immediate consequences.

Theorem 3.2. *Let $f(t)$ be continuous on $a \leq t \leq b$ and let $(\mathrm{S}_1^{\alpha,\beta} f)(t)$ exist on $a < t < b$, and let $f(a) = f(b)$. Then there exists a point $t_0 \in (a, b)$ such that $(\mathrm{S}_1^{\alpha,\beta} f)(t_0) \geq 0$ and a point $t_1 \in (a, b)$ such that $(\mathrm{S}_1^{\alpha,\beta} f)(t_1) \leq 0$.*

Proof. The case $f(t) \equiv f(a)$ is straightforward. Conversely, there exists either a point c such that $f(c) > f(a)$, or a point d such that $f(d) < f(a)$, or both. By Proposition 2.9, there exist points t_0 and t_1, $a < t_0 < c < t_1 < b$ or $a < t_1 < d < t_0 < b$, such that $(\mathrm{S}_1^{\alpha,\beta} f)(t_0) \geq 0$ and $(\mathrm{S}_1^{\alpha,\beta} f)(t_1) \leq 0$. $\square$

Theorem 3.3. *Let $f(t)$ be continuous on $a \leq t \leq b$, and let $(\mathrm{S}_1^{\alpha,\beta} f)(t)$ exist on $a < t < b$. Then there exist points $t_0, t_1 \in (a, b)$ such that*

$$(\mathrm{S}_1^{\alpha,\beta} f)(t_1) \leq \frac{f^\beta(b) - f^\beta(a)}{b^\alpha - a^\alpha} \leq (\mathrm{S}_1^{\alpha,\beta} f)(t_0).$$

Proof. Let $g(t) = \left[f^\beta(t) - f^\beta(a) - \frac{f^\beta(b)-f^\beta(a)}{b^\alpha-a^\alpha}(t^\alpha - a^\alpha)\right]^{\frac{1}{\beta}}$. By Theorem 3.2, as $g(a) = g(b) = 0$, there exists a point $t_0 \in (a,b)$ such that $(\mathrm{S}_1^{\alpha,\beta} g)(t_0) \geq 0$, and a point $t_1 \in (a,b)$ such that $(\mathrm{S}_1^{\alpha,\beta} g)(t_1) \leq 0$. Since

$$\begin{aligned}
(\mathrm{S}_1^{\alpha,\beta} g)(t_0) &= \lim_{t\to t_0} \frac{g^\beta(t) - g^\beta(2t_0 - t)}{2(t^\alpha - t_0^\alpha)} \\
&= \lim_{t\to t_0} \frac{f^\beta(t) - f^\beta(2t_0 - t) - \frac{f^\beta(b)-f^\beta(a)}{b^\alpha-a^\alpha}(t^\alpha - (2t_0 - t)^\alpha)}{2(t^\alpha - t_0^\alpha)} \\
&= (\mathrm{S}_1^{\alpha,\beta} f)(t_0) - \frac{f^\beta(b) - f^\beta(a)}{b^\alpha - a^\alpha}
\end{aligned}$$

and also

$$(\mathrm{S}_1^{\alpha,\beta} g)(t_1) = (\mathrm{S}_1^{\alpha,\beta} f)(t_1) - \frac{f^\beta(b) - f^\beta(a)}{b^\alpha - a^\alpha},$$

the result follows directly. $\square$

Definition 3.4. A function $f : \mathbb{R} \to \mathbb{R}$ is said to have the *Darboux property* (or *intermediate value property*) if, whenever $x, y \in \mathbb{R}$ and ϵ is any real number between $f(x)$ and $f(y)$, there exists a number t between x and y such that $f(t) = \epsilon$.

To a given real-valued function f we associate the set

$$C(f) = \{t : f \text{ is continuous at } t\}.$$

Remark 3.5. In Theorem 3.3 the continuity of $f(t)$ on an interval $[a,b]$ can be replaced by the weaker condition

$$f \in m\Sigma, \quad a, b \in C(f),$$

and the results still hold. The proof is similar taking into account that $C(f)$ is dense.

Theorem 3.6. *If $f \in m\Sigma$ is such that $\mathrm{S}_1^{\alpha,\beta} f$ has the Darboux property, then for each $\varphi, \psi \in C(f)$ such that $\varphi < \psi$ there is a $\vartheta \in (\varphi, \psi)$ such that*

$$\frac{f^\beta(\psi) - f^\beta(\varphi)}{\psi^\alpha - \varphi^\alpha} = (\mathrm{S}_1^{\alpha,\beta} f)(\vartheta).$$

Proof. Using Theorem 3.3 and taking into account Remark 3.5, we obtain that there exist points $t_0, t_1 \in (\varphi, \psi)$ such that

$$(\mathrm{S}_1^{\alpha,\beta} f)(t_1) \leq \frac{f^\beta(\psi) - f^\beta(\varphi)}{\psi^\alpha - \varphi^\alpha} \leq (\mathrm{S}_1^{\alpha,\beta} f)(t_0).$$

Since $\mathrm{S}_1^{\alpha,\beta} f$ has the Darboux property, we can ensure that there exists a $\vartheta \in (\varphi, \psi)$ such that

$$\frac{f^\beta(\psi) - f^\beta(\varphi)}{\psi^\alpha - \varphi^\alpha} = (\mathrm{S}_1^{\alpha,\beta} f)(\vartheta),$$

and we are done. $\square$

Theorem 3.7. *Let $f(t)$ and $(\mathrm{S}_1^{\alpha,\beta} f)(t)$ be continuous on $a < t < b$. Then $\frac{d^\beta f}{dt^\alpha}(t)$ exists and*

$$\frac{d^\beta f}{dt^\alpha}(t) = (\mathrm{S}_1^{\alpha,\beta} f)(t).$$

Proof. For ϵ sufficiently small such that $a < t+\epsilon < b$, we have by Theorem 3.3 that there exist t_0 and t_1 strictly between t and $t+\epsilon$ such that

$$(\mathrm{S}_1^{\alpha,\beta} f)(t_1) \leq \frac{f^\beta(t+\epsilon) - f^\beta(t)}{(t+\epsilon)^\alpha - t^\alpha} \leq (\mathrm{S}_1^{\alpha,\beta} f)(t_0).$$

By the continuity of $(\mathrm{S}_1^{\alpha,\beta} f)(t)$, there exists t_2 strictly between t and $t+\epsilon$ such that

$$(\mathrm{S}_1^{\alpha,\beta} f)(t_2) = \frac{f^\beta(t+\epsilon) - f^\beta(t)}{(t+\epsilon)^\alpha - t^\alpha}.$$

Taking $\epsilon \to 0$ gives the desired result. □

Theorem 3.8. *Let $(\mathrm{S}_1^{\alpha,\beta} f)(t)$ be continuous at a point t_0 and let $f(t)$ be continuous in a neighborhood of t_0. Then $\frac{d^\beta f}{dt^\alpha}(t_0)$ exists and*

$$\frac{d^\beta f}{dt^\alpha}(t_0) = (\mathrm{S}_1^{\alpha,\beta} f)(t_0).$$

Proof. For $\epsilon_1 > 0$, there exists a neighborhood V of t_0 such that if $t \in V$, $f(t)$ is continuous and

$$|(\mathrm{S}_1^{\alpha,\beta} f)(t) - (\mathrm{S}_1^{\alpha,\beta} f)(t_0)| < \epsilon_1.$$

By Theorem 3.3, there exist t_1 and t_2 strictly between t_0 and $t_0+\epsilon_2$ such that

$$(\mathrm{S}_1^{\alpha,\beta} f)(t_2) \leq \frac{f^\beta(t_0+\epsilon_2) - f^\beta(t_0)}{(t_0+\epsilon_2)^\alpha - t_0^\alpha} \leq (\mathrm{S}_1^{\alpha,\beta} f)(t_1),$$

with $(t_0+\epsilon_2) \in V$, $\epsilon_2 \neq 0$. Hence

$$(\mathrm{S}_1^{\alpha,\beta} f)(t_0) - \epsilon_1 < \frac{f^\beta(t_0+\epsilon_2) - f^\beta(t_0)}{(t_0+\epsilon_2)^\alpha - t_0^\alpha} < (\mathrm{S}_1^{\alpha,\beta} f)(t_0) + \epsilon_1.$$

The existence of $\frac{d^\beta f}{dt^\alpha}(t_0)$ follows and $\frac{d^\beta f}{dt^\alpha}(t_0) = (\mathrm{S}_1^{\alpha,\beta} f)(t_0)$. □

The following theorem is a generalization of [12, Theorem 7.3, p. 597].

Theorem 3.9. *If $f \in \sigma$, then for each $t \in \mathbb{R}$,*

$$\begin{aligned} &\liminf_{h\to 0} \frac{(S_1^{\alpha,\beta} f)(t+h) + (S_1^{\alpha,\beta} f)(t-h)}{2} \\ &\qquad \leq (S_1^{\alpha,\beta} f)(t) \leq \limsup_{h\to 0} \frac{(S_1^{\alpha,\beta} f)(t+h) + (S_1^{\alpha,\beta} f)(t-h)}{2}. \end{aligned} \tag{3.1}$$

Proof. Suppose that

$$(S_1^{\alpha,\beta} f)(t) > \limsup_{h\to 0} \frac{(S_1^{\alpha,\beta} f)(t+h) + (S_1^{\alpha,\beta} f)(t-h)}{2}.$$

Through a translation and the addition of an appropriate constant, we may assume that $t = 0$ and that

$$(S_1^{\alpha,\beta} f)(0) > 0 > \limsup_{h\to 0} \frac{(S_1^{\alpha,\beta} f)(h) + (S_1^{\alpha,\beta} f)(-h)}{2}. \tag{3.2}$$

Set $g(t) = \left[\frac{f^\beta(t) - f^\beta(-t)}{2}\right]^{\frac{1}{\beta}}$. Then $g \in \sigma$ and, setting $\tau = \frac{1}{2}(t - t_0)$,

$$\begin{aligned}
(\mathrm{S}_1^{\alpha,\beta} g)(t_0) &= \lim_{t\to t_0} \frac{g^\beta(t_0+\tau) - g^\beta(t_0-\tau)}{t^\alpha - t_0^\alpha} \\
&= \frac{1}{2}\lim_{t\to t_0} \frac{f^\beta(t_0+\tau) - f^\beta(-t_0-\tau) - f^\beta(t_0-\tau) + f^\beta(-t_0+\tau)}{t^\alpha - t_0^\alpha} \\
&= \frac{1}{2}[(\mathrm{S}_1^{\alpha,\beta} f)(t_0) + (\mathrm{S}_1^{\alpha,\beta} f)(-t_0)].
\end{aligned}$$

Since $(\mathrm{S}_1^{\alpha,\beta} g)(0) = (\mathrm{S}_1^{\alpha,\beta} f)(0)$, relation (3.2) implies that

$$(\mathrm{S}_1^{\alpha,\beta} g)(0) > 0 > \limsup_{h\to 0} (\mathrm{S}_1^{\alpha,\beta} g)(h). \tag{3.3}$$

Then there exists a $\delta > 0$ such that $(\mathrm{S}_1^{\alpha,\beta} g)(h) < 0$ whenever $0 < |h| < \delta$. For any $f \in m\Sigma$ we let

$$M_f = \left\{x : \Big|\limsup_{t\to x,\, t\in C(f)} f(t)\Big| = \infty\right\}$$

and define μ_f to be the real function

$$\mu_f(x) = \begin{cases} \limsup\limits_{t\to x,\, t\in C(f)} f(t), & x \notin M_f, \\ f(x), & x \in M_f. \end{cases}$$

If $g \in \sigma$ with $\mathrm{S}_1^{\alpha,\beta} g < 0$ a.e., then μ_g is continuous and nonincreasing on $(-\delta, \delta)$. Therefore, $(\mathrm{S}_1^{\alpha,\beta} g)(0) = (\mathrm{S}_1^{\alpha,\beta} \mu_g)(0) \leq 0$, which contradicts (3.3) and the right-hand inequality in (3.1) is established. The left-hand inequality is established in an analogous manner. □

Theorem 3.9 establishes that $S_1^{\alpha,\beta} f$ must satisfy a weaker "Darboux-like" condition at every point given by formula (3.1). It is also interesting that if the first-order Schwarz fractal derivative exists almost everywhere, then the fractal derivative exists almost everywhere.

Theorem 3.10. *Let $S_1^{\alpha,\beta} f$ be bounded on (a, b), with $\alpha \geq 1$, and let f be continuous on (a, b). Then f^β satisfies the Lipschitz condition on (a, b), i.e., there exists a constant M such that*

$$|f^\beta(x) - f^\beta(t)| \leq M|x - t|$$

for any $x, t \in (a, b)$.

Proof. Assume $x \neq t$, and, to be specific, $x < t$. Applying Theorem 3.3, it is apparent that no matter where x and t are in (a,b) there exist points p and q such that $x < p < t$, $x < q < t$, and

$$(S_1^{\alpha,\beta} f)(p) \leq \frac{f^\beta(x) - f^\beta(t)}{x^\alpha - t^\alpha} \leq (S_1^{\alpha,\beta} f)(q).$$

Since $(S_1^{\alpha,\beta} f)(x)$ is bounded on (a,b), choose

$$M = \max\Big(\big|\inf_{(a,b)} S_1^{\alpha,\beta} f\big|, \big|\sup_{(a,b)} S_1^{\alpha,\beta} f\big|\Big).$$

This implies that

$$-M \leq \frac{f^\beta(x) - f^\beta(t)}{x^\alpha - t^\alpha} \leq M.$$

Therefore,

$$|f^\beta(x) - f^\beta(t)| \leq M|x^\alpha - t^\alpha| \leq M'|x - t|. \qquad \square$$

Example 3.11. The function $f(t) = \sqrt{t}$ defined on $(0,1)$ satisfies $(S_1^{1,2} f)(t) = 1$ and is obviously continuous on the interval. Clearly, $f^2(t) = t$ is Lipschitz continuous; however, $f(t)$ is not Lipschitz continuous on $(0,1)$. As is well known, the function $f(t) = \sqrt{t}$ is uniformly continuous, Hölder continuous of class $C^{0,\nu}$ for $\nu \leq \frac{1}{2}$, and absolutely continuous on $[0,1]$.

Definition 3.12. A function f is said to satisfy *condition (F)* at a point c, if f crosses every straight line through $(c, f(c))$ at most a finite number of times in some neighborhood V_c.

In [4] the following existence theorem for the common symmetric derivative was proved.

Theorem 3.13. *Let f satisfy the Lipschitz condition on (a,b) and let condition (F) be satisfied for each $t \in (a,b)$. Then $S_1^{1,1} f$ exists and is bounded for $t \in (a,b)$.*

Since

$$(S_1^{\alpha,\beta} f)(t) = \frac{1}{\alpha |t|^{\alpha-1}} \cdot (S_1^{1,1} f^\beta)(t), \quad t \neq 0,$$

a simpler version of Theorem 3.13 can be obtained.

Theorem 3.14. *Let f^β satisfy the Lipschitz condition on (a,b) and let condition (F) be satisfied for each $t \in (a,b)$. If $\alpha = 1$ then $S_1^{\alpha,\beta} f$ exists and is bounded for $t \in (a,b)$; otherwise, $S_1^{\alpha,\beta} f$ exists and is bounded in any subinterval of (a,b) that does not contain zero as an accumulation point.*

In closing, a conjecture is stated. This is not to indicate that these are the only questions still unanswered but to point out a few ideas which can be further pursued.

Conjecture 3.15. If any straight line through the origin intersects an odd function f an infinite number of times in each neighborhood of the origin, then $(S_1^{\alpha,\beta} f)(0)$ does not exist.

4. Riemann fractal derivative

The study of the generalized Riemann derivative has attracted generations of physicists and mathematicians. One of the main areas benefiting from these developments is numerical analysis, since the use of generalized Riemann derivatives leads to the solution of a wider class of problems that are not solvable with classical tools. The Riemann fractal derivative can be seen as a peculiar version of these types of generalized derivatives that have been studied before in the literature (see e.g. [13]).

In this section, we will obtain a necessary condition for the determination of a local extreme in relation to the Riemann fractal derivative.

Now note that if $f \in C^2(t_0-\epsilon, t_0+\epsilon)$, with $\epsilon > 0$, and if we also assume that $t_0 \neq 0$ when $\alpha > 1$, and $f(t_0) \neq 0$ when $\beta \in (0,2) \setminus \{1\}$, then we have

$$
\begin{aligned}
&(S_2^{\alpha,\beta} f)(t_0)\\
&\quad = \lim_{t\to t_0} \frac{f^\beta(t) + f^\beta(2t_0 - t) - 2f^\beta(t_0)}{(t^\alpha - t_0^\alpha)^2}\\
&\quad = \lim_{t\to t_0} \frac{\beta|f(t)|^{\beta-1} f'(t) - \beta|f(2t_0-t)|^{\beta-1} f'(2t_0-t)}{2(t^\alpha - t_0^\alpha)\alpha|t|^{\alpha-1}}\\
&\quad = \lim_{t\to t_0} \frac{\begin{array}{l}\beta(\beta-1)|f(t)|^{\beta-2}(f'(t))^2 + \beta|f(t)|^{\beta-1}f''(t)\\ \quad + \beta(\beta-1)|f(2t_0-t)|^{\beta-2}(f'(2t_0-t))^2 + \beta|f(2t_0-t)|^{\beta-1}f''(2t_0-t)\end{array}}{2\alpha(2\alpha-1)|t|^{2\alpha-2} - 2\alpha(\alpha-1)|t_0|^\alpha|t|^{\alpha-2}}\\
&\quad = \frac{\beta(\beta-1)|f(t_0)|^{\beta-2}(f'(t_0))^2 + \beta|f(t_0)|^{\beta-1}f''(t_0)}{\alpha^2|t_0|^{2\alpha-2}}.
\end{aligned}
$$

and, on the other hand,

$$
\begin{aligned}
&\frac{d^1}{dt^\alpha}\left[\frac{d^\beta f}{dt^\alpha}\right](t_0)\\
&\quad = \frac{\beta}{\alpha}\frac{d^1[|t|^{1-\alpha}|f(t)|^{\beta-1}f'(t)]}{dt^\alpha}(t_0)\\
&\quad = \frac{\beta}{\alpha^2}|t_0|^{1-\alpha}\big\{(1-\alpha)|t_0|^{-\alpha}|f(t_0)|^{\beta-1}f'(t_0) + (\beta-1)|t_0|^{1-\alpha}|f(t)|^{\beta-2}(t_0)(f'(t_0))^2\\
&\qquad\qquad + |t_0|^{1-\alpha}|f(t_0)|^{\beta-1}f''(t_0)\big\}\\
&\quad = \frac{1-\alpha}{\alpha}|t_0|^{-\alpha}\frac{d^\beta f}{dt^\alpha}(t_0) + (\mathrm{S}_2^{\alpha,\beta} f)(t_0).
\end{aligned}
$$

Indeed, the special case when $\alpha = \beta = 1$ confirms that $(\mathrm{S}_2^{1,1} f)(t_0) = f''(t_0)$.

Theorem 4.1. *Let f be a continuous function in a neighborhood of t_0. If f has a local maximum (resp. minimum) at t_0, then $(\mathrm{S}_2^{\alpha,\beta} f)(t_0) \le 0$ (resp. $(\mathrm{S}_2^{\alpha,\beta} f)(t_0) \ge 0$).*

Proof. If f has a local maximum (resp. minimum) at t_0, then for a sufficiently small $\epsilon > 0$ one must have $f^\beta(t_0 \pm \epsilon) \le f^\beta(t_0)$ (resp. $f^\beta(t_0 \pm \epsilon) \ge f^\beta(t_0)$), hence it is a fact that $(\mathrm{S}_2^{\alpha,\beta} f)(t_0) \le 0$ (resp. $(\mathrm{S}_2^{\alpha,\beta} f)(t_0) \ge 0$). □

Now let us see that if f and g are Riemann fractal differentiable at t_0 and continuous in a neighborhood of t_0, then

$$
\begin{aligned}
&(\mathrm{S}_2^{\alpha,\beta} fg)(t_0)\\
&= \lim_{t\to t_0} \frac{(fg)^\beta(t) + (fg)^\beta(2t_0 - t) - 2(fg)^\beta(t_0)}{(t^\alpha - t_0^\alpha)^2}\\
&= \lim_{t\to t_0} \Bigg(\frac{(fg)^\beta(t) + f^\beta(2t_0 - t)g^\beta(t) - 2f^\beta(t_0)g^\beta(t) + (fg)^\beta(2t_0 - t) - 2(fg)^\beta(t_0)}{(t^\alpha - t_0^\alpha)^2}\\
&\qquad + \frac{2f^\beta(t_0)g^\beta(t) - f^\beta(2t_0 - t)g^\beta(t)}{(t^\alpha - t_0^\alpha)^2} \Bigg)\\
&= (\mathrm{S}_2^{\alpha,\beta} f)(t_0)g^\beta(t_0)\\
&\quad + \lim_{t\to t_0} \Bigg(\frac{(fg)^\beta(2t_0 - t) + f^\beta(2t_0 - t)g^\beta(t) - 2f^\beta(2t_0 - t)g^\beta(t_0)}{(t^\alpha - t_0^\alpha)^2}\\
&\qquad + \frac{2f^\beta(2t_0 - t)g^\beta(t_0) - 2f^\beta(2t_0 - t)g^\beta(t) - 2(fg)^\beta(t_0) + 2f^\beta(t_0)g^\beta(t)}{(t^\alpha - t_0^\alpha)^2} \Bigg)\\
&= (\mathrm{S}_2^{\alpha,\beta} f)(t_0)g^\beta(t_0) + f^\beta(t_0)(\mathrm{S}_2^{\alpha,\beta} g)(t_0)\\
&\quad + 2 \lim_{t\to t_0} \frac{(f^\beta(2t_0 - t) - f^\beta(t_0)) \cdot (g^\beta(t_0) - g^\beta(t))}{(t^\alpha - t_0^\alpha)^2}.
\end{aligned}
$$

Thus, if f and g are (β,α)-fractal differentiable at t_0 and $f, g \in \mathfrak{S}_2(t_0)$, then we obtain the following Leibniz rule for the Riemann fractal derivative:

$$
(\mathrm{S}_2^{\alpha,\beta} fg)(t_0) = (\mathrm{S}_2^{\alpha,\beta} f)(t_0)g^\beta(t_0) + f^\beta(t_0)(\mathrm{S}_2^{\alpha,\beta} g)(t_0) + 2\frac{d^\beta f}{dt^\alpha}(t_0) \cdot \frac{d^\beta g}{dt^\alpha}(t_0). \quad (4.1)
$$

As was shown in Proposition 2.2, the (β,α)-fractal differentiable functions form a Banach algebra. Nevertheless, in general the Schwarz fractal differentiable functions do not form a Banach algebra, even in the special situation when $\alpha = \beta = 1$.

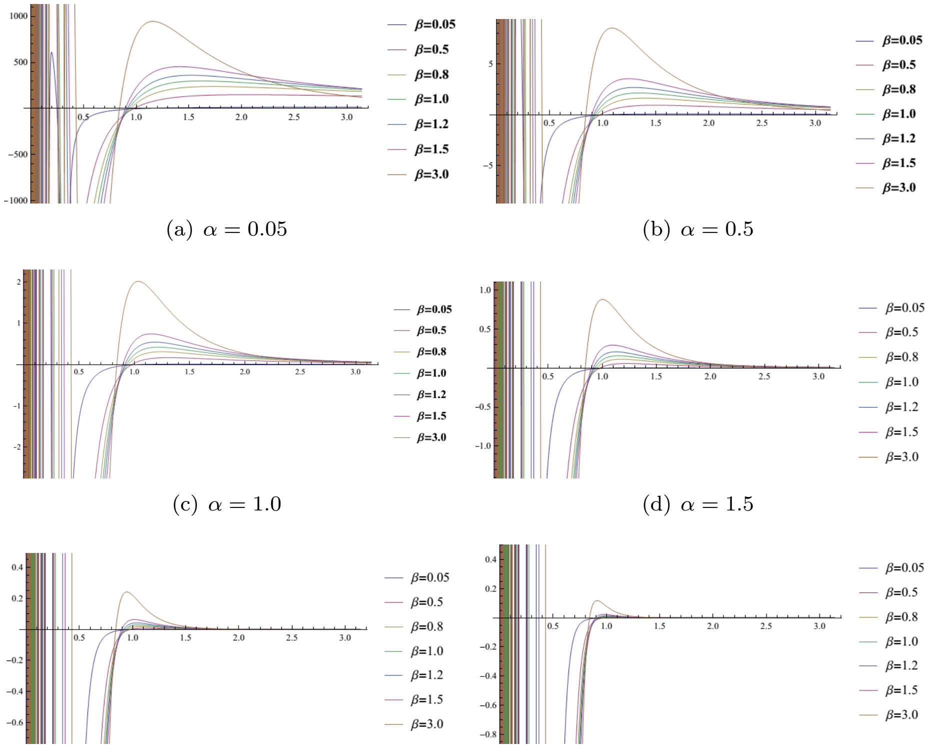
Example 4.2. Let

$$
f(t) = \begin{cases} \sin(1/t), & t \neq 0,\\ 0, & t = 0, \end{cases}
$$

and $g(t) = t$. It is clear that

$$
(\mathrm{S}_2^{1,1} f)(0) = (\mathrm{S}_2^{1,1} g)(0) = 0,
$$

while $(\mathrm{S}_2^{1,1} fg)(0) = \lim_{t\to 0} \frac{2\sin(1/t)}{t}$ does not exist. However, thanks to relation (4.1) we can state that the space $\mathfrak{S}_2(t_0) \cap \mathfrak{F}(t_0) \cap C(t_0 - \epsilon, t_0 + \epsilon)$, $\epsilon > 0$, does constitute a Banach algebra. Using *Wolfram Mathematica* 14.1, we obtained Figure 1, which illustrates the plots of the Riemann fractal derivative $\mathrm{S}_2^{\alpha,\beta} f$ for different values of α and β. The reader can also find an implementation in GeoGebra at https://www.geogebra.org/m/qbznujuj.



(a) $\alpha = 0.05$ (b) $\alpha = 0.5$

(c) $\alpha = 1.0$ (d) $\alpha = 1.5$

(e) $\alpha = 3.0$ (f) $\alpha = 5.0$

FIGURE 1. Plots of $\mathrm{S}_2^{\alpha,\beta} f$ for different values of α and β, where $f(t) = \sin(1/t)$ for $t \neq 0$ and $f(t) = 0$ for $t = 0$.

Remark 4.3. A trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

with bounded coefficients a_n, b_n can be summed by Riemann's method at a point $t_0 \neq 0$ to a number R if the function

$$G(t) = \left(\frac{a_0 t^2}{4} - \sum_{n=1}^{\infty} \frac{a_n \cos nt + b_n \sin nt}{n^2} \right)^{1/\beta}$$

has, at t_0, a Riemann fractal derivative equal to $\frac{R}{\alpha^2 |t_0|^{2(\alpha-1)}}$. We refer the reader to the preliminary work of Riemann [14].

5. Concluding remarks

The final extension of the basic concept is to briefly examine the first-order Schwarz derivative as it relates to functions of two variables. These ideas contribute to a generalization of this type of derivative in a multidimensional context. A suitable reference for the symmetric derivatives is the thesis [9].

Definition 5.1. A function $f : \mathbb{C} \to \mathbb{C}$ is *first-order Schwarz fractal differentiable* at a point $z_0 \in \mathbb{C}$ if the limit

$$(S_1^{\alpha,\beta} f)(z_0) = \lim_{z \to z_0} \frac{f^\beta(z_0 + \frac{1}{2}(z - z_0)) - f^\beta(z_0 - \frac{1}{2}(z - z_0))}{z^\alpha - z_0^\alpha}$$

exists.

The above definition is similar to the definition of Schwarz fractal differentiability in the case of a real argument, except that we allow z to take complex values.

Definition 5.2. Let the equation $u = f(x, y)$ define a function in a region R of the Euclidean plane. At a point (x, y) of $\mathbb{R}^2$ the *symmetric partial derivatives* of u with respect to x and of u with respect to y, written

$$\frac{\partial^* u}{\partial x} = S_{1,x}^{\alpha,\beta} f(x, y) \quad \text{and} \quad \frac{\partial^* u}{\partial y} = S_{1,y}^{\alpha,\beta} f(x, y),$$

respectively, are the limits of the difference quotients

$$S_{1,x}^{\alpha,\beta} f(x, y) = \begin{cases} \frac{1}{\alpha \|(x,y)\|^{\alpha-1}} \lim\limits_{\Delta x \to 0} \frac{f^\beta(x+\Delta x,y) - f^\beta(x-\Delta x,y)}{2\Delta x}, & (x, y) \neq 0, \\ \lim\limits_{\Delta x \to 0} \frac{f^\beta(\Delta x,0) - f^\beta(-\Delta x,0)}{(2\Delta x)^\alpha}, & (x, y) = 0, \end{cases}$$

$$S_{1,y}^{\alpha,\beta} f(x, y) = \begin{cases} \frac{1}{\alpha \|(x,y)\|^{\alpha-1}} \lim\limits_{\Delta y \to 0} \frac{f^\beta(x,y+\Delta y) - f^\beta(x,y-\Delta y)}{2\Delta y}, & (x, y) \neq 0, \\ \lim\limits_{\Delta y \to 0} \frac{f^\beta(0,\Delta y) - f^\beta(0,-\Delta y)}{(2\Delta y)^\alpha}, & (x, y) = 0. \end{cases}$$

We will refer to $S_{1,x}^{\alpha,\beta} f(x, y)$ and $S_{1,y}^{\alpha,\beta} f(x, y)$ as the first-order symmetric partial derivatives.

Definition 5.3. For any complex variable z and function f with $f(z) = u(x, y) + iv(x, y)$, the *symmetric Cauchy–Riemann conditions* are

$$\frac{\partial^* u^{\frac{1}{\beta}}}{\partial x} = \frac{\partial^* v^{\frac{1}{\beta}}}{\partial y} \quad \text{and} \quad \frac{\partial^* u^{\frac{1}{\beta}}}{\partial y} = -\frac{\partial^* v^{\frac{1}{\beta}}}{\partial x}.$$

Theorem 5.4. *If the first-order Schwarz fractal derivative $(S_1^{\alpha,\beta} f)(z)$ of a function f exists at a point $z_0 = x_0 + iy_0$, then the first-order symmetric partial derivatives with respect to x and y of each of the components u and v of $f^\beta = u + iv$ must exist at that point, and satisfy the symmetric Cauchy–Riemann conditions. Also,*

$$(S_1^{\alpha,\beta} f)(z_0) = \frac{\partial^* u^{\frac{1}{\beta}}}{\partial x}(x_0, y_0) + i \frac{\partial^* v^{\frac{1}{\beta}}}{\partial x}(x_0, y_0) = \frac{\partial^* v^{\frac{1}{\beta}}}{\partial y}(x_0, y_0) - i \frac{\partial^* u^{\frac{1}{\beta}}}{\partial y}(x_0, y_0).$$

Proof. Suppose the function f has a first-order Schwarz fractal derivative at $z_0 = x_0 + iy_0$ and $(S_1^{\alpha,\beta} f)(z_0) = a + bi$. Since

$$\begin{aligned} f^\beta(z) &= u(x,y) + iv(x,y), \\ \Delta^* f^\beta &= f^\beta(z_0 + \Delta z) - f^\beta(z_0 - \Delta z), \\ \Delta^* u &= u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0 - \Delta x, y_0 - \Delta y), \\ \Delta^* v &= v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0 - \Delta x, y_0 - \Delta y), \end{aligned}$$

when $z_0 \neq 0$ we have

$$(S_1^{\alpha,\beta} f)(z_0) = \lim_{\Delta z \to 0} \frac{1}{\alpha |z_0|^{\alpha-1}} \frac{\Delta^* u + i\Delta^* v}{2(\Delta x + i\Delta y)} = a + bi.$$

Therefore,

$$\frac{1}{\alpha |z_0|^{\alpha-1}} \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \operatorname{Re} \frac{\Delta^* u + i\Delta^* v}{2(\Delta x + i\Delta y)} = a$$

and

$$\frac{1}{\alpha |z_0|^{\alpha-1}} \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \operatorname{Im} \frac{\Delta^* u + i\Delta^* v}{2(\Delta x + i\Delta y)} = b.$$

On the other hand,

$$\begin{aligned} \frac{\partial^* u^{\frac{1}{\beta}}}{\partial x}(x_0, y_0) &= \frac{1}{\alpha |z_0|^{\alpha-1}} \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0 - \Delta x, y_0)}{2\Delta x} = a, \\ \frac{\partial^* v^{\frac{1}{\beta}}}{\partial x}(x_0, y_0) &= \frac{1}{\alpha |z_0|^{\alpha-1}} \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0 - \Delta x, y_0)}{2\Delta x} = b, \\ \frac{\partial^* v^{\frac{1}{\beta}}}{\partial y}(x_0, y_0) &= \frac{1}{\alpha |z_0|^{\alpha-1}} \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0 - \Delta y)}{2\Delta y} = a, \\ -\frac{\partial^* u^{\frac{1}{\beta}}}{\partial y}(x_0, y_0) &= -\frac{1}{\alpha |z_0|^{\alpha-1}} \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0 - \Delta y)}{2\Delta y} = b. \end{aligned}$$

Thus, the symmetric partial derivatives exist and

$$\frac{\partial^* u^{\frac{1}{\beta}}}{\partial x}(x_0, y_0) = \frac{\partial^* v^{\frac{1}{\beta}}}{\partial y}(x_0, y_0), \quad \frac{\partial^* u^{\frac{1}{\beta}}}{\partial y}(x_0, y_0) = -\frac{\partial^* v^{\frac{1}{\beta}}}{\partial x}(x_0, y_0).$$

The case $z_0 = 0$ is analyzed in a similar way; in this case, by definition, $i^\alpha = i$.

Then

$$(S_1^{\alpha,\beta} f)(z_0) = \frac{\partial^* u^{\frac{1}{\beta}}}{\partial x}(x_0, y_0) + i\frac{\partial^* v^{\frac{1}{\beta}}}{\partial x}(x_0, y_0) = \frac{\partial^* v^{\frac{1}{\beta}}}{\partial y}(x_0, y_0) - i\frac{\partial^* u^{\frac{1}{\beta}}}{\partial y}(x_0, y_0). \quad \square$$

In order to appreciate the generality of this construction, we obtain

$$S_1^{\alpha,\beta} f = \frac{1}{2}\left[\frac{\partial^*}{\partial x} - i\frac{\partial^*}{\partial y}\right]\left[(\operatorname{Re} f^\beta)^{\frac{1}{\beta}} + i(\operatorname{Im} f^\beta)^{\frac{1}{\beta}}\right]$$

and the version of the Cauchy–Riemann operator would be as follows:

$$\overline{S}_1^{\alpha,\beta} f = \frac{1}{2}\left[\frac{\partial^*}{\partial x} + i\frac{\partial^*}{\partial y}\right]\big[(\operatorname{Re} f^\beta)^{\frac{1}{\beta}} + i(\operatorname{Im} f^\beta)^{\frac{1}{\beta}}\big].$$

Therefore, if $S_1^{\alpha,\beta} f$ exists at a point $z \in \mathbb{C}$, then $(\overline{S}_1^{\alpha,\beta} f)(z) = 0$. Note that, in general, these versions of the Cauchy–Riemann operator and its complex conjugate do not behave as linear operators unless $\beta = 1$, which is to be expected, since these derivatives are not linear.

A straightforward consequence of Theorem 5.4 is the following:

Theorem 5.5. *Let u and v be real and single-valued functions of x and y which, together with their first-order partial derivatives, are continuous at a point (x_0, y_0). If the symmetric partial derivatives with respect to the real components of the function $f^\beta = u + iv$ satisfy the symmetric Cauchy–Riemann conditions at that point, then the first-order Schwarz fractal derivative $(\mathrm{S}_1^{\alpha,\beta} f)(z_0)$ exists, where $z_0 = x_0 + iy_0$ and $z_0 \neq 0$ if $\alpha > 1$.*

Proof. Since the first-order partial derivatives exist and the symmetric partial derivatives with respect to the real components of the function $f^\beta = u + iv$ satisfy the symmetric Cauchy–Riemann conditions, we obtain that the partial derivatives satisfy the Cauchy–Riemann conditions. Hence $\frac{df^\beta}{dz}(z_0)$ exists. The existence of $\frac{df^\beta}{dz}(z_0)$ implies that $(\mathrm{S}_1^{\alpha,\beta} f)(z_0)$ also exists as long as $z_0 \neq 0$ if $\alpha > 1$. Note that when $z_0 \neq 0$ we have

$$\begin{aligned}
\frac{\partial^* u^{\frac{1}{\beta}}}{\partial x}(x_0, y_0) &= \frac{1}{\alpha|z_0|^{\alpha-1}} \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0 - \Delta x, y_0)}{2\Delta x} \\
&= \frac{1}{\alpha|z_0|^{\alpha-1}} \frac{\partial u}{\partial x}(x_0, y_0), \\
\frac{\partial^* v^{\frac{1}{\beta}}}{\partial x}(x_0, y_0) &= \frac{1}{\alpha|z_0|^{\alpha-1}} \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0 - \Delta x, y_0)}{2\Delta x} \\
&= \frac{1}{\alpha|z_0|^{\alpha-1}} \frac{\partial v}{\partial x}(x_0, y_0), \\
\frac{\partial^* v^{\frac{1}{\beta}}}{\partial y}(x_0, y_0) &= \frac{1}{\alpha|z_0|^{\alpha-1}} \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0 - \Delta y)}{2\Delta y} \\
&= \frac{1}{\alpha|z_0|^{\alpha-1}} \frac{\partial v}{\partial y}(x_0, y_0), \\
-\frac{\partial^* u^{\frac{1}{\beta}}}{\partial y}(x_0, y_0) &= -\frac{1}{\alpha|z_0|^{\alpha-1}} \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0 - \Delta y)}{2\Delta y} \\
&= -\frac{1}{\alpha|z_0|^{\alpha-1}} \frac{\partial u}{\partial y}(x_0, y_0),
\end{aligned}$$

whence

$$(\mathrm{S}_1^{\alpha,\beta} f)(z_0) = \frac{1}{\alpha|z_0|^{\alpha-1}} \frac{df^\beta}{dz}(z_0).$$

In the particular point $z_0 = 0$, when $\alpha > 1$, we obtain

$$\begin{aligned}
\frac{\partial^* u^{\frac{1}{\beta}}}{\partial x}(0,0) &= \lim_{\Delta x \to 0} \frac{u(\Delta x, 0) - u(-\Delta x, 0)}{(2\Delta x)^\alpha} \\
&= \lim_{\Delta x \to 0} \frac{u(\Delta x, 0) - u(-\Delta x, 0)}{2\Delta x} \cdot \frac{1}{(2\Delta x)^{\alpha-1}}, \\
\frac{\partial^* v^{\frac{1}{\beta}}}{\partial x}(0,0) &= \lim_{\Delta x \to 0} \frac{v(\Delta x, 0) - v(-\Delta x, 0)}{(2\Delta x)^\alpha} \\
&= \lim_{\Delta x \to 0} \frac{v(\Delta x, 0) - v(-\Delta x, 0)}{2\Delta x} \cdot \frac{1}{(2\Delta x)^{\alpha-1}}, \\
\frac{\partial^* v^{\frac{1}{\beta}}}{\partial y}(0,0) &= \lim_{\Delta y \to 0} \frac{v(0, \Delta y) - v(0, -\Delta y)}{(2\Delta y)^\alpha} \\
&= \lim_{\Delta y \to 0} \frac{v(0, \Delta y) - v(0, -\Delta y)}{2\Delta y} \cdot \frac{1}{(2\Delta y)^{\alpha-1}}, \\
-\frac{\partial^* u^{\frac{1}{\beta}}}{\partial y}(0,0) &= -\lim_{\Delta y \to 0} \frac{u(0, \Delta y) - u(0, -\Delta y)}{(2\Delta y)^\alpha} \\
&= -\lim_{\Delta y \to 0} \frac{u(0, \Delta y) - u(0, -\Delta y)}{2\Delta y} \cdot \frac{1}{(2\Delta y)^{\alpha-1}}.
\end{aligned}$$

Therefore, the existence of the partial derivatives at 0 does not imply the existence of the symmetric partial derivatives at 0, so we cannot ensure the existence of $(\mathrm{S}_1^{\alpha,\beta} f)(0)$ when $\alpha > 1$. ☐

It is important to note that the existence of the first-order symmetric partial derivatives is not strong enough to obtain the existence of the first-order Schwarz fractal derivative. For this reason, in the previous theorem the existence and continuity of the ordinary partial derivatives were taken as assumptions. The use of a weaker condition remains an open question of interest.

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