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ENDPOINT ESTIMATES FOR HIGHER-ORDER GAUSSIAN RIESZ TRANSFORMS

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Abstract. We show that, contrary to the behavior of the higher-order Riesz transforms studied so far on the atomic Hardy space $\mathcal{H}^1(\mathbb{R}^n,\gamma)$ associated with the Ornstein–Uhlenbeck operator with respect to the n-dimensional Gaussian measure γ, the new Gaussian Riesz transforms are bounded from $\mathcal{H}^1(\mathbb{R}^n,\gamma)$ to $L^1(\mathbb{R}^n,\gamma)$, for any order and any dimension n. We also prove that the classical Gaussian Riesz transforms of higher order are bounded from an appropriate subspace of $\mathcal{H}^1(\mathbb{R}^n,\gamma)$ into $L^1(\mathbb{R}^n,\gamma)$, extending T. Bruno (2019) to the first-order case.

1. Introduction

For $x \in \mathbb{R}^n$, let $d\gamma(x) = \pi^{-n/2}e^{-|x|^2}\,dx$ be the n-dimensional non-standard Gaussian measure and let $\mathcal{L}$ be the closure on $L^2(\gamma)$ of the Ornstein–Uhlenbeck differential operator given by

$$L = -\frac{1}{2}\Delta + x\cdot\nabla = \sum_{i=1}^{n}\delta_i^*\delta_i,$$

where

$$\delta_i = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_i}$$

and

$$\delta_i^* = -\frac{1}{\sqrt{2}}e^{|x|^2}\frac{\partial}{\partial x_i}\left(e^{-|x|^2}\cdot\right)$$

is the formal adjoint of δ_i on $L^2(\gamma)$. This operator L is defined on the space $C_c^\infty(\mathbb{R}^n)$ of smooth and compactly supported functions on $\mathbb{R}^n$. It is well known that $\mathcal{L}$ is an unbounded positive self-adjoint operator on $L^2(\gamma)$. Its spectrum is discrete, composed of non-negative integers as eigenvalues, whose eigenfunctions are the normalized n-dimensional Hermite polynomials $\{h_\alpha\}_{\alpha\in\mathbb{N}_0^n}$, which turn out

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to be an orthonormal basis for $L^2(\gamma)$. That is, $\mathcal{L}h_\alpha = |\alpha|h_\alpha$ with $|\alpha| = \sum_{i=1}^n \alpha_i$ and

$$\mathrm{Dom}(\mathcal{L}) = \left\{ f \in L^2(\gamma) : \sum_{\alpha\in\mathbb{N}_0^n} |\alpha|^2 \, |\langle f, h_\alpha\rangle|^2 < \infty \right\}.$$

Let us observe also that $C_c^\infty(\mathbb{R}^n) \subseteq \mathrm{Dom}\,(\mathcal{L})$ and $\mathcal{L} = L$ on $C_c^\infty(\mathbb{R}^n)$.

We define two types of higher-order Gaussian Riesz transforms, known in the literature as the "old" and the "new" ones. First, let us note that these transforms can be spectrally defined as follows: for a multi-index $\alpha \in \mathbb{N}_0^n \setminus \{(0,\ldots,0)\}$, the "old" Gaussian Riesz transforms of order α are given by

$$R_\alpha = D^\alpha \mathcal{L}^{-\frac{|\alpha|}{2}},$$

where $D^\alpha = \delta_1^{\alpha_1} \cdots \delta_n^{\alpha_n}$, and the "new" ones have the form

$$R_\alpha^* = D^{*\alpha} (\mathcal{L} + I)^{-\frac{|\alpha|}{2}},$$

where $D^{*\alpha} = \delta_1^{*\alpha_1} \cdots \delta_n^{*\alpha_n}$ and I denotes the identity operator on $L^2(\gamma)$.

Our main aim is to analyze the continuity of these singular integrals on $L^1(\gamma)$. It is well known, as in the classical setting for the Laplacian operator, that these transforms are not bounded on $L^1(\gamma)$. Moreover, as noted in [11] the old Riesz transforms are also not bounded from the Gaussian atomic Hardy space $\mathcal{H}^1(\gamma)$ (given in [10]) into $L^1(\gamma)$, for $n > 1$. A satisfactory answer to this issue was provided by T. Bruno in [3], in the case of the first-order old Gaussian Riesz transforms, which are bounded from a certain subspace $X^1(\gamma) \subseteq \mathcal{H}^1(\gamma)$ into $L^1(\gamma)$. Now we shall complete this study for higher-order old Gaussian Riesz transforms by using smaller subspaces as we increase the order of the transform, and at the same time we prove the boundedness of the new ones from $\mathcal{H}^1(\gamma)$ into $L^1(\gamma)$.

To that end, we will first need the concept of atom to introduce the corresponding atomic Hardy space $\mathcal{H}^1(\gamma)$ later. Given $r \in (1,\infty]$, a Gaussian $(1,r)$-atom is either the constant function 1 or a function $a \in L^r(\gamma)$ supported in an admissible ball B (see its definition in Section 2) such that

$$\int a\, d\gamma = 0 \quad \text{and} \quad \|a\|_r \le \gamma(B)^{\frac{1}{r}-1}.$$

From now on, the symbol $\|\cdot\|_r$ denotes the norm in $L^r(\gamma)$. In the latter case, we say that the atom a is associated to the ball B.

The space $\mathcal{H}^{1,r}(\gamma)$ is then the vector space of all functions $f \in L^1(\gamma)$ that admit a decomposition of the form $\sum_j \lambda_j a_j$, where the a_j are Gaussian $(1,r)$-atoms and the series associated to the sequence of complex numbers $\{\lambda_j\}$ is absolutely convergent. The norm of f in $\mathcal{H}^{1,r}(\gamma)$ is defined as the infimum of $\sum_j |\lambda_j|$ over all these representations of f.

In [10] and [11] the spaces $\mathcal{H}^{1,r}(\gamma)$ were defined and proved to coincide for all $1 < r \le \infty$, with equivalent norms. So any one of them is called $\mathcal{H}^1(\gamma)$. In view of these facts, we shall refer to the atoms in $\mathcal{H}^1(\gamma)$ as $\mathcal{H}^1$-atoms.

In Section 2 we will introduce a new Hardy space $X^k(\gamma)$ for each $k \in \mathbb{N}$. The sequence of these new Hardy spaces form a strictly decreasing chain such that $X^{k+1}(\gamma) \subsetneq X^k(\gamma) \subsetneq \mathcal{H}^1(\gamma)$. And these spaces will be suitable to the boundedness of R_α when $p = 1$, where the parameter k is related to the order α as we can see below.

We state our two main theorems which take care of the boundedness of these Gaussian Riesz transforms on $L^1(\gamma)$.

Theorem 1.1. *R_α is bounded from $X^k(\gamma)$ to $L^1(\gamma)$ for any multi-index α with $k = |\alpha|$ and any dimension.*

Theorem 1.2. *R^*_α is bounded from $\mathcal{H}^1(\gamma)$ to $L^1(\gamma)$ for any multi-index α and any dimension.*

The article is organized as follows. In Section 2, we introduce some notation, definitions and properties of $\mathcal{L}$, whereas in Section 3 we establish the definitions of atoms and Hardy-type spaces in the Gaussian framework. In order to prove our main theorems in Section 5, we first present several auxiliary results in Section 4.

2. Preliminaries

For the operator $\mathcal{L}$ introduced above, and for every $z \in \mathbb{C}$, we define

$$\mathcal{L}^z = \sum_{j=1}^{\infty} j^z \mathcal{P}_j, \quad \operatorname{Dom}(\mathcal{L}^z) = \left\{ f \in L^2(\gamma) : \sum_{j=1}^{\infty} j^{2\operatorname{Re} z} \|\mathcal{P}_j f\|_2^2 < \infty \right\},$$

where $\mathcal{P}_j$ is the orthogonal projection onto the Wiener chaos space of order j, i.e., $\mathcal{P}_j f = \sum_{|\alpha|=j} \langle f, h_\alpha \rangle h_\alpha$ with h_α the normalized Hermite polynomial of degree $|\alpha| = j$ and $\alpha \in \mathbb{N}_0^n$. Let us remark that $\mathcal{L}^1 = \mathcal{L}$.

For $f \in \operatorname{Dom}(\mathcal{L})$, we have

$$\mathcal{L} f(x) = \sum_{j=1}^{\infty} j\, \mathcal{P}_j f(x).$$

Recall that the family of orthonormalized Hermite polynomials $\{h_\alpha\}_{\alpha \in \mathbb{N}_0^n}$ is an orthonormal basis for $L^2(\gamma)$ (see, for instance, [14]). Thus, we can rephrase

$$\operatorname{Dom}(\mathcal{L}) = \left\{ f \in L^2(\gamma) : \mathcal{L} f \in L^2(\gamma) \right\}.$$

If $\operatorname{Re} z < 0$, the operator $\mathcal{L}^z$ turns out to be bounded on $L^2(\gamma)$ and we have $\operatorname{Dom}(\mathcal{L}^z) = L^2(\gamma)$. Meanwhile, if $\operatorname{Re} z \geq 0$, then $C_c^\infty(\mathbb{R}^n) \subseteq \operatorname{Dom}(\mathcal{L}^z)$ by the decomposition $\mathcal{L}^z = \mathcal{L}^{z-N} \mathcal{L}^N$ with $N = [\operatorname{Re} z] + 1$.

Let Π_0 be the orthogonal projection

$$\Pi_0 : L^2(\gamma) \to \ker(\mathcal{L})^\perp = \left\{ f \in L^2(\gamma) : \int_{\mathbb{R}^n} f \, d\gamma = 0 \right\}.$$

In terms of the spectral resolution $\{\mathcal{P}_j\}_{j \geq 0}$, $\Pi_0 = I - \mathcal{P}_0$ since

$$\mathcal{P}_0 : L^2(\gamma) \to \ker(\mathcal{L}) \equiv \mathbb{C},$$

with

$$\mathcal{P}_0 f = \int_{\mathbb{R}^n} f \, d\gamma.$$

We shall denote the space $\Pi_0(L^2(\gamma))$ by $L^2_0(\gamma)$.

Lemma 2.1. *For every positive integer k we have*

$$\mathcal{L}^k\mathcal{L}^{-k}f = \Pi_0 f \quad \forall f \in L^2(\gamma), \qquad \mathcal{L}^{-k}\mathcal{L}^k f = \Pi_0 f \quad \forall f \in \mathrm{Dom}(\mathcal{L}^k).$$

Proof. Let us see that for $f \in L^2(\gamma)$, $\mathcal{L}^{-k}f \in \mathrm{Dom}(\mathcal{L}^k)$. From the definitions of $\mathcal{P}_j$ and $\mathcal{L}^{-k}$ and the orthonormality of $\{h_\alpha\}_{\alpha\in\mathbb{N}_0^n}$, we have

$$\mathcal{P}_j(\mathcal{L}^{-k}f) = \frac{1}{j^k}\mathcal{P}_j f$$

and thus

$$\begin{aligned}\sum_{j=1}^{\infty} j^{2k}\|\mathcal{P}_j(\mathcal{L}^{-k}f)\|_2^2 &= \sum_{j=1}^{\infty} j^{2k}\frac{1}{j^{2k}}\|\mathcal{P}_j f\|_2^2\\ &= \sum_{j=1}^{\infty}\|\mathcal{P}_j f\|_2^2\\ &\le \sum_{j=0}^{\infty}\|\mathcal{P}_j f\|_2^2\\ &= \|f\|_2^2 < \infty.\end{aligned}$$

Hence, $\mathcal{L}^{-k}f \in \mathrm{Dom}(\mathcal{L}^k)$ and from the definition of $\mathcal{L}^k$ and $\mathcal{L}^{-k}$ we get that, for $f \in L^2(\gamma)$,

$$\mathcal{L}^k\mathcal{L}^{-k}f = \sum_{j=1}^{\infty}\mathcal{P}_j f = f - \int_{\mathbb{R}^n} f\,d\gamma = \Pi_0 f.$$

Similarly, for $f \in \mathrm{Dom}(\mathcal{L}^k)$, we get

$$\mathcal{L}^{-k}\mathcal{L}^k f = \Pi_0 f.$$

□

In particular, from the above result, we have

$$\mathcal{L}^k\mathcal{L}^{-k}f = f \quad \forall f \in L_0^2(\gamma), \qquad \mathcal{L}^{-k}\mathcal{L}^k f = f \quad \forall f \in \mathrm{Dom}(\mathcal{L}^k)\cap L_0^2(\gamma).$$

Let us introduce several function spaces that will play an important role in the definition of special atoms and investigate the relationship among them.

Recall that $\mathcal{L}$ is an elliptic operator. Thus, for a given bounded open subset Ω of $\mathbb{R}^n$, a positive integer k and a real constant c, every solution u of the equation

$$\mathcal{L}^k u = c\mathcal{X}_\Omega$$

is smooth in Ω by elliptic regularity. Here, $\mathcal{X}_\Omega$ denotes the characteristic function of Ω.

We will now introduce some function spaces related to the solutions of the integer powers of $\mathcal{L}$.

Definition 2.2. Suppose that k is a positive integer and that Ω is a bounded open subset of $\mathbb{R}^n$. We say that a function u is *k-quasi-harmonic* on Ω if $\mathcal{L}^k u$ is constant on Ω (in the sense of distributions, hence in the classical sense, since u is smooth by elliptic regularity). We shall denote by $q_k^2(\Omega)$ the space of k-quasi-harmonic functions on Ω that belong to $L^2(\gamma)$.

The subspace of $q_k^2(\Omega)$ of all functions v such that $\mathcal{L}^k v = 0$ on Ω will be denoted by $h_k^2(\Omega)$.

Definition 2.3. Given a compact subset $K \subseteq \mathbb{R}^n$, we say that a function v is *k-quasi-harmonic* on K if v is the restriction to K of a function in $q_k^2(\Omega)$ for some bounded open set Ω containing K. We shall denote by $q_k^2(K)$ the space of all k-quasi-harmonic functions on K. The subspace of all functions which are restrictions to K of functions in $h_k^2(\Omega)$ will be denoted by $h_k^2(K)$.

Remark 2.4. Let K be a compact subset of $\mathbb{R}^n$. Then
$$q_k^2(K)^\perp = \left\{v \in L^2(\gamma) : \mathcal{L}^{-k} v \in L_0^2(K,\gamma)\right\},$$
where $L_0^2(K,\gamma)$ is the space of functions $w \in L^2(\gamma)$ with $\operatorname{supp} w \subseteq K$ and $\int w\, d\gamma = 0$.

The proof of this remark can be found in [12] for the Laplace–Beltrami operator and $K = \overline{B}$ with B a ball. Although it follows the same lines as the ones in [12, Proposition 3.3 (i)], we will state it here for the sake of completeness.

Proof of Remark 2.4. First let us prove $q_k^2(K)^\perp \subseteq \{v \in L^2(\gamma) : \mathcal{L}^{-k} v \in L_0^2(K,\gamma)\}$. Let $v \in q_k^2(K)^\perp$. In order to prove that the support of $\mathcal{L}^{-k}v$ is a subset of K it suffices to show that $\langle \mathcal{L}^{-k} v, \mathcal{X}_B\rangle = 0$ for every ball $B \subseteq \mathbb{R}^n \setminus K$. Since $\mathcal{L}$ is self-adjoint, we have
$$\langle \mathcal{L}^{-k} v, \mathcal{X}_B\rangle = \langle v, \mathcal{L}^{-k}\mathcal{X}_B\rangle.$$
On the other hand, notice that $\mathcal{L}^{-k}\mathcal{X}_B \in q_k^2(K)$. Indeed, there exists a bounded open subset Ω of $\mathbb{R}^n$ with $K \subseteq \Omega$ and $\mathcal{L}^k\mathcal{L}^{-k}\mathcal{X}_B = \mathcal{X}_B - \gamma(B)$ on Ω; in particular, with $\mathcal{X}_B = 0$ on K. Thus the last inner product in the above equality vanishes.

Now we prove that the function $\mathcal{L}^{-k}v$ has average zero with respect to γ. Indeed, since $\operatorname{supp}(\mathcal{L}^{-k}v) \subseteq K$ and $\mathcal{L}$ is self-adjoint, we have
$$\int_{\mathbb{R}^n} \mathcal{L}^{-k} v\, d\gamma = \langle \mathcal{L}^{-k} v, \mathcal{X}_\Omega\rangle = \langle v, \mathcal{L}^{-k}\mathcal{X}_\Omega\rangle.$$
Since $v \in q_k^2(K)^\perp$ and $\mathcal{L}^{-k}\mathcal{X}_\Omega \in q_k^2(K)$, this last integral vanishes, as required.

Next, let us prove the other inclusion. We assume that $\mathcal{L}^{-k}v \in L_0^2(K,\gamma)$ for $v \in L^2(\gamma)$. Then $v \in \operatorname{Dom}(\mathcal{L}^k)$ and $v = \mathcal{L}^k\mathcal{L}^{-k}v$. If we take $w \in q_k^2(K)$, there exists a bounded open subset $\Omega \subset \mathbb{R}^n$ with $K \subseteq \Omega$ such that $\mathcal{L}^k w$ is constant on Ω. Therefore w turns out to be smooth on Ω, and
$$\langle v, w\rangle = \langle \mathcal{L}^k\mathcal{L}^{-k} v, w\rangle = \langle \mathcal{L}^{-k} v, \mathcal{L}^k w\rangle = 0.$$
The last equality holds since $\mathcal{L}^k w$ is constant on Ω and $\mathcal{L}^{-k}v \in L_0^2(K,\gamma)$. □

Everything stated henceforth is understood in the sense of distributions, unless otherwise specified. Let u be a function in $\operatorname{Dom}(\mathcal{L}^k)$ that vanishes on the complement of $\overline{B}$ where B is a ball in $\mathbb{R}^n$. Then $\mathcal{L}^k u$ is in $L^2(\gamma)$ and vanishes on $\mathbb{R}^n \setminus \overline{B}$. For every ball B we introduce two operators $\mathcal{L}_B^k$ and $\mathcal{L}_{B,\mathrm{Dir}}^k$ defined as the restriction of $\mathcal{L}^k$ (in the distribution sense) to
$$\operatorname{Dom}(\mathcal{L}_B^k) := \{u \in \operatorname{Dom}(\mathcal{L}^k) : \operatorname{supp} u \subseteq \overline{B}\},$$
$$\operatorname{Dom}(\mathcal{L}_{B,\mathrm{Dir}}^k) := \left\{u \in W_0^{2k-1,2}(B) : \mathcal{L}^k u \in L^2(\gamma),\ \operatorname{supp}(\mathcal{L}^k u) \subseteq B\right\},$$

respectively. Here, for $j \in \mathbb{N}$, the closure of $C_c^\infty(B) = \{u \in C_c^\infty(\mathbb{R}^n) : \operatorname{supp} u \subseteq B\}$ will be denoted by $W_0^{j,2}(B)$ with respect to the norm

$$\|u\|_{W_0^{j,2}(B)} = \left(\sum_{|\alpha| \leq j} \|D^\alpha u\|_{L^2(B)}^2 \right)^{1/2}.$$

Notice that $W_0^{j,2}(B) = W_0^{j,2}(B,\gamma)$ since B is a bounded set.

The following lemma gives an identification of the domain of $\mathcal{L}_B^k$ for any $k \in \mathbb{N}$. The case $k = 1$ can be found in [3, Lemma 2.6].

Lemma 2.5. *Let B be a ball in $\mathbb{R}^n$. Then $\operatorname{Dom}(\mathcal{L}_B^k) = W_0^{2k,2}(B)$ with equivalence of norms.*

Proof. As said above, for the case $k = 1$ we have the result by T. Bruno in [3]. For the general case we use induction on k since

$$\operatorname{Dom}(\mathcal{L}_B^k) = \{u \in \operatorname{Dom}(\mathcal{L}_B) : \mathcal{L}u \in \operatorname{Dom}(\mathcal{L}_B^{k-1})\}.$$

Indeed, let us assume $\operatorname{Dom}(\mathcal{L}_B^k) = W_0^{2k,2}(B)$. Then

$$\operatorname{Dom}(\mathcal{L}_B^{k+1}) = \{u \in \operatorname{Dom}(\mathcal{L}_B) : \mathcal{L}u \in \operatorname{Dom}(\mathcal{L}_B^k) = W_0^{2k,2}(B)\}.$$

Thus, given $u \in \operatorname{Dom}(\mathcal{L}_B^{k+1})$, we have $\mathcal{L}u = f$ with $f \in W_0^{2k,2}(B)$ and $\operatorname{supp} u \subseteq \overline{B}$, so we get $u \in W_0^{2k+2,2}(B)$ (cf. [4, Theorem 6.5]). Conversely, taking into account that $D^\alpha L = LD^\alpha - 2|\alpha|D^\alpha$ and that B is a bounded set, if $u \in W_0^{2k+2,2}(B)$ then $\mathcal{L}u \in W_0^{2k,2}(B)$, and from this we get $u \in \operatorname{Dom}(\mathcal{L}_B^{k+1})$. So the case $k+1$ is also true. $\square$

In the spirit of [3, Lemma 2.7], we can obtain the following lemma regarding several properties of the function spaces previously defined, as well as the operators $\mathcal{L}_B^k$ and $\mathcal{L}_{B,\mathrm{Dir}}^k$.

Lemma 2.6. *The following statements hold.*

(i) *Both spaces $q_k^2(B)$ and $h_k^2(B)$ are closed subspaces of $L^2(B)$;*
(ii) *$\mathcal{L}^k$ is a Banach space isomorphism between $\operatorname{Dom}(\mathcal{L}_B^k)$ and $h_k^2(\overline{B})^\perp$;*
(iii) *$h_k^2(\overline{B})^\perp = h_k^2(B)^\perp$;*
(iv) *$\operatorname{Ran}(\mathcal{L}_B^k) = h_k^2(B)^\perp$;*
(v) *$q_k^2(\overline{B})^\perp = q_k^2(B)^\perp$;*
(vi) *$\operatorname{Dom}(\mathcal{L}_B^k) \subseteq \operatorname{Dom}(\mathcal{L}_{B,\mathrm{Dir}}^k)$.*

Proof. Let us first prove item (i). Clearly $h_k^2(B)$ is a subspace of $q_k^2(B)$ of co-dimension one. Indeed, we have the vector space decomposition

$$q_k^2(B) = h_k^2(B) \oplus \mathbb{C}(\mathcal{L}^{-k}\psi|_B),$$

where $\psi \in L_0^2(\gamma) \cap C_c^\infty(\mathbb{R}^n)$ is such that $\psi = 1$ on B, and $\mathbb{C}(\varphi) = \{c\varphi : c \in \mathbb{C}\}$. Let $u \in q_k^2(B)$. Then $\mathcal{L}^k u = c = c\psi$ on B. From the definition of ψ we have $\mathcal{L}^k\mathcal{L}^{-k}\psi = \psi$. If we take $v = u - c\mathcal{L}^{-k}\psi|_B$ we have that $\mathcal{L}^k v = \mathcal{L}^k u - c\mathcal{L}^k\mathcal{L}^{-k}\psi = 0$ on B. Thus $v \in h_k^2(B)$ and $u = v + c\mathcal{L}^{-k}\psi|_B$. Since $\psi \in L_0^2(\gamma)$, we have that the sum is direct.

Let us prove that $q_k^2(B)$ and $h_k^2(B)$ are closed subspaces of $L^2(B)$. Due to the above decomposition, it is enough to prove that $h_k^2(B)$ is closed. Indeed, let $\{v_j\}_{j\in\mathbb{N}}$ be a sequence on $h_k^2(B)$ that converges in $L^2(B)$ to $v \in L^2(B)$. Then $\mathcal{L}^k v_j$ converges to $\mathcal{L}^k v$ in the sense of distributions. Thus $\mathcal{L}^k v = 0$ on B, whence $v \in h_k^2(B)$.

We now consider item (ii). We first observe that $\mathcal{L}^k(\mathrm{Dom}(\mathcal{L}_B^k)) \subseteq h_k^2(\overline{B})^\perp$. Let $f \in \mathrm{Dom}(\mathcal{L}_B^k)$ and fix $v \in h_k^2(\overline{B})$. Then there exist an open set $\Omega \supseteq \overline{B}$ and a function $w \in h_k^2(\Omega)$ such that $w|_{\overline{B}} = v$. We can construct a function $\tilde{v} \in C_c^\infty(\mathbb{R}^n) \cap h_k^2(\mathbb{R}^n)$ such that $\tilde{v} = w$ on $\overline{B}$ and $\operatorname{supp}\tilde{v} \subseteq \Omega$. Observe that

$$\int_{\overline{B}} v\mathcal{L}^k f\, d\gamma = \int_{\overline{B}} w\mathcal{L}^k f\, d\gamma = \int_\Omega \tilde{v}\mathcal{L}^k f\, d\gamma = \int_{\mathbb{R}^n} \tilde{v}\mathcal{L}^k f\, d\gamma = \int_{\mathbb{R}^n} \mathcal{L}^k\tilde{v}\, f\, d\gamma = 0,$$

where we have used that

$$\begin{aligned}
\int_{\mathbb{R}^n} \tilde{v}\mathcal{L}^k f\, d\gamma &= \int_{\mathbb{R}^n} \left(\sum_\alpha \langle \tilde{v}, h_\alpha\rangle h_\alpha\right)\left(\sum_\beta |\beta|^k \langle f, h_\beta\rangle h_\beta\right) d\gamma \\
&= \sum_\alpha \sum_\beta \langle \tilde{v}, h_\alpha\rangle |\beta|^k \langle f, h_\beta\rangle \langle h_\alpha, h_\beta\rangle \\
&= \sum_\alpha |\alpha|^k \langle \tilde{v}, h_\alpha\rangle \langle f, h_\alpha\rangle \\
&= \int_{\mathbb{R}^n} \mathcal{L}^k \tilde{v} f\, d\gamma.
\end{aligned}$$

At this point, we shall prove that $\mathcal{L}_B^k$ is injective on $\mathrm{Dom}(\mathcal{L}_B^k)$. Indeed, if $f \in \mathrm{Dom}(\mathcal{L}_B^k)$ and $\mathcal{L}^k f = 0$, we have $\mathcal{L}(\mathcal{L}^{k-1} f) = 0$, so $\mathcal{L}^{k-1} f$ is a constant L^2-function with support contained in $\overline{B}$. Therefore $\mathcal{L}^{k-1} f = 0$. By proceeding recursively we get that $f = 0$.

As the next step, we will prove that $\mathcal{L}^k$ maps $\mathrm{Dom}(\mathcal{L}_B^k)$ onto $h_k^2(\overline{B})^\perp$. Fix $v \in h_k^2(\overline{B})^\perp$ and define $\tilde{v} = v\mathcal{X}_{\overline{B}}$. Then, we have $f = \mathcal{L}^{-k}(\tilde{v}) \in \mathrm{Dom}(\mathcal{L}^k)$, since $\mathcal{L}^{-k} : L^2(\gamma) \to \mathrm{Dom}(\mathcal{L}^k)$. We also have

$$\int_{\mathbb{R}^n} \tilde{v}\, d\gamma = \int_{\overline{B}} v\, d\gamma = 0,$$

since $1 \in h_k^2(\overline{B})$. Thus $\tilde{v} \in L_0^2(\gamma)$ and consequently

$$\mathcal{L}^k f = \mathcal{L}^k \left(\mathcal{L}^{-k}\tilde{v}\right) = \Pi_0 \tilde{v} = \tilde{v}.$$

We now fix a test function $\phi \in C_c^\infty(\overline{B}^c) \cap L_0^2(\gamma)$. Then we can write

$$\langle \phi, f\rangle = \left\langle \mathcal{L}^k(\mathcal{L}^{-k}\phi), \mathcal{L}^{-k}\tilde{v}\right\rangle = \left\langle \mathcal{L}^{-k}\phi, \tilde{v}\right\rangle = \int_{\overline{B}} \mathcal{L}^{-k}\phi v\, d\gamma = 0,$$

since $\mathcal{L}^{-k}\phi \in h_k^2(\overline{B})$. By the arbitrariness of ϕ we can conclude that f is constant on $\overline{B}^c$, that is, $f(x) = c$ for every $x \in \overline{B}^c$. The function $g = f - c$ belongs to $\mathrm{Dom}(\mathcal{L}^k)$, $\operatorname{supp} g \subseteq \overline{B}$ and

$$\mathcal{L}^k g = \mathcal{L}^k(f - c) = \mathcal{L}^k f = \tilde{v} = v.$$

Consequently, we have proved that the operator $\mathcal{L}^k$ is a bijection between $\mathrm{Dom}(\mathcal{L}^k_B)$ and $h^2_k(\overline{B})^\perp$. We also have that $\mathcal{L}^k$ is continuous since $\mathrm{Dom}(\mathcal{L}^k_B) = W_0^{2k,2}(\overline{B})$.

Let $T = \mathcal{L}^k : \mathrm{Dom}(\mathcal{L}^k_B) \to h^2_k(\overline{B})^\perp$. We are going to show that T^{-1} is continuous from $h^2_k(\overline{B})^\perp$ to $\mathrm{Dom}(\mathcal{L}^k_B)$. In order to prove that, we shall see that its graph is closed. Let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence in $h^2_k(\overline{B})^\perp$ such that $f_j \to f$ when $j \to \infty$ on $L^2(\overline{B})$ and $T^{-1}f_j \to g$, where $g \in W_0^{2k,2}(\overline{B})$. By the surjectivity of T^{-1} we have that there exists $\tilde{f} \in h^2_k(\overline{B})^\perp$ such that $g = T^{-1}\tilde{f}$. By the continuity of T we get

$$f_j = T\left(T^{-1}f_j\right) \to Tg = \tilde{f};$$

then we conclude that $f = \tilde{f}$. By the closed graph theorem we obtain that T^{-1} is continuous.

The proofs of items (iii) to (vi) follow similar lines as in [3, Lemma 2.7] and we shall omit them. □

3. Hardy-type spaces and atoms

As mentioned in the introduction, defining suitable atoms for the old Gaussian Riesz transforms requires considering a family of balls on which the Gaussian measure γ is doubling.

Given an Euclidean ball $B = B(c_B, r_B)$, where c_B is its center and $r_B > 0$ its radius, we will say that B is an *admissible ball* if $r_B \le m(|c_B|)$, where the function $m : \mathbb{R}_0^+ \to \mathbb{R}^+$ is defined as

$$m(s) = \begin{cases} 1, & 0 \le s \le 1, \\ \frac{1}{s}, & s > 1. \end{cases}$$

The collection of these balls will be denoted by $\mathscr{B}_1$. For any constant $c > 0$, by cB we will denote the ball with same center as B and c times its radius.

As it is well known (see, for instance, [10, Proposition 2.1 and Remark 2.2]), the Gaussian measure is doubling on $\mathscr{B}_1$, that is, for any ball $B \in \mathscr{B}_1$,

$$\gamma(2B) \le D_\gamma \gamma(B),$$

with doubling constant $D_\gamma > 0$ independent of B.

We are now in a position to define, for every $k \in \mathbb{N}$, an X^k-atom and its corresponding atomic Hardy-type space $X^k(\gamma)$ in this setting.

Definition 3.1. Given $k \in \mathbb{N}$, we say that a function $a \in L^2(\gamma)$ is an X^k*-atom* if a is supported on an admissible ball $B \in \mathscr{B}_1$ and satisfies

(a) $\|a\|_{L^2(\gamma)} \le \omega_k(r_B)\gamma(B)^{-1/2}$, with $\omega_1(r_B) = \omega_2(r_B) = 1$ and $\omega_k(r_B) = r_B^{k-2}$ for $k \ge 3$;

(b) $a \in q^2_k(\overline{B})^\perp$.

Remark 3.2. Since $\mathcal{X}_{2B} \in q^2_k(\overline{B})$, observe that condition (b) implicitly says that $\int a\, d\gamma = 0$. On the other hand, since $\omega_k(r_B) \le 1$ for every k, we get from (a) that

$$\|a\|_{L^2(\gamma)} \le \gamma(B)^{-1/2}. \tag{3.1}$$

Remark 3.3. It is easy to see that every X^k-atom is also an X^j-atom for every $1 \leq j \leq k$.

Remark 3.4. Condition (a) implies that each atom is in $L^1(w_k\gamma)$ with

$$w_k(x) = \begin{cases} 1, & k = 1, 2; \\ 1 + |x|^{k-2}, & k \geq 3. \end{cases}$$

Indeed, for any $k \geq 1$,

$$\begin{aligned} \|a\|_{L^1(w_k\gamma)} &\leq \|w_k \mathcal{X}_B\|_{L^2(\gamma)} \|a\|_{L^2(\gamma)} \\ &\leq \|w_k \mathcal{X}_B\|_{L^2(\gamma)} \omega_k(r_B) \gamma(B)^{-1/2} \\ &\leq 1 + 2^{k-2}. \end{aligned} \tag{3.2}$$

This will allow us to deduce the boundedness of the old Gaussian Riesz transforms once the uniform boundedness on atoms has been established, taking into account that R_α with $|\alpha| = k$ are bounded from $L^1(w_k\gamma)$ into $L^{1,\infty}(\gamma)$ (see [5]). We also remark that, to obtain the uniform boundedness of these old Gaussian Riesz transforms on atoms, we will use the weaker condition (3.1).

Remark 3.5. We shall point out that if a is an X^k-atom, then $\frac{a}{|||\mathcal{L}^{-k}|||_2}$ is an $\mathcal{H}^1$-atom supported in $\overline{B}$. Here $|||\cdot|||_2$ stands for the norm on the Banach space of bounded linear operators defined on $L^2(\gamma)$.

Indeed, since $a \in q_k^2(\overline{B})^\perp$, by Remark 2.4, $\mathcal{L}^{-k}a \in L_0^2(\overline{B}, \gamma)$. Moreover, by (3.1)

$$\|\mathcal{L}^{-k}a\|_{L^2(\gamma)} \leq \left|\left|\left|\mathcal{L}^{-k}\right|\right|\right|_2 \|a\|_{L^2(\gamma)} \leq \left|\left|\left|\mathcal{L}^{-k}\right|\right|\right|_2 \gamma(B)^{-1/2}.$$

Since we are expecting a strong endpoint type result for the old Gaussian Riesz operator, we shall consider a subset of $L^1(w_k\gamma)$, given below.

Definition 3.6. Given $k \in \mathbb{N}$, the Hardy space $X^k(\gamma)$ is defined by

$$X^k(\gamma) := \left\{ f \in L^1(\gamma) : f = \sum_{j\in\mathbb{N}} \lambda_j a_j,\ a_j \text{ an } X^k\text{-atom } \forall j \in \mathbb{N},\ \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1 \right\}.$$

The norm for this space is given by

$$\|f\|_{X^k(\gamma)} := \inf\left\{ \|\{\lambda_j\}\|_{\ell^1} : f = \sum_{j\in\mathbb{N}} \lambda_j a_j,\ a_j \text{ an } X^k\text{-atom } \forall j \in \mathbb{N} \right\}.$$

As claimed, $X^k(\gamma) \subseteq L^1(w_k\gamma)$, since for $f \in X^k(\gamma)$ and from (3.2) we get

$$\|f\|_{L^1(w_k\gamma)} \leq \sum_{j=1}^{\infty} |\lambda_j| \|a_j\|_{L^1(w_k\gamma)} \leq \left(1 + 2^{k-2}\right) \sum_{j=1}^{\infty} |\lambda_j|.$$

4. Auxiliary results

We shall see next that the operator $\mathcal{L}^{-k}$ preserves the support of X^k-atoms, which will be one of the key ingredients in the proof of Theorem 1.1 in the following section. We refer to [3, Proposition 2.5] for the case $k = 1$.

In the following, by $A \lesssim B$ we shall understand that there exists a positive constant C such that $A \leq CB$, where C may change in each occurrence.

Proposition 4.1. *Let a be an X^k-atom supported on an admissible ball B. Then,* $\operatorname{supp}(\mathcal{L}^{-k}a) \subseteq \overline{B}$ *and*

$$\|\mathcal{L}^{-k}a\|_{L^2(\gamma)} \lesssim r_B^{2k}\gamma(B)^{-1/2}.$$

Proof. Let a be an X^k-atom. By items (iv) and (v) of Lemma 2.6 we obtain

$$a \in q_k^2(\overline{B})^\perp = q_k^2(B)^\perp \subseteq h_k^2(B)^\perp = \operatorname{Ran}(\mathcal{L}_B^k).$$

Then there exists $f \in \operatorname{Dom}(\mathcal{L}_B^k) \subseteq \operatorname{Dom}(\mathcal{L}_{B,\mathrm{Dir}}^k)$, by Lemma 2.6 (vi), such that $\mathcal{L}_{B,\mathrm{Dir}}^k f = \mathcal{L}_B^k f = a$. We have that $f = \mathcal{L}_{B,\mathrm{Dir}}^{-k}a$, as $\mathcal{L}_{B,\mathrm{Dir}}^k$ is one-to-one in $\operatorname{Dom}(\mathcal{L}_B^k)$. Consequently, $\operatorname{supp}\mathcal{L}_{B,\mathrm{Dir}}^{-k}a = \operatorname{supp} f \subseteq \overline{B}$, and further $\mathcal{L}_{B,\mathrm{Dir}}^{-k}a = \mathcal{L}^{-k}a$. Therefore, since $\mathcal{L}_{B,\mathrm{Dir}}$ has a discrete spectrum (cf. [8, Theorem 10.13]), so does $\mathcal{L}_{B,\mathrm{Dir}}^k$, and thus

$$\left\|\mathcal{L}^{-k}a\right\|_{L^2(\gamma)} = \left\|\mathcal{L}_{B,\mathrm{Dir}}^{-k}a\right\|_{L^2(\gamma)} \leq \left(\lambda_{\mathrm{Dir}}^\gamma(B)\right)^{-k}\|a\|_{L^2(\gamma)} \leq \left(\lambda_{\mathrm{Dir}}^\gamma(B)\right)^{-k}\gamma(B)^{-1/2},$$

where we have used (3.1) and $\lambda_{\mathrm{Dir}}^\gamma(B)$ denotes the first eigenvalue of $\mathcal{L}_{B,\mathrm{Dir}}$. From this point on, we can proceed as in the proof of [3, Proposition 2.5]. □

In order to prove Theorem 1.1 we will require the following lemma, proved in [10].

Lemma 4.2 ([10, Lemma 7.1]). *Let $B = B(c_B, r_B)$ be a ball in $\mathbb{R}^n$. For $y \in B$, set $r_{B,y} = \frac{r_B}{2|y|}$ for $y \neq 0$ and $r_{B,y} = \infty$ for $y = 0$.*

(i) *If $r_{B,y} \geq 1$, then $4|x-ry| \geq |x-c_B|$ for every $r \in [0,1]$ and every $x \in (2B)^c$;*
(ii) *if $r_{B,y} < 1$, then $4|x-ry| \geq |x-c_B|$ for every $r \in [1-r_{B,y}, 1]$ and every $x \in (2B)^c$;*
(iii) *for every $\delta > 0$, there exist positive constants C_1 and C_2 such that*

$$\frac{1}{(1-r^2)^{n/2}}\int_{(2B)^c} e^{-\frac{\delta|x-c_B|^2}{1-r^2}}\,dx \leq C_1\varphi_\delta\left(\frac{r_B}{\sqrt{1-r^2}}\right) \leq C_1 e^{-\frac{C_2 r_B^2}{1-r^2}},$$

where $\varphi_\delta(s) = (1+s)^{n-2}e^{-\delta s^2} \leq c_n e^{-C_2 s^2}$ for $s > 0$.

The following result will be useful in the proof of Theorem 1.2, as it takes care of the derivatives of $\mathcal{L}^{k/2}$ for odd orders k. When $k = 1$, the proof can be found in [3, Lemma 2.8].

Lemma 4.3. *Let α be a multi-index with $|\alpha| = k$, where k is an odd positive integer. Then*

$$\left\|D^\alpha\mathcal{L}^{k/2}f\right\|_{L^1((4B)^c,\gamma)} \lesssim r_B^{-2k}\|f\|_{L^1(B,\gamma)}$$

for every ball $B \in \mathscr{B}_1$ and every $f \in L^1(\gamma)$ such that $\operatorname{supp} f \subseteq \overline{B}$.

Proof. For k odd, we have that $\mathcal{L}^{k/2}$ has the following kernel (see, for instance, [3, p. 1612]):

$$K_{\mathcal{L}^{k/2}}(x,y) = \frac{1}{\pi^{\frac{n}{2}}\Gamma\left(-\frac{k}{2}\right)}\int_0^1(-\log r)^{-\frac{k}{2}-1}\left(\frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\frac{n}{2}}} - e^{-|y|^2}\right)\frac{dr}{r}.$$

For $|\alpha| = k$, by using the chain rule and the definition of the n-dimensional Hermite polynomials, we have

$$D_x^\alpha \left(e^{-\frac{|rx-y|^2}{1-r^2}}\right) = \frac{r^k}{(1-r^2)^{\frac{k}{2}}} H_\alpha\left(\frac{rx-y}{\sqrt{1-r^2}}\right) e^{-\frac{|rx-y|^2}{1-r^2}},$$

and taking into account that $-|rx-y|^2 = -|x-ry|^2 + (1-r^2)\left(|x|^2 - |y|^2\right)$, we get the following kernel:

$$\begin{aligned}
&D_x^\alpha K_{\mathcal{L}^{k/2}}(x,y) \\
&\quad = \frac{1}{\pi^{\frac{n}{2}}\Gamma\left(-\frac{k}{2}\right)} \int_0^1 \frac{(-\log r)^{-\frac{k}{2}-1}}{(1-r^2)^{\frac{n}{2}}} D_x^\alpha\left(e^{-\frac{|rx-y|^2}{1-r^2}}\right)\frac{dr}{r} \\
&\quad = \frac{1}{\pi^{\frac{n}{2}}\Gamma\left(-\frac{k}{2}\right)} \int_0^1 \frac{(-\log r)^{-\frac{k}{2}-1} r^{k-1}}{(1-r^2)^{\frac{n+k}{2}}} H_\alpha\left(\frac{rx-y}{\sqrt{1-r^2}}\right) e^{-\frac{|rx-y|^2}{1-r^2}}\, dr \\
&\quad = \frac{e^{|x|^2-|y|^2}}{\pi^{\frac{n}{2}}\Gamma\left(-\frac{k}{2}\right)} \int_0^1 \frac{r^{k-1}}{(-\log r)^{\frac{k+2}{2}}(1-r^2)^{\frac{n+k}{2}}} H_\alpha\left(\frac{rx-y}{\sqrt{1-r^2}}\right) e^{-\frac{|x-ry|^2}{1-r^2}}\, dr.
\end{aligned}$$

Let $\ell \in \mathbb{N}_0$ be a non-negative integer such that $k = 2\ell + 1$. By using again the definition of Hermite polynomials on $\mathbb{R}^n$, it is well known that

$$|H_\alpha(u)| \lesssim \sum_{j=0}^{\ell} |u|^{2j+1}$$

for $u \in \mathbb{R}^n$. Thus,

$$|D_x^\alpha K_{\mathcal{L}^{k/2}}(x,y)| \lesssim \sum_{j=0}^{\ell} e^{|x|^2-|y|^2} \int_0^1 \frac{r^{k-1}|rx-y|^{2j+1}}{(-\log r)^{\frac{k+2}{2}}(1-r^2)^{\frac{n+k+1}{2}+j}} e^{-\frac{|x-ry|^2}{1-r^2}}\, dr.$$

Fix $f \in L^1(\gamma)$ with $\operatorname{supp} f \subseteq \overline{B}$ for some ball $B = B(c_B, r_B) \in \mathscr{B}_1$. Then,

$$\begin{aligned}
\|D_x^\alpha \mathcal{L}^{k/2} f\|_{L^1((4B)^c,\gamma)} &\leq \int_{(4B)^c}\int_B |D_x^\alpha K_{\mathcal{L}^{k/2}}(x,y)||f(y)|\, dy\, e^{-|x|^2}\, dx \\
&\lesssim \sum_{j=0}^{\ell} \int_{(4B)^c}\int_B\int_0^1 \frac{r^{k-1}|rx-y|^{2j+1}}{(-\log r)^{\frac{k+2}{2}}(1-r^2)^{\frac{n+k+1}{2}+j}} \\
&\qquad\qquad \times e^{-\frac{|x-ry|^2}{1-r^2}}\, dr\, |f(y)|\, d\gamma(y)\, dx \\
&:= \sum_{j=0}^{\ell} \int_B I_j(y)|f(y)|\, d\gamma(y),
\end{aligned}$$

where

$$\begin{aligned}I_j(y) &= \int_0^1 \frac{r^{k-1}}{(-\log r)^{\frac{k+2}{2}}(1-r^2)^{\frac{n+k+1}{2}+j}} \int_{(4B)^c} |rx-y|^{2j+1} e^{-\frac{|x-ry|^2}{1-r^2}}\, dx\, dr \\ &:= \int_0^1 J(y,r)\, dr = \int_0^{\frac{1}{2}} J(y,r)\, dr + \int_{\frac{1}{2}}^1 J(y,r)\, dr \\ &:= I_{j,1}(y) + I_{j,2}(y)\end{aligned}$$

for each $j = 0, 1, \ldots, \ell$.

If we prove that

$$I_{j,i}(y) \lesssim r_B^{-2k}, \quad j = 0, 1, \ldots, \ell, \;\; i = 1, 2, \tag{4.1}$$

we can conclude that

$$\|D_x^\alpha \mathcal{L}^{k/2} f\|_{L^1((4B)^c,\gamma)} \lesssim \ell r_B^{-2k} \|f\|_{L^1(B,\gamma)} \approx r_B^{-2k} \|f\|_{L^1(B,\gamma)}.$$

In order to estimate (4.1) for $I_{j,1}(y)$, let us consider $v = x - ry$ so we get $rx - y = rv + (r^2-1)y$ and $dx = dv$. Since $y \in B \in \mathscr{B}_1$, we get $|y| \leq |c_B| + r_B \leq \frac{2}{r_B}$. Then

$$|rx-y| \leq r|v| + (1-r^2)|y| \leq |v| + |y| \leq |v| + \frac{2}{r_B} \leq 2\frac{|v|+1}{r_B}.$$

We apply this and use that

$$1 - r^2 \geq \frac{3}{4} \quad \text{and} \quad \frac{r^{k-1}}{(-\log r)^{\frac{k+2}{2}}} \leq \frac{2^{1-k}}{(\log 2)^{\frac{k+2}{2}}}$$

for $0 < r \leq \frac{1}{2}$, to get

$$\begin{aligned}I_{j,1}(y) &\lesssim \int_0^{\frac{1}{2}} \frac{r^{k-1}}{(-\log r)^{\frac{k+2}{2}}} \int_{\mathbb{R}^n} \frac{(|v|+1)^{2j+1}}{r_B^{2j+1}} e^{-|v|^2}\, dv\, dr \\ &\lesssim \frac{1}{r_B^{2\ell+1}} \int_{\mathbb{R}^n} e^{-|v|^2/2}\, dv \lesssim \frac{1}{r_B^k}.\end{aligned}$$

Clearly, since $r_B \leq 1$, we also have $I_{j,1}(y) \lesssim r_B^{-2k}$ for every $j = 0, 1, \ldots, \ell$.

When $\frac{1}{2} < r < 1$, we have that $-\log r \approx 1 - r^2$. By splitting

$$|rx-y| = |r(x-ry) - (1-r^2)y| \leq |x-ry| + (1-r^2)|y|,$$

we have

$$\begin{aligned}&I_{j,2}(y) \\ &\lesssim \int_{\frac{1}{2}}^1 \frac{1}{(1-r^2)^{\frac{k+2}{2}}(1-r^2)^{\frac{n+k}{2}}} \int_{(4B)^c} \left(\frac{|rx-y|}{\sqrt{1-r^2}}\right)^{2j+1} e^{-\frac{|x-ry|^2}{1-r^2}}\, dx\, dr \\ &\lesssim \int_{\frac{1}{2}}^1 \frac{1}{(1-r^2)^{\frac{n}{2}+k+1}} \int_{(4B)^c} \left[\left(\frac{|x-ry|}{\sqrt{1-r^2}}\right)^{2j+1} + \left(\sqrt{1-r^2}|y|\right)^{2j+1}\right] e^{-\frac{|x-ry|^2}{1-r^2}}\, dx\, dr \\ &\lesssim \int_{\frac{1}{2}}^1 \frac{1 + \left(\sqrt{1-r^2}|y|\right)^{2j+1}}{(1-r^2)^{k+1}} \left(\frac{1}{(1-r^2)^{\frac{n}{2}}} \int_{(4B)^c} e^{-\frac{|x-ry|^2}{2(1-r^2)}}\, dx\right) dr. \end{aligned} \tag{4.2}$$

Now, let $r_{B,y}$ be the number defined in Lemma 4.2 and consider the cases $r_{B,y} \geq 1$ and $r_{B,y} < 1$.

If $r_{B,y} \geq 1$, for any $x \in (4B)^c$ and $r \in \left(\frac{1}{2}, 1\right)$ we have that $|x - ry| \geq \frac{1}{4}|x - c_B|$ by Lemma 4.2 (i). Hence, from Lemma 4.2 (iii),

$$\begin{aligned}\frac{1}{(1-r^2)^{\frac{n}{2}}} \int_{(4B)^c} e^{-\frac{|x-ry|^2}{2(1-r^2)}}\, dx &\leq \frac{1}{(1-r^2)^{\frac{n}{2}}} \int_{(4B)^c} e^{-c\frac{|x-c_B|^2}{1-r^2}}\, dx \\ &= C_1 \varphi_c\left(\frac{r_B}{\sqrt{1-r^2}}\right) \leq C_1 e^{-C_2 \frac{r_B^2}{1-r^2}},\end{aligned}$$

where $\varphi_c(s) = (1+s)^{n-2} e^{-cs^2}$.

This leads to

$$\begin{aligned}I_{j,2}(y) &\lesssim \int_{\frac{1}{2}}^{1} \frac{1 + \left(\sqrt{1-r^2}|y|\right)^{2j+1}}{(1-r^2)^{k+1}} e^{-C_2 \frac{r_B^2}{1-r^2}}\, dr \\ &\lesssim \int_{\frac{1}{2}}^{1} \frac{1 + \left(\sqrt{1-r^2}|y|\right)^{2j+1}}{(1-r^2)^{k-\frac{1}{2}}} e^{-C_2 \frac{r_B^2}{1-r^2}} \frac{2r}{(1-r^2)^{\frac{3}{2}}}\, dr.\end{aligned}$$

Setting $s = r_B/\sqrt{1-r^2}$, $ds/r_B = r dr/(1-r^2)^{\frac{3}{2}}$, and recalling that $r_B|y| \leq \frac{1}{2}$, we get

$$\begin{aligned}I_{j,2}(y) &\lesssim \int_{\frac{1}{2}}^{1} \frac{1 + \left(\sqrt{1-r^2}|y|\right)^{2j+1}}{\left(\sqrt{1-r^2}\right)^{2k-1}} e^{-C_2 \frac{r_B^2}{1-r^2}} \frac{r}{(1-r^2)^{\frac{3}{2}}}\, dr \\ &\lesssim \int_0^\infty \frac{1 + \left(\frac{r_B}{s}|y|\right)^{2j+1}}{\left(\frac{r_B}{s}\right)^{2k-1}} e^{-C_2 s^2} \frac{ds}{r_B} \\ &= \frac{1}{r_B^{2k}} \int_0^\infty \frac{s^{2j+1} + (r_B|y|)^{2j+1}}{s^{2j+1}} s^{2k-1} e^{-C_2 s^2} ds \\ &\lesssim \frac{1}{r_B^{2k}} \int_0^\infty \left(s^{2k-1} + s^{2(k-j-1)}\right) e^{-C_2 s^2} ds \lesssim \frac{1}{r_B^{2k}},\end{aligned}$$

since $k - j - 1 \geq \frac{k-1}{2} \geq 0$ for every $0 \leq j \leq \ell = \frac{k-1}{2}$. This gives estimate (4.1) when $r_{B,y} \geq 1$.

We now study the case $r_{B,y} < 1$. Set $I_{j,2}(y) := I_{j,2,1}(y) + I_{j,2,2}(y)$, splitting the integral over $\left(\frac{1}{2}, 1 - r_{B,y}\right)$ and $(1 - r_{B,y}, 1)$, respectively. For the second interval, we can apply Lemma 4.2 (ii) and (iii), and proceed as in the previous case.

On the interval $\left(\frac{1}{2}, 1 - r_{B,y}\right)$, we can proceed as in (4.2) to get

$$I_{j,2,1}(y) \lesssim \int_{\frac{1}{2}}^{1-r_{B,y}} \frac{1 + (1-r^2)^{j+\frac{1}{2}}|y|^{2j+1}}{(1-r^2)^{k+1}} \left(\frac{1}{(1-r^2)^{\frac{n}{2}}} \int_{(4B)^c} e^{-\frac{|x-ry|^2}{2(1-r^2)}}\, dx\right) dr.$$

We perform the change of variables

$$v = \frac{x - ry}{\sqrt{2(1-r^2)}}, \quad dv = \frac{dx}{(2(1-r^2))^{\frac{n}{2}}},$$

in order to obtain

$$\begin{aligned}
I_{j,2,1}(y) &\approx \int_{\frac{1}{2}}^{1-r_{B,y}} \frac{1+(1-r^2)^{j+\frac{1}{2}}|y|^{2j+1}}{(1-r^2)^{k+1}} \int_{\mathbb{R}^n} e^{-|v|^2}\,dv\,dr \\
&\approx \int_{\frac{1}{2}}^{1-r_{B,y}} \frac{1+(1-r)^{j+\frac{1}{2}}|y|^{2j+1}}{(1-r)^{k+1}}\,dr \\
&\leq \frac{1}{r_{B,y}^k} + |y|^{2j+1}\int_{\frac{1}{2}}^{1-r_{B,y}} (1-r)^{j-k-\frac{1}{2}}\,dr \\
&\lesssim \frac{1}{r_{B,y}^k} + \frac{|y|^{2j+1}}{r_{B,y}^{k-j-\frac{1}{2}}} \\
&\approx \frac{|y|^k}{r_B^k} + \frac{|y|^{k+j+\frac{1}{2}}}{r_B^{k-j-\frac{1}{2}}} \\
&= \left(\frac{|y|}{r_B}\right)^k \Big(1+(|y|r_B)^{j+\frac{1}{2}}\Big).
\end{aligned}$$

Since $|y| \leq |c_B| + r_B \leq \frac{2}{r_B}$ and $|y|r_B \leq 2$, we get

$$I_{j,2,2}(y) \lesssim r_B^{-2k}.$$

The proof is now concluded. □

5. Proofs of Theorems 1.1 and 1.2

For proving the main results, we may need the integral representations of both Gaussian Riesz transforms R_α and R_α^*. Given a multi-index $\alpha \in \mathbb{N}_0^n \setminus \{(0,\ldots,0)\}$, we have

$$R_\alpha f(x) = \text{p.v.} \int_{\mathbb{R}^n} k_\alpha(x,y) f(y)\,dy,$$

and

$$R_\alpha^* f(x) = \text{p.v.} \int_{\mathbb{R}^n} k_\alpha^*(x,y) f(y)\,dy,$$

where

$$k_\alpha(x,y) = c_{n,\alpha} \int_0^1 r^{|\alpha|-1} \left(\frac{-\log r}{1-r^2}\right)^{|\alpha|/2-1} H_\alpha\left(\frac{y-rx}{\sqrt{1-r^2}}\right) \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{n/2+1}}\,dr$$

and

$$k_\alpha^*(x,y) = c_{n,\alpha} e^{|x|^2-|y|^2} \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{|\alpha|/2-1} H_\alpha\left(\frac{x-ry}{\sqrt{1-r^2}}\right) \frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{n/2+1}}\,dr,$$

respectively.

The operator R_α turns out to be bounded on $L^p(\gamma)$ for $1<p<\infty$ (see, for instance, [7, 9, 13]). For the first-order Gaussian Riesz transforms $R_{e_i}^*$, $i=1,\ldots,n$, the $L^p(\gamma)$ boundedness was obtained in [6] for $1<p<\infty$. By Meyer's multiplier theorem, the new higher-order Gaussian Riesz transforms are also bounded on $L^p(\gamma)$, as can be proved similarly to [14, Corollary 9.14].

Proof of Theorem 1.1. First we are going to prove that the old higher-order Gaussian Riesz transforms R_α with $|\alpha| = k$ are uniformly bounded on $L^1(\gamma)$ when applied to every X^k-atom. Once this is done, the boundedness of R_α from $X^k(\gamma)$ into $L^1(\gamma)$ follows from the fact that these transforms are bounded from $L^1(w_k\gamma)$ into $L^{1,\infty}(\gamma)$ (see [5]). Indeed, by following steps closely related to those in, for instance, [2], we obtain that for $f = \sum_j \lambda_j a_j$, where each a_j is an X^k-atom and $\sum_j |\lambda_j| < \infty$, the series defining f converges in $L^1(w_k\gamma)$ and therefore

$$R_\alpha f = \lim_{\ell\to\infty} R_\alpha\left(\sum_{j=1}^{\ell} \lambda_j a_j\right)$$

in $L^{1,\infty}(\gamma)$.

Then there exists an increasing function $\psi : \mathbb{N} \to \mathbb{N}$ such that

$$R_\alpha f(x) = \lim_{\ell\to\infty} R_\alpha\left(\sum_{j=1}^{\psi(\ell)} \lambda_j a_j\right)(x) \quad \text{a.e. } x \in \mathbb{R}^n.$$

Thus,

$$|R_\alpha f(x)| = \lim_{\ell\to\infty}\left|\sum_{j=1}^{\psi(\ell)} \lambda_j R_\alpha a_j(x)\right| \le \sum_{j=1}^{\infty} |\lambda_j||R_\alpha a_j(x)|.$$

From this,

$$\|R_\alpha f\|_{L^1(\gamma)} \le \sum_{j=1}^{\infty} |\lambda_j|\|R_\alpha a_j\|_{L^1(\gamma)} \le C\sum_{j=1}^{\infty} |\lambda_j|,$$

where C is an absolute constant independent of the atoms on which any $f \in X^k(\gamma)$ is decomposed as we shall see it in what follows. Then, $\|R_\alpha f\|_{L^1(\gamma)} \le C\|f\|_{X^k(\gamma)}$.

Let us then prove that

$$\|R_\alpha a\|_{L^1(\gamma)} \le C \tag{5.1}$$

for every X^k-atom a. For k even, let us call $k = 2j$. Then the old Gaussian Riesz transforms of even order now become $\nabla^{2j}\mathcal{L}^{-j}$ with $\nabla^{2j} = \sum_{|\alpha|=2j} D^\alpha$.

Let a be an X^k-atom. Then, by Remark 3.3, a is also an X^j-atom. Since a is supported in a critical ball B, by Proposition 4.1 we have that $\mathcal{L}^{-j}a$ is supported in $\overline{B}$. By applying Hölder's inequality and using the boundedness of $\nabla^{2j}\mathcal{L}^{-j}$ on $L^2(\gamma)$, we get that

$$\|\nabla^{2j}\mathcal{L}^{-j}a\|_{L^1(\gamma)} \le \|\nabla^{2j}\mathcal{L}^{-j}a\|_{L^2(\gamma)}\gamma(B)^{1/2} \lesssim \|a\|_{L^2(\gamma)}\gamma(B)^{1/2} \lesssim 1,$$

where we have also used Remark 3.2. This takes care of the boundedness from $X^k(\gamma)$ to $L^1(\gamma)$ of the old k-th order Gaussian Riesz transforms for k even.

We now turn to the proof of the boundedness of this operator for k odd. Given an X^k-atom a, notice that

$$\|R_\alpha a\|_{L^1(\gamma)} \le \|R_\alpha a\|_{L^1(4B,\gamma)} + \|R_\alpha a\|_{L^1((4B)^c,\gamma)}.$$

For the first term on the right-hand side, we apply again Hölder's inequality, together with the $L^2(\gamma)$-boundedness of R_α. This yields

$$\|R_\alpha a\|_{L^1(4B,\gamma)} \le \|R_\alpha a\|_{L^2(4B,\gamma)}\gamma(4B)^{1/2} \lesssim \|a\|_{L^2(\gamma)}\gamma(4B)^{1/2} \lesssim 1,$$

where in the last inequality we have used that a is an X^k-atom and γ is doubling over admissible balls.

To take care of the term $\|R_\alpha a\|_{L^1((4B)^c,\gamma)}$, we write $D^\alpha \mathcal{L}^{-k/2}a = D^\alpha \mathcal{L}^{k/2}(\mathcal{L}^{-k}a)$. Since a is an X^k-atom, we get $\operatorname{supp}\mathcal{L}^{-k}a \subseteq \overline{B}$ and

$$\gamma(B)^{1/2}\|\mathcal{L}^{-k}a\|_{L^2(\gamma)} \lesssim r_B^{2k},$$

by Proposition 4.1. Using Lemma 4.3 and the inequality above with $f = \mathcal{L}^{-k}a$, we get

$$\begin{aligned}\|D^\alpha \mathcal{L}^{-k/2}a\|_{L^1((4B)^c,\gamma)} &= \left\|D^\alpha \mathcal{L}^{k/2}f\right\|_{L^1((4B)^c,\gamma)}\\ &\lesssim r_B^{-2k}\|f\|_{L^1(B,\gamma)} \lesssim r_B^{-2k}\gamma(B)^{1/2}\|\mathcal{L}^{-k}a\|_{L^2(\gamma)} \lesssim 1.\end{aligned}$$

This proves (5.1) for every $k \in \mathbb{N}$. □

Proof of Theorem 1.2. Let us point out here that in order to prove the boundedness of the new Gaussian Riesz transforms from $\mathcal{H}^1(\gamma)$ into $L^1(\gamma)$, we will not be able to apply [10, Theorem 6.1 (ii) and Remark 6.2], for if we consider $m(x,y)$ being the kernel associated to the old Gaussian Riesz transforms, $m^*(x,y)$ does not represent the kernel associated to the new higher-order ones. So, given an $\mathcal{H}^1$-atom a, in order to prove this result we will proceed as in the proof of the boundedness of the old higher-order Gaussian Riesz transforms of odd order, by splitting the norm in the following way:

$$\|R_\alpha^* a\|_{L^1(\gamma)} \le \|R_\alpha^* a\|_{L^1(2B,\gamma)} + \|R_\alpha^* a\|_{L^1((2B)^c,\gamma)}.$$

Then, the first term in the sum is bounded by the $L^2(\gamma)$-norm of $R_\alpha^* a$ times $\gamma^{1/2}(2B)$, and we use the continuity of R_α^* on $L^2(\gamma)$ and the fact that γ is a doubling measure over $\mathscr{B}_1$.

Now, for taking care of the second term in the above sum, we use the fact that the atom a has its support contained in B and it has average zero over that ball. This yields

$$\begin{aligned}&\|R_\alpha^* a\|_{L^1((2B)^c,\gamma)}\\ &\quad\le C\int_B |a(y)| \int_{(2B)^c}\left|\int_0^1 \frac{\lambda_\alpha(r)}{(1-r^2)^{n/2+1}}\left(F_\alpha(x,y,r) - F_\alpha(x,c_B,r)\right)dr\right| dx\,d\gamma(y),\end{aligned}$$

where

$$\lambda_\alpha(r) = \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\alpha|}{2}-1} \quad \text{and} \quad F_\alpha(x,y,r) = H_\alpha\left(\frac{x-ry}{\sqrt{1-r^2}}\right)e^{-\frac{|x-ry|^2}{1-r^2}}.$$

By applying the mean value theorem to the function F_α in the variable y, we get that

$$\|R_\alpha^* a\|_{L^1((2B)^c,\gamma)} \lesssim \nu_s\|a\|_{L^1(\gamma)} \le \nu_s\|a\|_{L^2(\gamma)}\gamma^{1/2}(B) \lesssim \nu_s,$$

where

$$\nu_s = \sup_{B\in\mathscr{B}_1} \sup_{y\in B} r_B \int_{(2B)^c} s(x,y)\,dx$$

and

$$s(x,y) = c_n \int_0^1 \frac{\lambda_\alpha(r)}{(1-r^2)^{n/2+1}} |\nabla_y F_\alpha(x,y,r)|\,dr.$$

Now we prove the finiteness of ν_s. For every $i=1,\dots,n$,

$$\partial_{y_i} F_\alpha(x,y,r) = 2r\left(\alpha_i H_{\alpha-e_i}\left(\frac{x-ry}{\sqrt{1-r^2}}\right) - \frac{x_i - ry_i}{\sqrt{1-r^2}} H_\alpha\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\right)\frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{\sqrt{1-r^2}},$$

which leads to

$$|\nabla_y F_\alpha(x,y,r)| \lesssim \left|P\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\right| \frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{\sqrt{1-r^2}},$$

where P is a polynomial of degree $|\alpha|$. Therefore,

$$|\nabla_y F_\alpha(x,y,r)| \lesssim \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{\sqrt{1-r^2}}.$$

Consequently,

$$\begin{aligned}\nu_s &\lesssim \sup_{B\in\mathscr{B}_1} \sup_{y\in B} r_B \int_{(2B)^c} \int_0^1 \frac{\lambda_\alpha(r)}{(1-r^2)^{3/2}} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{n/2}}\,dr\,dx \\ &= \sup_{B\in\mathscr{B}_1} \sup_{y\in B} r_B \int_0^1 \frac{\lambda_\alpha(r)}{(1-r^2)^{3/2}} \int_{(2B)^c} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{n/2}}\,dx\,dr.\end{aligned}$$

In order to see that $\nu_s < \infty$, it will be enough to show that, for every ball $B\in\mathscr{B}_1$ and every $y\in B$,

$$I = \int_0^1 \frac{\lambda_\alpha(r)}{(1-r^2)^{3/2}} \int_{(2B)^c} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{n/2}}\,dx\,dr \lesssim \frac{1}{r_B}.$$

Let $r_{B,y}$ be the number defined in Lemma 4.2 and assume that $r_{B,y} \ge 1$. By applying items (i) and (iii) in Lemma 4.2, and the fact that $\lambda_\alpha(r) \le C_{\alpha,n} r^{-1/2}$ for any $r\in(0,1/2)$, we get

$$\begin{aligned}I &\lesssim \int_0^1 \frac{\lambda_\alpha(r)}{(1-r^2)^{3/2}} e^{-C_2\frac{r_B^2}{1-r^2}}\,dr \\ &\lesssim \int_0^{1/2} \lambda_\alpha(r)\,dr + \int_{1/2}^1 \frac{e^{-\frac{C_2 r_B^2}{2(1-r^2)}}}{(1-r^2)^{3/2}}\,dr \qquad (5.2) \\ &\le \sqrt{2}C_{\alpha,n} + \frac{2}{r_B}\int_0^\infty e^{-C_2u^2}\,du \lesssim 1 + \frac{1}{r_B} \lesssim \frac{1}{r_B}.\end{aligned}$$

We now turn our attention to the case $r_{B,y} < 1$. We split the integral I into three parts, I_1, I_2 and I_3, as follows:

$$I_1 = \int_0^{1/2} \frac{\lambda_\alpha(r)}{(1-r^2)^{3/2}} \int_{(2B)^c} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{n/2}}\, dx\, dr,$$

$$I_2 = \int_{1/2}^{1-r_{B,y}} \frac{\lambda_\alpha(r)}{(1-r^2)^{3/2}} \int_{(2B)^c} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{n/2}}\, dx\, dr,$$

$$I_3 = \int_{1-r_{B,y}}^{1} \frac{\lambda_\alpha(r)}{(1-r^2)^{3/2}} \int_{(2B)^c} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{n/2}}\, dx\, dr.$$

We shall estimate every term above separately. We apply item (ii) in Lemma 4.2 and the change of variable $z = |x - c_B|/\sqrt{1-r^2}$ to have

$$\begin{aligned} I_1 &\lesssim \int_0^{1/2} \lambda_\alpha(r) \int_{\mathbb{R}^n} \frac{e^{-\frac{|x-c_B|^2}{2(1-r^2)}}}{(1-r^2)^{n/2}}\, dx\, dr \\ &= \left(\int_0^{1/2} \lambda_\alpha(r)\, dr\right)\left(\int_{\mathbb{R}^n} e^{-\frac{|z|^2}{2}}\, dz\right) \lesssim 1 \le \frac{1}{r_B}, \end{aligned}$$

where we have again used the integrability of λ_α as in (5.2).

On the other hand, since $\lambda_\alpha(r) \le C_\alpha$ for $r \in [1/2, 1]$, and $1 - r^2 \approx 1 - r$ on $[1/2, 1 - r_{B,y}]$, by repeating the argument for I_1 on the inner integral, we get

$$\begin{aligned} I_2 &\lesssim \int_{1/2}^{1-r_{B,y}} \frac{1}{(1-r)^{3/2}} \int_{\mathbb{R}^n} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{n/2}}\, dx\, dr \\ &\lesssim \left(\int_{r_{B,y}}^{1/2} u^{-3/2}\, du\right)\left(\int_{\mathbb{R}^n} e^{-\frac{|z|^2}{2}}\, dz\right) \lesssim \frac{1}{r_{B,y}^{1/2}} = \left(\frac{2|y|}{r_B}\right)^{1/2} \lesssim 1 + \frac{1}{r_B} \lesssim \frac{1}{r_B}, \end{aligned}$$

since $|y|^{1/2} \le r_B^{1/2} + r_B^{-1/2}$.

Finally, the bound for I_3 follows, as in the case $r_{B,y} \ge 1$, by using item (ii) in Lemma 4.2 and proceeding as in the estimate for the second term in (5.2).

Taking into account that these new higher-order Gaussian Riesz transforms are also bounded from $L^1(\gamma)$ into $L^{1,\infty}(\gamma)$ (see [1]), and proceeding as before when we have other types of continuity, these operators extend boundedly to the whole atomic space $\mathcal{H}^1(\gamma)$. $\square$

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