ON THE DIFFERENTIABILITY OF FUNCTIONS
WHICH ARE OF BOUNDED VARIATION
IN TONELLI'S SENSE (*)

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1. In this note we generalize some of the results of our note [1]. These generalizations are not difficult but seem to be of interest in applications.

We shall consider complex — valued measurable functions $f(x) = f(x_1, x_2, \ldots, x_n)$ of n real variables defined for simplicity over the whole n — dimensional Euclidean space E_n, and we write

$$|| f ||_p = \left\{ \int | f(x) |^p \, dx \right\}^{1/p},$$

where the integral is extended over E_n. By (f, g) we mean $\int f g \, dx$, assuming that the integral converges absolutely, and by $f * g$ the convolution of f and g.

We shall also consider completely additive functions $\mu(E)$ of Borel subsets E of E_n. By (f, μ) we mean $\int f \, d\mu$ and by $f * \mu$ the convolution $\int f(x - y) \, d\mu(y)$. By D we denote the class of infinitely differentiable functions in E_n with compact support. Finally, C will stand for a constant depending only on the dimension and the parameters displayed.

A locally integrable function is said to have first derivatives $f_j(x)$, $j = 1, 2, \ldots, n$, in the sense of distributions if

$$(\varphi, f_j) = - \left(\frac{\partial \varphi}{\partial x_j}, f \right)$$

for all $\varphi \in D$. We shall also occasionally write $\dfrac{\partial f}{\partial x_j}$ for $f_j(x)$.

Similarly we say that f has measures μ_j for first derivatives if

$$(\varphi, \, d\,\mu_j) \; = \; - \left(\frac{\partial \, \varphi}{\partial \, x_j} \, , \, f \right) ,$$

for all $\varphi \, \varepsilon \, D$.

We recall now some notions introduced in [1]. We considered there classes $T_u{}^p(x_0)$ and $t_u{}^p(x_0)$ of functions. A function f belongs to $T_u{}^p(x_0)$, where $1 \le p \le \infty$ and $u \ge - n/p$, if there exists a polynomial $P(x)$ of degree strictly less than u (in particular, $P = 0$ if $u \le 0$) such that

$$\left\{ \frac{1}{\rho^n} \int\limits_{|\,x - x_0\,| \,\le\, \rho} |\, f(x) - P(x)\,|^p\, dx \right\}^{1/p} \; \le \; M\, \rho^u \; , \quad 0 < \rho < \infty. \quad [1]$$

The polynomial P is uniquely determined by f. In [1] we also introduced a norm in the space $T_u{}^p(x_0)$. The norm of an $f \, \varepsilon \, T_u{}^p(x_0)$, which we denoted as $T_u{}^p(x_0, f)$, is defined as the sum of the norm of f in L^p, the absolute values of the coefficients of the Taylor expansion of P at x_0, and the greatest lower bound of the constants M for which [1] holds.

The space $t_u{}^p(x_0)$ consists of those functions in $T_u{}^p(x_0)$ for which there exists a polynomial $Q(x)$ such that

$$\left\{ \frac{1}{\rho^n} \int\limits_{|\,x - x_0 \,\le\, \rho} |\, f(x) - Q(x)\,|^p\, dx \right\}^{1/p} \; \le \; M\, \rho^u$$

and, in addition, the expression on the left is $o\,(\rho^u)$ as ρ tends to 0. Here $Q(x)$ is unique, and is equal to the polynomial $P(x)$ of [1] if u is distinct from 0, 1, 2, $\ldots$ The degree of Q is equal to the integral part of u for $u \ge 0$.

In the present note we study an extension of the classes $T_u{}^p(x_0)$ and $t_u{}^p(x_0)$ in the case $p = 1$. Let $u \ge - n/p$. We denote by $S_u(x_0)$ the class of countably additive finite functions of Borel subsets of E_n for which there exists a polynomial $P(x)$ of degree strictly less than u (in particular $P \equiv 0$ if $u \le 0$) such that

$$\frac{1}{\rho^n} \int\limits_{|\,x - x_0\,| \,\le\, \rho} |\, d\, \{\, \mu - P(x)\, \lambda \,\}\,| \; \le \; M\, \rho^u \; , \quad 0 < \rho < \infty. \quad [2]$$

Here $P(x)\,\lambda$ stands for the indefinite integral of $P(x)$ with respect to the Lebesgue measure λ, and the integral denotes the total

variation of the set function $\mu - P(x) \lambda$ in $|x - x_0| \leq \rho$. The norm of μ in $S_u(x_0)$ is analogously defined, namely, as the sum of the total variation of μ, the absolute values of the coefficients of the Taylor expansion of $P(x)$ at x_0, and the greatest lower bound of the constants M for which [2] holds.

The subclass $s_u(x_0)$ of $S_u(x_0)$ is obtained in the same way as $t_u{}^p(x_0)$ was singled out from $T_u{}^p(x_0)$.

The classical theorem about Lebesgue sets of integrable functions is valid also for set functions μ and asserts that for almost all x_0 we have

$$\frac{1}{\rho^n} \int\limits_{|x - x_0| \leq \rho} |\, d \{ \mu - \mu'(x_0) \lambda \} \,| = o(1) \ , \quad \rho \to 0,$$

where $\mu'(x_0)$ is the derivative of μ at x_0. In other words, μ belongs to $s_0(x_0)$ for almost all x_0.

It is well known (see [2]) that a locally integrable function whose first derivatives are set functions μ_j of finite total variation coincides almost everywhere with a function of bounded variation in the sense of Tonelli, and conversely. The main result of this note concerns the differentiability of functions of bounded variation in Tonelli's sense.

THEOREM. *Let $f(x_1, x_2, \ldots, x_n)$ be a function of bounded variation in the sense of Tonelli. Then*

(i) *there is a constant a such that $f - a$ belongs to $L^{\frac{n}{n-1}}$;*

(ii) $\mu_j = \dfrac{\partial f}{\partial x_j}$ *is a countably additive set function of finite total variation, and if $\mu_j =$ belongs to $S_u(x_0)$, $u \geq -n$, $u \neq -1$, $j = 1$, $2, \ldots, n$, then $f - a \in T_{u+1}^{\frac{n}{n-1}}(x_0)$ and*

$$T_{u+1}^{\frac{n}{n-1}}(x_0 , f - a) \leq C \sum_{j=1}^{n} S_u(x_0 , \mu_j);$$

(iii) *if $\mu_j = \dfrac{\partial f}{\partial x_j}$ belongs to $s_u(x_0)$, $u \geq -n$, $u \neq -1$, $j = 1$, $2, \ldots, n$, then $f - a \in t_{u+1}^{\frac{n}{n-1}}(x_0)$.*

If f is of bounded variation in Tonelli's sense then, as we have pointed out, its derivatives in the sense of distributions are countably additive set functions. Therefore, as we also pointed out, these derivatives belong to $s_0(x_0)$ for almost all x_0. Therefore, after the subtraction of an appropriate constant, f will belong to $t_1(x_0)$ for almost all x_0.

In other words, *a function $f(x)$ which is of bounded variation in the sense of Tonelli has a first differential almost everywhere, provided the remainder is estimated in the metric $L^{\frac{n}{n-1}}$.*

2. In the case when the $\mu_j = \dfrac{\partial f}{\partial x_j}$ are absolutely continuous with respect to Lebesgue measure, our theorem is contained in Theorem 11 of [1]. Our proof will consist in reducing the present case to that one.

In what follows we will regularize functions and measures by convoluting them with a kernel

$$\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi\left(\frac{x}{\varepsilon}\right),$$

where φ is a function in D, with support in $|x| \leq 1$, of integral equal to 1, with the additional property that for any polynomial $P(x)$ of degree not exceding a fixed integer m the identity

$$\varphi_\varepsilon * P = P$$

holds. It is not difficult to construct such a function φ (see [1], Lemma 2.6).

LEMMA 1. *Let μ be a completely additive set function belonging to $S_u(x_0)$, $u \geq -n$. Let $f_\varepsilon = \varphi_\varepsilon * \mu$, where φ is a function as described above with the property that $\varphi_\varepsilon * P = P$ for all polynomials P of degree not exceeding u. Then $f_\varepsilon \varepsilon T_u^1(x_0)$ and*

$$T_u^1(x_0, f_\varepsilon) \leq C_{\varphi, u} S(x_0, \mu).$$

Proof. For simplicity of notation we will assume that $x_0 = 0$ and we will write $S_u(x_0, \mu) = S_u(\mu)$, etc. Suppose first that $u > 0$ and let r be the largest integer strictly less than u. The function f_ε is infinitely differentiable and we denote by P_ε the Taylor polynomial of f of degree r. We then have

$$f_\varepsilon(x) - P_\varepsilon(x) = \int \left[\varphi_\varepsilon(x-y) - \sum_{|\alpha| \leq r} \frac{x^\alpha}{\alpha!} \varphi_\varepsilon^{(\alpha)}(-y) \right] d\mu(y).$$

Here α stands for the multiple index $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, where the α_j are non-negative integers, $\alpha! = \alpha_1! \, \alpha_2! \, \ldots \, \alpha_n!$,

$$| \alpha | = \alpha_0 + \alpha_2 + \ldots + \alpha_n \, , \ x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n},$$

and

$$\varphi^{(\alpha)} = \left(\frac{\partial}{\partial x} \right)^\alpha \varphi = \left(\frac{\partial}{\partial x_1} \right)^{\alpha_1} \left(\frac{\partial}{\partial x_2} \right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n} \right)^{\alpha_n} \varphi$$

On account of the property $\varphi_\varepsilon * P = P$ the last integral can be written

$$\int \left[\varphi_\varepsilon (x - y) - \sum_{|\alpha| \leq r} \frac{x^\alpha}{\alpha!} \varphi_\varepsilon^{(\alpha)} (-y) \right] d \, [\mu (y) - P (y) \, \lambda (y)],$$

where P is the Taylor polynomial associated with μ. Integrating with respect to x we obtain

$$\int_{|x| \leq \rho} | f_\varepsilon (x) - P_\varepsilon (x) | \, dx \leq \int \left[\int_{|x| \leq \rho} \Big| \varphi_\varepsilon (x - y) - \right.$$

$$\left. - \sum_{|\alpha| \leq r} \frac{x^\alpha}{\alpha!} \varphi_\varepsilon^{(\alpha)} (-y) \Big| \, dx \right] | \, d \, [\mu (y) - \lambda (y) \, P (y) | \,] = I$$

say.

If $\rho \geq \varepsilon$, then

$$I \leq \int \int [| \varphi_\varepsilon (x - y) | \, dx] | \, d \, [\mu (y) - P (y) \, \lambda)] | +$$

$$+ \sum_{|\alpha| \leq r} \int_{|y| \leq \varepsilon} | \varphi_\varepsilon^{(\alpha)} (-y) | \int_{|x| \leq \rho} | x^\alpha | \, dx \, | d \, [\mu (y) - P (y) \, \lambda (y)] | . \quad]3]$$

Since the support of $\varphi_\varepsilon (x)$ is contained in $| x | \leq \varepsilon$, we have $\varphi_\varepsilon (x - y) = 0$ if $| x | \leq \rho$ and $| y | \geq 2 \, \rho \geq 2 \, \varepsilon$, and consequently the first integral above is dominated by

$$\int_{|y| \leq 2 \rho} | \, d \, [\mu (y) - P (y) \, \lambda] | \int | \varphi_\varepsilon (x) | \, dx \leq$$

$$\leq C_\varphi \, S_u (\mu) \, (2 \, \rho)^{n+u} = C_{\varphi, u} \, S_u (\mu) \, \rho^{n+u}.$$

On the other hand, $| \varphi_\varepsilon^{(\alpha)} (-y) | \leq C_{\varphi, u} \, \varepsilon^{-n-|\alpha|}$, and thus the terms of the sum in [3] are dominated by

$$C_{\varphi, u} \, S_u (\mu) \, \rho^{|\alpha|+n} \, \varepsilon^{-n-|\alpha|} \, \varepsilon^{n+u} \leq C_{\varphi, u} \, S_u (\mu) \, \rho^{n+u}$$

Consequently,

$$I \leq C_{\varphi,u} S_u(\mu) \, \rho^{n+u} \text{ , if } \rho \geq \varepsilon.$$

If $\rho < \varepsilon$, using Lagrange's remainder formula we obtain, with $0 < \theta = \theta(x, y) < 1$,

$$I = \int \left[\int_{|x| \leq \rho} \sum_{|\alpha|=r+1} \left| \frac{x^\alpha}{\alpha!} \, \varphi_\varepsilon^{(\alpha)} (\theta x - y) \right| d x \right] \mid d \left[\mu(y) - P(y) \lambda(y) \right] \mid \leq$$

$$\leq C_{\varphi,u} \, \rho^{r+1+n} \, \varepsilon^{-n-r-1} \int_{|y| \leq 2\varepsilon} \mid d \left[\mu - P(y) \lambda \right] \mid \leq$$

$$\leq C_{\varphi,u} \, S_u(\mu) \, \rho^{r+1+n} \, \varepsilon^{-n-r-1} \, (2 \, \varepsilon)^{n+u} \leq$$

$$\leq C_{\varphi,u} \, S_u(\mu) \, \rho^{n+u} \left(\frac{\rho}{\varepsilon} \right)^{r+1-u} \leq C_{\varphi,u} \, S_u(\mu) \, \rho^{n+u}.$$

Thus

$$I \leq C_{\varphi,u} \, S_u(\mu) \, \rho^{n+u}$$

for all ρ.

We now estimate the coefficients of P_ε in terms of $S_u(\mu)$. If $P_\varepsilon(x) = \Sigma \, a_\alpha^{(\varepsilon)} \, x^\alpha$, then

$$a_\alpha^{(\varepsilon)} = \frac{1}{\alpha!} \int \varphi_\alpha^{(\varepsilon)} (-y) \, d \mu(y) = \frac{1}{\alpha!} \int \varphi_\varepsilon^{(\alpha)} (-y) \, d \left[\mu(y) - P(y) \lambda(y) \right] +$$

$$+ \frac{1}{\alpha!} \int \varphi_\varepsilon^{(\alpha)} (-y) \, P(y) \, d y .$$

Thus

$$\left| a_\alpha^{(\varepsilon)} \right| \leq \frac{1}{\alpha!} \, C_{\varphi,u} \, \varepsilon^{-n-|\alpha|} \int_{|y| \leq \varepsilon} \mid d \left[\mu - P(y) \lambda \right] \mid + \frac{1}{\alpha!} \left| \left(\frac{\partial}{\partial x} \right)^\alpha P(0) \right| \leq$$

$$\leq C_{\varphi,u} \, \varepsilon^{-n-|\alpha|} \, S_u(u) \, \varepsilon^{n+u} + S_u(\mu) \leq C_{\varphi,u} \, S_u(\mu) .$$

Finally, since

$$\int \mid f_\varepsilon \mid d x \leq C_\varphi \int \mid d \mu \mid \leq C_\varphi \, S_u(\mu) ,$$

collecting results we obtain

$$T_u^1 (f_\varepsilon) \leq C_{\varphi,u} \, S_u(\mu).$$

If $u \leq 0$, we merely have to show that

$$\int\limits_{|x| \leq \rho} | f_\varepsilon(x) | \, dx \leq C_{\varphi, u} \, S_u(\mu) \, \rho^{n+u}.$$

We have

$$\int\limits_{|x| \leq \rho} | f_\varepsilon(x) | \, dx \leq \int\limits_{|x| \leq \rho} \left| \int \varphi_\varepsilon (x-y) \, d\mu(y) \right| \, dx \leq$$

$$\leq \int | d\mu(y) | \int\limits_{|x| \leq \rho} | \varphi_\varepsilon(x-y) | \, dx \qquad [4]$$

If $\rho > \varepsilon$ the last integral is dominated by

$$\int\limits_{|y| \leq 2\rho} | d\mu(y) | \int | \varphi_\varepsilon(x) | \, dx \leq C_\varphi \, S_u(\mu) \, \rho^{n+u}.$$

If, on the other hand, $\rho \leq \varepsilon$ then, since $|\varphi_\varepsilon(x)| \leq C_\varphi \, \varepsilon^{-n}$, the last integral in [4] is dominated by

$$C_\varphi \left(\frac{\rho}{\varepsilon} \right)^n \int\limits_{|y| \leq 2\varepsilon} | d\mu(y) | \leq C_\varphi \, S_u(\mu) \left(\frac{\rho}{\varepsilon} \right)^n (2\,\varepsilon)^{n+u} \leq$$

$$\leq C_{\varphi, u} \, S_u(\mu) \, \rho^{u+n}.$$

This completes the proof of Lemma 1.

3. LEMMA 2. *Let φ be a function in D such that $\int \varphi(x) \, dx = 1$ and let $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi\left(\dfrac{x}{\varepsilon}\right)$, $\varepsilon > 0$.*

(i) *If μ is a countably additive set function such that $T_u^1(x_0, \mu * \varphi_\varepsilon) \leq M$ for all ε, then $\mu \in S_u(x_0)$ and $S_u(x_0, \mu) \leq M$.*

(ii) *If $f \in L^p$, $1 \leq p \leq \infty$, and $T_u^p(x_0, f * \varphi_\varepsilon) \leq M$ for all ε, then $f \in T_u^p(x_0)$ and $T_u^p(x_0, f) \leq M$.*

Proof. Let $P_\varepsilon(x)$ be the Taylor polynomial of $f_\varepsilon = f * \varphi_\varepsilon$, or $f = \mu * \varphi_\varepsilon$, of degree r, where r us the largest integer strictly less than u if $u > 0$, or $P(x) \equiv 0$ if $u \leq 0$. Then our assumptions imply that the coefficients of P_ε are bounded, and we can select a sequence $\varepsilon = \varepsilon_k$ such that $P_{\varepsilon_k}(x)$ converges to a polynomial $P(x)$

uniformly on bounded sets. Let now ρ and δ be positive. Then, assuming $x_0 = 0$, in case (i) we have

$$\int\limits_{|x| \leq \rho} | \, d \, [\mu - P(x) \, \lambda] \, | \leq \overline{\lim_{k \to \infty}} \int\limits_{|x| \leq \rho + \delta} | \, [(\mu - P \, \lambda) * \varphi_{\varepsilon_k}] \, (x) \, | \, dx$$

$$= \overline{\lim_{k \to \infty}} \int\limits_{|x| \leq \rho + \delta} | \, f_{\varepsilon_k} - (P * \varphi_{\varepsilon_k}) \, | \, dx \leq \overline{\lim_{k \to \infty}} \int\limits_{|x| \leq \rho + \delta} | \, f_{\varepsilon_k} - P_{\varepsilon_k} \, | \, dx$$

$$+ \overline{\lim_{k \to \infty}} \int\limits_{|x| \leq \rho + \delta} | \, P_{\varepsilon_k} - (P * \varphi_{\varepsilon_k}) \, | \, dx \, ,$$

and similarly in case (ii) we have

$$\left[\int\limits_{|x| \leq \rho} | \, f(x) - P(x) \, |^p \, dx \right]^{1/p} \leq \overline{\lim_{k \to \infty}} \left[\int\limits_{|x| \leq \rho + \delta} | \, f_{\varepsilon_k} - P_{\varepsilon_k} \, |^p \, dx \right]^{1/p} +$$

$$+ \overline{\lim_{k \to \infty}} \left[\int\limits_{|x| \leq \rho + \delta} | \, P_{\varepsilon_k} - (P * \varphi_{\varepsilon_k}) \, |^p \, dx \right]^{1/p} .$$

Now $P * \varphi_{\varepsilon_k}$ and P_{ε_k} converge to P uniformly on bounded sets and

$$\left[\int\limits_{|x| \leq \rho} | f_\varepsilon - P_\varepsilon |^p \, dx \right]^{1/p} \leq M \, (\rho + \delta)^{\frac{n}{p} + u}, \quad \varepsilon < \delta$$

Thus, passing to the limit we get

$$\int\limits_{|x| \leq \rho} | \, d \, (\mu - P \, \lambda) \, | \leq M \, (\rho + \delta)^{n + u}$$

and

$$\left[\int\limits_{|x| \leq \rho} | f - P |^p \, dx \right]^{1/p} \leq M \, (\rho + \delta)^{\frac{n}{p} + u},$$

in the cases (i) and (ii) respectively. Making $\delta \to 0$ in the preceding inequalities we conclude that $\mu \, \varepsilon \, S_u(x_0)$ and $f \, \varepsilon \, T_u^p(x_0)$.

To prove that $S_u(x_0, \mu) \leq M$ and $T_u^p(x_0, f) \leq M$ we denote by $a_\alpha^{(k)}$ the coefficients of $P_{\varepsilon_k}(x)$ and by a_α those of P. Then, given $\delta > 0$ we can find $\rho_0 > 0$ and $\eta > 0$ such that

$$S_u(x_0, \mu) \leq \int | \, d \, \mu \, | + \Sigma \, | \, a_\alpha \, | + \frac{1}{(\rho_0 + \eta)^{n+u}} \int\limits_{|x_0| \leq \rho_0} | \, d \, (\mu - P\lambda) \, | + \delta$$

in case (i), or

$$T_u^p(x_0, f) \leq \left[\int |f|^p \, dx \right]^{1/p} + \Sigma \, |a_\alpha| +$$

$$+ \frac{1}{\rho_0^{u+n/p}} \left[\int\limits_{|x| \leq \rho_0} |f - P|^p \, dx \right]^{1/p} + \delta$$

in case (ii). Since $\lim |a_\alpha^{(k)}| = |c_\alpha|$, and, in case (i),

$$\underline{\lim} \int |f_{\epsilon_k}| \, dx \geq \int |d\mu|, \qquad \underline{\lim} \int\limits_{|x| \leq \rho_0 + \eta} |f_{\epsilon_k} - P_{\epsilon_k}| \, dx \geq$$

$$\geq \int\limits_{|x| \leq \rho_0} |d(\mu - P\lambda)|$$

$$\int |f_{\epsilon_k}| \, dx + \Sigma \, |c_\alpha^{(k)}| + \frac{1}{(\rho_0 + \eta)^{n+u}} \int\limits_{|x| \leq \rho_0 + \eta} |f_{\epsilon_k} - P_{\epsilon_k}| \, dx \leq M$$

passing to the limit we find that $S_u(x_0, \mu) \leq M + \delta$. Since δ is arbitrary it follows that $S_u(x_0, \mu) \leq M$. Similarly we find that $T_u^p(x_0, f) \leq M$ in the case (ii).

4. The lemma that follows is essentially a result of Gagliardo and Nirenberg (see [3]), but we give a proof which is slightly different from theirs.

LEMMA 3. *Let* μ_j, $j = 1, 2, \ldots, n$, *be countably additive set functions such that, in the sense of distributions,*

$$\frac{\partial \mu_j}{\partial x_i} = \frac{\partial \mu_i}{\partial x_j}$$

for all i *and* j. *Let*

$$k_j(x) = -\frac{1}{\omega_n} \frac{x_j}{|x|^n},$$

where ω_n *is the surface area of the unit sphere* $|x| = 1$. *Let*

$$F = \sum_{j=1}^n k_j * \mu_j.$$

Then $\dfrac{\partial F}{\partial x_j} = \mu_j$, $F \, \varepsilon \, L^{\frac{n}{n-1}}$, *and*

$$\| F \|_{\frac{n}{n-1}} \leq \left[\prod_{j=1}^n \int |d\mu_j| \right]^{1/n}.$$

Proof. Let φ be a non-negative and even (i. e. $\varphi(-x) = \varphi(x)$) function in D. Then, if $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi\left(\dfrac{x}{\varepsilon}\right)$ we set

$$F_\varepsilon(x) = \varphi_\varepsilon * (\Sigma\, \mu_j * k_j) = \sum_{j=1}^{n} (\mu_j * \varphi_\varepsilon) * k_j$$

We will show that

$$\frac{\partial F_\varepsilon}{\partial x_j} = \mu_j * \varphi_\varepsilon \quad, \quad \| F_\varepsilon \|_{\frac{n}{n-1}} \leq \left(\prod_{j=1}^{n} \left\| \frac{\partial F}{\partial x_j} \right\|_1\right)^{1/n},$$

and a passage to the limit will yield the desired result, as we shall see.

First, let us observe that $\mu_j * \varphi_\varepsilon$ tends to 0 at infinity. Further,

$$\frac{\partial F_\varepsilon}{\partial x_i} = \sum_{j=1}^{n} \left[\frac{\partial}{\partial x_i}(\mu_j * \varphi_\varepsilon)\right] * [k_j](x) =$$

$$= \sum_{j=1}^{n} \left[\frac{\partial}{\partial x_j}(\mu_i * \varphi_\varepsilon)\right] * [k_j](x).$$

Integrating by parts the integrals that give the external convolutions over the region between the spheres of radii δ and $1/\delta$ with center at the singularity of k_j, and observing that

$$\Sigma\, \frac{\partial}{\partial x_j} k_j = 0\,,$$

and letting δ tend to 0, we easily see that

$$\frac{\partial F_\varepsilon}{\partial x_j}(x) = (\mu_j * \varphi_\varepsilon)(x).$$

Let now $k_j^N(x) = k_j(x)$ if $|x| \leq N$, $k_j^N(x) = 0$ otherwise, and write

$$F_\varepsilon(x) = \Sigma\left(\frac{\partial F_\varepsilon}{\partial x_j} * k_j^N\right) + \Sigma\, \frac{\partial F_\varepsilon}{\partial x_j} * (k_j - k_j^N)$$

Since $\dfrac{\partial F_\varepsilon}{\partial x_j} = \mu_j * \varphi_\varepsilon$ tends to 0 at infinity, the first sum tends to 0 at infinity; since the $\dfrac{\partial F_\varepsilon}{\partial x_j}$ are integrable, the second sum tends to 0 as $N \longrightarrow \infty$. Consequently F_ε tends to 0 at infinity.

Now we will show that if $G(x)$ is any continuously differentiable function tending to 0 at infinity and having integrable derivatives, then

$$\| G \|_{\frac{n}{n-1}} \leq \left[\frac{n}{\prod_{1}} \left\| \frac{\partial G}{\partial x_j} \right\|_1 \right]^{1/n}$$

This we will do by induction on the number of variables. For $n = 1$ the statement is obvious. Assuming the inequality to hold for functions of $n - 1$ variables we have, on the one hand,

$$\left[\int | G(x_1, x_2, \ldots, x_n) |^{\frac{n-1}{n-2}} \, dx_2 \ldots dx_n \right]^{\frac{n-2}{n-1}} \leq$$

$$\leq \left[\prod_{j=2}^{n} \Phi_j(x_1) \right]^{1/(n-1)}$$

where

$$\Phi_j(x_1) = \int \left| \frac{\partial G}{\partial x_j} (x_1, x_2, \ldots, x_n) \right| dx_2 \ldots dx_n,$$

and

$$| G(x_1, x_2, \ldots, x_n) | \leq \int \left| \frac{\partial G}{\partial x_1} (x_1, x_2, \ldots, x_n) \right| dx_1 = \psi(x_2, \ldots, x_n)$$

on the other. From the last inequality we obtain

$$| G(x_1, x_2, \ldots, x_n) |^{\frac{n}{n-1}} \leq \psi(x_2, \ldots, x_n)^{\frac{1}{n-1}} | G(x_1, x_2, \ldots, x_n) |$$

Integrating with respect to $x_2, \ldots, x_n$ and using Holder's inequality with the pair of conjugate exponents $n - 1$ and $(n - 1)/(n - 2)$ we get

$$\int | G(x_1, x_2, \ldots, x_n) |^{\frac{n}{n-1}} \, dx_2 \ldots, dx_n \leq$$

$$\leq \left[\int \psi(x_2, \ldots, x_n) \, dx_2 \ldots dx_n \right]^{\frac{1}{n-1}}$$

$$\left[\int G(x_1, \ldots, x_n)^{\frac{n-1}{n-2}} dx_2 \ldots dx_n \right]^{\frac{n-2}{n-1}} \leq$$

$$\leq \left\| \frac{\partial G}{\partial x_1} \right\|_1^{\frac{1}{n-1}} \left[\prod_{j=2}^{n} \Phi_j(x_1) \right]^{\frac{1}{n-1}} .$$

Integrating this inequality with respect to x_1 and applying to the integral of the product on the right Holder's inequality with exponents $(n-1)$ we obtain

$$\int |G(x_1, x_2, \ldots, x_n)|^{\frac{n-1}{n}} \, dx_1 \, dx_2 \ldots dx_n \leq$$

$$\leq \left\| \frac{\partial G}{\partial x_1} \right\|_1^{\frac{1}{n-1}} \prod_{j=2}^{n} \left[\int \Phi_j(x_1) \, dx_1 \right]^{\frac{1}{n-1}} = \left(\prod_{j=1}^{n} \left\| \frac{\partial G}{\partial x_j} \right\|_1 \right)^{\frac{1}{n-1}},$$

which is the desired result.

Returning to our function F_ε we thus have

$$|| F_\varepsilon ||_{\frac{n}{n-1}} \leq \left(\prod_{1}^{n} \left\| \frac{\partial F_\varepsilon}{\partial x_j} \right\|_1 \right)^{\frac{1}{n}} = \left(\prod_{1}^{n} || \mu_j * \varphi_\varepsilon ||_1 \right)^{\frac{1}{n}} \leq$$

$$\leq \left(\prod_{1}^{n} \int | \, d\mu_j | \right)^{\frac{1}{n}},$$

that is, $|| F_\varepsilon ||_{\frac{n}{n-1}}$ is bounded. By restricting ε to an appropriate sequence tending to zero F_ε will converge weakly to a limit F in $L^{\frac{n}{n-1}}$ (we exclude here the case $n = 1$ where, to begin with, the theorem is obvious), for which also

$$|| F ||_{\frac{n}{n-1}} \leq \left[\prod_{1}^{n} \int | \, d\mu_j | \right]^{\frac{1}{n}}.$$

Let now ψ be an arbitrary function from D. Then, since φ_ε is even,

$$\int F_\varepsilon \frac{\partial \psi}{\partial x_j} \, dx = -\int \frac{\partial F_\varepsilon}{\partial x_j} \psi \, dx =$$

$$- \int (\mu_j * \varphi_\varepsilon) \, \psi \, dx = -\int (\varphi_\varepsilon * \psi) \, d\mu_j,$$

and letting ε tend to 0 we obtain

$$\int F \frac{\partial \psi}{\partial x_j} \, dx = -\int \psi \, d\mu_j,$$

which shows that $\dfrac{\partial F}{\partial x_j} = \mu_j$. Furthermore,

$$\int F_\varepsilon \psi \, dx = \int \psi \left[\varphi_\varepsilon * \sum (\mu_j * k_j) \right] dx = \int (\psi * \varphi_\varepsilon) \left(\sum \mu_j * k_j \right) dx,$$

and passing to the limit again we find that

$$\int F \, \psi \, dx = \int \psi \, (\Sigma \, \mu_j * k_j) \, dx \, ,$$

whence $F = \Sigma \, \mu_j * k_j$. This completes the proof of the lemma.

5. We can now prove parts (i) and (ii) of the theorem.

Let f be locally integrable and let $\dfrac{\partial f}{\partial x_j} = \mu_j$ in the sense of distributions, where the μ_j are countably additive set functions. Then $\dfrac{\partial \mu_i}{\partial x_j} = \dfrac{\partial \mu_j}{\partial x_i}$ and according to Lemma 3 the function

$$F = \Sigma \, \mu_j * k_j$$

has the property that $\dfrac{\partial F}{\partial x_j} = \mu_j$ and $F \, \varepsilon \, L^{\frac{n}{n-1}}$. Consequently $\dfrac{\partial}{\partial x_j} (f - F) = 0$ for $j = 1, 2, \ldots, n$, and $f - F$ is (essentially) a constant. This establishes part (i) of the theorem.

Suppose now that $\mu_j \, \varepsilon \, S_u(x_0)$ and let φ be the function of Lemma 1. Set $F_\varepsilon = F * \varphi_\varepsilon$. Then $\dfrac{\partial F_\varepsilon}{\partial x_j} = \mu_j * \varphi_\varepsilon$ and, according to Lemma 1,

$$T_u{}^1 \left(x_0 \, , \, \frac{\partial F_\varepsilon}{\partial x_j} \right) = T_u{}^1 \, (x_0, \mu_j * \varphi_\varepsilon) \leq C_{\varphi, \, u} \, S_u \, (x_0 \, , \mu_j) \, .$$

Now Theorem 11 of [1] asserts that

$$T^{\frac{n}{n-1}}_{u+1} \, (x_0 \, , F_\varepsilon) \leq C \sum_{j=1}^{n} T_u{}^1 \left(x_0, \frac{\partial F_\varepsilon}{\partial x_j} \right) \leq C_{\varphi, \, u} \sum_{j=1}^{n} S_u \, (x_0 \, , \mu_j) \, ,$$

and from Lemma 2 it follows that

$$T^{\frac{n}{n-1}}_{u+1} \, (x_0 \, , F) \leq C_{\varphi, \, u} \sum_{j=1}^{n} S_u \, (x_0 \, , \mu_j) \, ,$$

which proves part (ii) of the theore$m$.

6. For the proof of part (iii) we need two more lemmas.

LEMMA 4. *Let*

$$\left(\frac{\partial}{\partial x} \right)^\alpha f = \mu \, \varepsilon \, s_u \, (x_0) \, , \quad \left(\frac{\partial}{\partial x} \right)^\beta f = \nu \, \varepsilon \, s_v \, (x_0) \, ,$$

where the derivatives are taken in the sense of distributions, μ and ν are countably additive functions and $|\alpha| + [u] = |\beta| + [v]$. Let P and Q be the Taylor polynomials associated with μ and ν respectively. Then

$$\left(\frac{\partial}{\partial x}\right)^{\beta} P = \left(\frac{\partial}{\partial x}\right)^{\alpha} Q.$$

Proof. Let φ be a function from D with support in $|x| \leq 1$ and with integral equal to 1, and assume for simplicity that $x_0 = 0$. Then

$$\int f(x) \left(\frac{\partial}{\partial x}\right)^{\alpha+\beta+\gamma} \varphi_\varepsilon(-x)\, dx = (-1)^{|\alpha|} \int \left(\frac{\partial}{\partial x}\right)^{\beta+\gamma} \varphi_\varepsilon(-x)\, d\mu =$$

$$= (-1)^{|\alpha|} \int \left(\frac{\partial}{\partial x}\right)^{\beta+\gamma} \varphi_\varepsilon(-x)\, d[\mu - P(x)\lambda] +$$

$$+ (-1)^{|\alpha|} \int P(x) \left(\frac{\partial}{\partial x}\right)^{\beta+\gamma} \varphi_\varepsilon(-x)\, dx =$$

$$= (-1)^{|\alpha|} \int \left(\frac{\partial}{\partial x}\right)^{\beta+\gamma} \varphi_\varepsilon(-x)\, d[\mu - P(x)\lambda] +$$

$$+ (-1)^{|\alpha+\beta+\gamma|} \int \varphi_\varepsilon(-x) \left(\frac{\partial}{\partial x}\right)^{\beta+\gamma} P\, dx =$$

$$= (-1)^{|\alpha|} \int \left(\frac{\partial}{\partial x}\right)^{\beta+\gamma} \varphi_\varepsilon(-x)\, d[\mu - P(x)\lambda] +$$

$$+ (-1)^{|\alpha+\beta+\gamma|} \left(\frac{\partial}{\partial x}\right)^{\beta+\gamma} P(0) + o(1)$$

as $\varepsilon \to 0$. Since $\varphi_\varepsilon(-x)$ is supported by $|x| \leq \varepsilon$ and

$$\left| \left(\frac{\partial}{\partial x}\right)^{\beta+\gamma} \varphi_\varepsilon(-x) \right| \leq C\, \varepsilon^{-n-|\beta|-|\gamma|},$$

integral in the last expression is dominated by

$$C\, \varepsilon^{-n-|\beta|-|\gamma|} \int\limits_{|x| \leq \varepsilon} |\, d[\mu - P(x)\lambda]\,| =$$

$$= C\, \varepsilon^{-n-|\beta|-|\gamma|}\, o\,(\varepsilon^{n+u}) = o\,(1),$$

provided $|\beta| + |\gamma| \leq [u]$, or $|\gamma| \leq [u] - |\beta|$. Thus

$$(-1)^{|\alpha+\beta+\gamma|} \int f(x) \left(\frac{\partial}{\partial x}\right)^{\alpha+\beta+\gamma} \varphi_\varepsilon(-x)\, dx = \left(\frac{\partial}{\partial x}\right)^{\beta+\gamma} P(0) + o(1),$$

and similarly

$$(-1)^{|\alpha + \beta + \gamma|} \int f(x) \left(\frac{\partial}{\partial x}\right)^{\alpha + \beta + \gamma} \varphi_\varepsilon(-x) \cdot dx = \left(\frac{\partial}{\partial x}\right)^{\alpha + \gamma} Q(0) + o(1),$$

provided $|\gamma| \leq [v] - |\alpha|$. Consequently we have

$$\left(\frac{\partial}{\partial x}\right)^\gamma \left[\left(\frac{\partial}{\partial x}\right)^\beta P - \left(\frac{\partial}{\partial x}\right)^\alpha Q\right](0) = 0$$

for $|\gamma|$ not exceeding the number $[u] - |\beta| = [v] - |\alpha|$, which is not less than the degree of both

$$\left(\frac{\partial}{\partial x}\right)^\beta P \quad \text{and} \quad \left(\frac{\partial}{\partial x}\right)^\alpha Q,$$

and the lemma follows.

7. LEMMA. *Let f be a continuously differentiable function and let*

$$A(\rho) = \int\limits_{|x| \leq \rho} \left[\sum_{j=1}^n \left|\frac{\partial f}{\partial x_j}\right|\right] dx$$

(i) *If $f(0) = 0$, then*

$$\int\limits_{|x| \leq \rho} |f(x)| \, dx \leq \rho \, \frac{A(\rho)}{n} + \rho^n \int_0^\rho \frac{A(t)}{t^n} \, dt.$$

(ii) *If $f(x) = 0$ for $|x| \geq 1$, then for $0 \leq \rho \leq 1$ we have*

$$\int\limits_{|x| \leq \rho} |f(x)| \, dx \leq \frac{n-1}{n} \, \rho^n \int_\rho^1 \frac{A(t)}{t^n} \, dt + \frac{A(1)}{n} \, \rho^n.$$

Proof. Let $x' = \dfrac{x}{|x|}$. Then, if $f(0) = 0$,

$$|f(x)| \leq \int_0^{|x|} \sum \left|\frac{\partial f}{\partial x_j}\right| (t x') \, dt,$$

$$\int\limits_{|x| \leq \rho} |f(x)| \, dx \leq \int\limits_{|x| \leq \rho} dx \int_0^{|x|} \sum \left|\frac{\partial f}{\partial x_j}\right| (t x') \, dt =$$

$$= \int dx' \int_0^\rho s^{n-1} \, ds \int_0^s \sum \left|\frac{\partial f}{\partial x_j}\right| (t x') \, dt,$$

where $d\,x'$ is the surface area element of $|\,x\,| = 1$. Changing the order of integration we obtain

$$\int\limits_{|x| \leq \varrho} |\,f\,(x)\,|\,d\,x \leq \int_0^\varrho \frac{\varrho^n - t^n}{n}\,d\,t \int \Sigma \left| \frac{\partial f}{\partial x_j} \right| (t\,x')\,d\,x' =$$

$$= \int_0^\varrho \frac{\varrho^n - t^n}{n}\,\frac{d\,A\,(t)}{t^{n-1}} \leq \frac{\varrho^n}{n} \int_0^\varrho \frac{d\,A\,(t)}{t^{n-1}} \leq$$

$$\leq \frac{\varrho\,A\,(\varrho)}{n} + \varrho^n \int_0^\varrho \frac{A\,(t)}{t^n}\,d\,t\,.$$

If $f\,(x) = 0$ for $|\,x\,| \geq 1$ we have for $|\,x\,| \leq 1$:

$$|\,f\,(x)\,| \leq \int_{|x|}^1 \Sigma \left| \frac{\partial f}{\partial x_j} \right| (t\,x')\,d\,t =$$

$$= \int\limits_{|x| \leq \varrho} |\,f\,(x)\,|\,d\,x \leq \int\limits_{|x| \leq \varrho} d\,x \int_{|x|}^1 \Sigma \left| \frac{\partial f}{\partial x_j} \right| (t\,x')\,d\,t =$$

$$= \int d\,x' \int_0^\varrho s^{n-1}\,d\,s \int_s^1 \Sigma \left| \frac{\partial f}{\partial x_j} \right| (t\,x')\,d\,t,$$

and changing the order of integration we get

$$\int\limits_{|x| \leq \varrho} |\,f\,(x)\,|\,d\,x \leq \int_0^\varrho \frac{t^n}{n}\,d\,t \int \Sigma \left| \frac{\partial f}{\partial x_j} \right| (t\,x')\,d\,x' +$$

$$+ \frac{\varrho^n}{n} \int_\varrho^1 d\,t \int \Sigma \left| \frac{\partial f}{\partial x_j} \right| (t\,x')\,d\,x' =$$

$$= \int_0^\varrho \frac{t}{n}\,d\,A\,(t) + \frac{\varrho^n}{n} \int_\varrho^1 \frac{d\,A\,(t)}{t^{n-1}} \leq$$

$$\leq \frac{n-1}{n}\,\varrho^n \int_\varrho^1 \frac{A\,(t)\,d\,t}{t^n} + \frac{A\,(1)}{n}\,\varrho^n\,.$$

This completes the proof of the lemma.

8. We will now prove part (iii) of the theorem, first considering the case $u > -1$. Since $\mu_j \,\varepsilon\, s_u(x_0) \subset S_u(x_0)$, part (i) of the theorem asserts that $f - a \,\varepsilon\, T^{\frac{n}{n-1}}_{u+1} (x_0)$ for a suitable a, and thus there

exists a polynomial P of degree stricly less than $u + 1$ such that

$$\left[\frac{1}{\varrho^n} \int\limits_{|x| \le \varrho} | f(x) - a - P(x) |^{\frac{n}{n-1}} d x \right]^{\frac{n-1}{n}} \le C \, \varrho^{u+1}$$

Let Q_j be the Taylor polynomial of μ_j. According to Lemma 4 we have $\dfrac{\partial Q_j}{\partial x_i}(x) = \dfrac{\partial Q_i}{\partial x_j}(x)$, and therefore there exists a polynomial Q, $Q(0) = 0$, such that $Q_j = \dfrac{\partial Q}{\partial x_j}$. We will prove that

$$\left[\frac{1}{\varrho^n} \int\limits_{|x| \le \varrho} | f(x) - a - P(0) - Q(x) |^{\frac{n}{n-1}} d x \right]^{\frac{n-1}{n}} = o\, (\varrho^{u+1})$$
$$(\varrho \rightarrow 0)$$

and this will show that $f \in t^{\frac{n}{n-1}}_{u+1}(x_0)$. For this purpose it will be enough to show that

$$\frac{1}{\varrho^n} \int\limits_{|x| \le \varrho} | f(x) - a - P(0) - Q(x) | \, d x = o\, (\varrho^{u+1})$$

for if $\psi(x)$ is function in D such that $\psi(x) = 1$ for $|x| \le 1$ and $\psi(x) = 0$ for $|x| \ge 2$, and if we write

$$\psi_\varepsilon(x) = \psi\left(\frac{x}{\varepsilon} \right),$$

then, setting $\overline{f}(x) = f(x) - a - P(0) - Q(x)$ we have

$$\left[\frac{1}{\varrho^n} \int\limits_{|x| \le \varrho} | \overline{f}(x) |^{\frac{n}{n-1}} d x \right]^{\frac{n-1}{n}} \le \frac{1}{\varrho^{n-1}} \left[\int | \overline{f}(x) \, \psi_\varrho(x) |^{\frac{n}{n-1}} d x \right]^{\frac{n-1}{n}}$$

and, by Lemma 3, the right-hand side does not exceed

$$\frac{1}{\varrho^{n-1}} \left[\prod_1^n \left\| \frac{\partial}{\partial x_j} (\overline{f}\, \psi_\varrho) \right\|_1 \right]^{\frac{1}{n}} \le \frac{1}{\varrho^{n-1}} \frac{1}{n} \sum_1^n \left\| \frac{\partial}{\partial x_j} (\overline{f}\, \psi_\varrho) \right\|_1 \le$$

$$\le \frac{1}{\varrho^{n-1}} \frac{1}{n} \sum \left\| \psi_\varrho \frac{\partial \overline{f}}{\partial x_j} \right\|_1 + \frac{1}{\varrho^{n-1}} \frac{1}{n} \sum \left\| \overline{f} \frac{\partial \psi_\varrho}{\partial x_j} \right\|_1,$$

where, of course, the derivatives are meant as measures. Since ψ_ϱ is bounded and vanishes for $|x| \ge 2 \, \varrho$, the terms of the first sum

on the right are dominated by

$$\frac{C}{\varrho^{n-1}} \int\limits_{|x| \leq 2\varrho} \left| \frac{\partial \bar{f}}{\partial x_j} \right| d x = \frac{C}{\varrho^{n-1}} \int\limits_{|x| \leq 2\varrho} | d [\mu_j - Q_j \lambda] | = o\left(\varrho^{u+1}\right).$$

Similarly, since $\left| \frac{\partial}{\partial x_j} \psi_\varrho(x) \right| \leq \frac{C}{\varrho}$, we have

$$\frac{1}{\varrho^{n-1}} \left\| \bar{f} \frac{\partial \psi_\varrho}{\partial x_j} \right\|_1 \leq \frac{C}{\varrho^n} \cdot \int\limits_{|x| \leq 2\varrho} | \bar{f}(x) | d x.$$

It is therefore enough to show that

$$\frac{1}{\varrho^n} \int\limits_{|x| \leq \varrho} | \bar{f}(x) | d x = o\left(\varrho^{u+1}\right).$$

Let now φ be a function like the one in Lemma 1 with the property that $\varphi_\varepsilon * P = P$ for all polynomials of degree not exceeding $u + 1$, and write $\bar{f}_\varepsilon = \bar{f} * \varphi_\varepsilon$. Then $| \varphi_\varepsilon(x) | \leq \frac{C}{\varepsilon^n}$ and

$$| \bar{f}_\varepsilon(0) | = \left| \int \bar{f}(x) \varphi_\varepsilon(-x) d x \right| \leq \frac{C}{\varepsilon^n} \int\limits_{|x| \leq \varepsilon} | \bar{f}(x) | d x.$$

Now, according to part (ii) of the theorem, $f - a \ \varepsilon \ T_{u+1}^{\frac{n}{n-1}}(x_0)$ and thus

$$\frac{C}{\varepsilon^n} \int\limits_{|x| \leq \varepsilon} | \bar{f}(x) | d x \leq C \left[\frac{1}{\varepsilon^n} \int\limits_{|x| \leq \varepsilon} | \bar{f}(x) |^{\frac{n}{n-1}} d x \right]^{\frac{n-1}{n}} =$$

$$= O \left\{ \varepsilon^{\min (1,\, u+1)} \right\} = o(1)$$

from which we conclude that $\bar{f}_\varepsilon(0) \longrightarrow 0$ with ε.

Using now part (i) of Lemma 5 we find that

$$\frac{1}{\varrho^n} \int\limits_{|x| \leq \varrho} | \bar{f}_\varepsilon(x) - \bar{f}_\varepsilon(0) | d x \leq \frac{1}{n\, \varrho^{n-1}} A_\varepsilon(\varrho) + \int_0^\varrho \frac{A_\varepsilon(t)}{t^n} d t, \quad [5]$$

where, according to Lemma 1,

$$A_\varepsilon(t) = \int\limits_{|x| \leq t} \sum_{j=1}^{n} \left| \frac{\partial \overline{f}_\varepsilon}{\partial x_j} \right| d x = \int\limits_{|x| \leq t} \sum_{j=1}^{n} |(\mu_j * \varphi_\varepsilon) - Q_j(x)| d x \leq$$

$$\leq t^{n+u} \sum_{j=1}^{n} T_u^1(x_0, \mu_j * \varphi_\varepsilon) \leq C_{\varphi,u} \, t^{n+u} \sum_{j=1}^{n} S_u(x_0, \mu_j).$$

If we now let here ε tend to 0, since $\overline{f}_\varepsilon(0) \rightarrow 0$ we have

$$\int\limits_{|x| \leq \rho} |\overline{f}_\varepsilon(x) - \overline{f}_\varepsilon(0)| d x \longrightarrow \int\limits_{|x| \leq \rho} |\overline{f}(x)| d x .$$

On the other hand, if we set

$$A(t) = \sum_{j=1}^{n} \int\limits_{|x| \leq t} |d [\mu_j - Q_j(x) \lambda]|$$

we have $A_\varepsilon(t) \longrightarrow A(t)$ at all points of continuity of $A(t)$, and since $A_\varepsilon(t) \leq C \, t^{n+u}$ we can pass to the limit in [5] getting

$$\frac{1}{\rho^n} \int\limits_{|x| \leq \rho} |\overline{f}(x)| d x \leq \frac{1}{n \, \rho^{n-1}} A(\rho) + \int_0^\rho \frac{A(t)}{t^n} d t$$

Since $\mu_j \varepsilon \, s_u(x_0)$ we have $A(t) = o(t^{n+u})$ as $t \longrightarrow 0$, and the preceding inequality implies that

$$\frac{1}{\rho^n} \int\limits_{|x| \leq \rho} |\overline{f}(x)| d x = o(\rho^{u+1}) .$$

This completes the proof of part (iii) of the theorem in the case $u > -1$.

9. If $-n \leq u < -1$ we may assume without loss of generality that $f(x) = 0$ in $|x| > \dfrac{1}{2}$ and argue as above setting $\overline{f}(x) = = f(x)$ and $\overline{f}_\varepsilon = \overline{f} * \varphi_\varepsilon$ and using the second inequality of Lemma 5 instead of the first. In this way we get

$$\frac{1}{\rho^n} \int\limits_{|x| \leq \rho} |\overline{f}_\varepsilon(x)| d x \leq \frac{n-1}{n} \int_\rho^1 \frac{A_\varepsilon(t)}{t^n} d t + \frac{A_\varepsilon(1)}{n}$$

instead of [5], and a passage to the limit gives

$$\frac{1}{\varrho^n} \int\limits_{|x| \leq \varrho} |\bar{f}(x)| \, dx = \frac{1}{\varrho^n} \int |f(x)| \, dx =$$

$$= \frac{n-1}{n} \int_\varrho^1 \frac{A(t)}{t^n} \, dt + O(1).$$

Since $A(t) = o(t^{n+u})$ and $u < -1$, this implies that

$$\frac{1}{\varrho^n} \int\limits_{|x| \leq \varrho} |f(x)| \, dx = o(\varrho^{u+1}) + O(1) = O(\varrho^{u+1})$$

as $\varrho \rightarrow 0$ and completes the proof of part (iii) of the theorem.

Remarks. a) Since the proof of part (iii) of the theorem is rather long, it may be worth noting that in the case when u is a non-negative integer (and in particular when $u = 0$, which is the most interesting special case) part (iii) is almost everywhere a consequence of part (ii), on account of the known fact (see [1]) that if $f \varepsilon T_u^q(x)$, $1 \leq q \leq \infty$, $u = 0, 1, 2, \ldots$, for all x in a set E of positive measure, then $f \varepsilon t_u^q(x)$ for almost every x in E.

b) The theorem of this paper admits of an extension to the case when all derivatives of a given order are measures while the derivatives of lower orders are functions. This extension is however an immediate consequence of our theorem and Theorem 11 of [1] and need not be stated here explicitly.

BIBLIOGRAPHY

[1] A. P. Calderón and A. Zygmund, Local properties of solutions of elliptic partial differential equations. To appear in *Studia Mathematica.*

[2] K. Krickeberg, Distributionen, Funktionen beschränkter Variation und Lebesguescher Inhalt nichtparametrische Flächen, *Annali di Matematica* (IV) 44, 1957, pp. 105-134.

[3] L. Nirenberg, On elliptic partial differential equations, *Annali della Scuola Normale Superiore di Pisa*, vol. 13, pp. 116-162 (1959).