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STABILITY OF THE RITZ PROJECTION IN WEIGHTED $W^{1,1}$

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Abstract. We prove stability in weighted $W^{1,1}$ spaces for standard finite element approximations of the Poisson equation in convex polygonal or polyhedral domains, when the weight belongs to Muckenhoupt's class A_1 and the family of meshes is quasi-uniform.

1. Introduction

The Ritz projection is the best approximation in the norm of the Sobolev space $W_0^{1,2}(\Omega)$ (see Section 2 for notation), or equivalently, the finite element approximation of the solution to the Poisson equation. As a consequence, its stability in that norm follows immediately. However, stability in other norms is a difficult problem that has been the subject of many papers, mostly dealing with the case of $W^{1,\infty}(\Omega)$ (see, for instance, the books [1, 2] or the articles [3, 6] and references therein). More recently, motivated by the numerical approximation of singular problems, attention was turned in [5] to weighted $W^{1,p}(\Omega)$ norms with weights belonging to Muckenhoupt's classes.

The result in that paper was improved by a much stronger result in [4] where it was proved that, for a convex polytope in $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$, the gradient of the Ritz projection over quasi-uniform meshes is pointwise controlled by the Hardy–Littlewood maximal operator of the gradient of the original function. This estimate immediately implies the stability of the Ritz projection in $W_w^{1,p}(\Omega)$ whenever $1 < p < \infty$ and $w \in A_p$ (as well as in other spaces where the maximal operator is bounded – see examples in [4]). The cases of $W^{1,1}(\Omega)$ and $W_w^{1,1}(\Omega)$ for $w \in A_1$ were left by the authors of [4] as open problems. The aim of this short note is to show that these results can be obtained by a modification of their proof.

2. Notation and preliminaries

As usual, we will write $A \lesssim B$ to mean $A \leq CB$ for a positive constant C independent of A, B and other relevant quantities.

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The Hardy–Littlewood maximal operator is defined as

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where the supremum is taken over all cubes containing x.

A weight w is a non-negative measurable function defined in $\mathbb{R}^n$, and it is said to belong to Muckenhoupt's class A_1 iff $Mw(x) \lesssim w(x)$ almost everywhere.

The spaces $L^1(\Omega)$ and $W^{1,p}(\Omega)$ are the usual Lebesgue and Sobolev spaces, and $W_0^{1,p}(\Omega)$ is the subspace of functions of $W^{1,p}(\Omega)$ vanishing at the boundary. The weighted spaces associated to the measure $w(x)\,dx$ will be denoted by $L^1_w(\Omega)$ and $W^{1,p}_w(\Omega)$.

In what follows, we briefly recall the notation from [4] that we will use below. For $K, \gamma > 0$ (that can be appropriately chosen), $\varphi_1 : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\varphi_1(x) = c_1(|x|^2 + K^2)^{-\frac{n+\gamma}{2}},$$

where c_1 is such that $\int_\Omega \varphi_1(x)\, dx = 1$. For $\varepsilon > 0$ and $z \in \Omega$, $\varphi_\varepsilon = \varepsilon^{-n}\varphi_1(x/\varepsilon)$ and $\varphi_{\varepsilon,z} = \varphi_\varepsilon(z - x)$.

Let $\mathbb{T} = \{\mathcal{T}_h\}_{h>0}$ be a family of conforming and quasi-uniform triangulations of Ω, where $h > 0$ is the mesh size of $\mathcal{T}_h$. For $h > 0$ and $z \in \Omega$ such that $z \in \mathring{T}$ for some $T \in \mathcal{T}_h$, there exists a function $\delta_z \in C_0^\infty(T)$ such that

$$\int_T \delta_z(x)P(x)\, dx = P(z) \quad \forall P \in \mathbb{P}_k, \qquad \|D^m \delta_z\|_{L^\infty(\Omega)} \le h^{-n-m}, \quad m \in \mathbb{N}_0.$$

For $l \in \{1, \dots, n\}$, the regularized Green's function is $g_z \in W_0^{1,2}(\Omega)$ such that

$$\langle \nabla g_z, \nabla v\rangle_{L^2(\Omega)} = \langle \delta_z, \partial_l v\rangle_{L^2(\Omega)} \quad \forall v \in W_0^{1,2}(\Omega).$$

For $k \in \mathbb{N}$, the Lagrange space of degree k is

$$\mathcal{L}^1_k(\mathcal{T}_h) = \{f \in C(\overline{\Omega}) : f|_T \in \mathbb{P}_k \ \forall T \in \mathcal{T}_h\},$$

where $\mathbb{P}_k$ is the space of polynomials of degree at most k. Then, $V_h = \mathcal{L}^1_k(\mathcal{T}_h) \cap W_0^{1,1}(\Omega)$ and the Ritz projection $R_h : W_0^{1,1} \to V_h$ is defined by

$$\langle \nabla R_h u, \nabla \psi\rangle_{L^2(\Omega)} = \langle \nabla u, \nabla \psi\rangle_{L^2(\Omega)} \quad \forall \psi \in V_h.$$

3. Stability in weighted $W_0^{1,1}(\Omega)$

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$ be a convex polytope and $\mathbb{T} = \{\mathcal{T}_h\}_{h>0}$ be a family of conforming and quasi-uniform triangulations of Ω. For every $u \in W_0^{1,1}(\Omega)$ and every weight $w \in A_1$, there holds*

$$\|\nabla R_h u\|_{L^1_w(\Omega)} \lesssim \|\nabla u\|_{L^1_w(\Omega)}.$$

Proof. Fix $l \in \{1, \dots, n\}$. Using the previous notation, simple computations show that

$$\partial_l R_h u(z) = \langle \delta_z, \partial_l u(z)\rangle_{L^2} + \langle \nabla(R_h g_z - g_z), \nabla u\rangle_{L^2}$$

(see [1, equation (8.2.3)] or [4, Step 1]). Therefore,

$$\|\partial_l R_h u(z)\|_{L^1_w(\Omega)} \le \|\langle \delta_z, \partial_l u(z)\rangle_{L^2}\|_{L^1_w(\Omega)} + \|\langle \nabla(R_h g_z - g_z), \nabla u\rangle_{L^2}\|_{L^1_w(\Omega)}. \quad (3.1)$$

The first term on the right-hand side of (3.1) is

$$\begin{aligned}\int_\Omega\int_\Omega |\delta_z(x)\,\partial_l u(x)|\,dx\,w(z)\,dz &= \int_\Omega\int_T \delta_z(x)w(z)\,dz\,|\partial_l u(x)|\,dx\\ &\lesssim \int_\Omega Mw(x)\,|\partial_l u(x)|\,dx\\ &\lesssim \int_\Omega w(x)\,|\partial_l u(x)|\,dx\\ &\lesssim \|\nabla u\|_{L^1_w(\Omega)},\end{aligned}$$

where we have used Fubini's theorem, the properties of δ_z, and the fact that $w \in A_1$.

To bound the second term on the right-hand side of (3.1), recall that, by [4, Proposition 4.4], there are appropriate choices of the parameters K, γ in the definition of φ_1 such that

$$\mathcal{G}_h := \sup_{z\in\Omega} \|\varphi_{h,z}^{-1}\nabla(R_h g_z - g_z)\|_{L^\infty(\Omega)} \lesssim 1. \tag{3.2}$$

Also, observe that

$$\int_\Omega \varphi_{h,z}(x)w(z)\,dz = (\varphi_h * w)(x) \lesssim Mw(x) \lesssim w(x), \tag{3.3}$$

because φ_h is a radial and decreasing function (see [7, Theorem 2.2 in Section 2.2]) and $w \in A_1$.

Therefore, using (3.2), (3.3), Fubini's theorem, and the fact that $w \in A_1$, we may write

$$\begin{aligned}\int_\Omega\int_\Omega |\nabla(R_h g_z - g_z)(x)\nabla u(x)|\,dx\,w(z)\,dz &\lesssim \mathcal{G}_h \int_\Omega\int_\Omega \varphi_{h,z}(x)|\nabla u(x)|\,dx\,w(z)\,dz\\ &\lesssim \int_\Omega |\nabla u(x)| \int_\Omega \varphi_{h,z}(x)w(z)\,dz\,dx\\ &\lesssim \|\nabla u\|_{L^1_w(\Omega)}.\end{aligned}$$

This concludes the proof. □

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