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ON OPERATIONS OF CONVOLUTION TYPE AND ORTHONORMAL SYSTEMS ON COMPACT ABELIAN GROUPS

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INTRODUCTION. This paper is divided into three sections: the present one (which contains the motivation of the others) and the following parts I and II. Part II is devoted to the study of certain Banach algebras, to which one is naturally led when trying to solve the problem of introducing a convolution in a general finite measure space. Part I deals with necessary and sufficient conditions on an orthonormal system of measurable functions on a compact abelian group G, to be the image of the character group $G^\wedge$, under measurable transformations on the original group G.

Preliminary results. 1. Given two finite measure spaces (X_i, Σ_i, μ_i), $i = 1, 2$, $(\mu_i\ (X_i) < \infty)$, we say that they are B-isomorphic if there exists a σ-isomorphism between the Boolean algebras Σ_1/N_1 and Σ_2/N_2, where N_i denotes the sets of measure zero of Σ_i. If besides, the σ-isomorphism between the measure algebras Σ_i/N_i preserves measure we say that the spaces are m-isomorphic. Following D. Maharam's paper ([8]) we call a finite measure algebra *homogeneous* if any two principal ideals admit minimal σ-basis of the same power. The result of Maharam which interests us is the following: a) A finite measure space (X, Σ, μ), is m-isomorphic to the measure theoretic union of a denumerable set of spaces of the form $P_n = (\prod_{1 \leq i < \gamma_n} [0,1)_i, \Sigma_n, k_n. \mu_n)$ and a purely atomic space. The γ_n are infinite ordinal numbers (and the leading ordinals of their cardinal classes) and verify $\gamma_n <_{n+1}$; Σ_n represents the σ-field of Baire sets of the compact groups $\prod_{1 \leq i < \gamma n} [0,1)_i$; μ_n is the normalized Haar measure and k_n a real number such that $0 \leq k_n \leq 1$. Given

(X, Σ, μ) the γ_n and k_n are uniquely determined. b) If (X, Σ, μ) is homogeneous (and non atomic) the Maharam's representation is reduced to only one product of unit circles.

2. If we try to introduce a convolution type operation on a finite measure space (X, Σ, μ) we can do it as follows. Supposing that our space has no atoms, it is m-isomorphic to $\cup P_n$, and therefore there is induced a natural isomorphism τ between both L^1-spaces. Since each P_n is a compact group it has a convolution defined in the ordinary way. Representing by χ characteristic functions, we define $f^* g$, for $f, g \in L^1 (X, \Sigma, \mu)$ as:

$$f\,(^*)\; g = \tau^{-1} \left(\sum_n \left(\tau(f)\, \chi_{P_n} * \tau(g)\, \chi_{P_n} \right) \right) \tag{1}$$

If (X, Σ, μ) also has atoms no problem arises because they are easier to manage.

This suggests to study locally compact spaces obtained as union of locally compact abelian groups and to define convolution type operations in analogous fashion as (1). We do this in part II and we see that, as it might be expected, these spaces admit a formal treatment like a common locally compact abelian group. However the Bochner theorem splits into two parts and is the main difference with the theory developed in [1] or [7].

3. Since an infinite product of unit circles, from the measure theoretic point of view, can be replaced by product of a set with the same power of copies of the two-element group, we see that there are several ways of introducing a convolution operation like (1). To see how they are related we can restrict ourselves to study the same problem for a fixed compact abelian group.

The first observation we need is contained in the next lemma.

Lemma 1. Any compact non-finite group is m-isomorphic to a product of unit circles.

Proof. It is necessary to prove that a compact group G is homogeneous. Given two homogeneous sets A and B of positive measure, contained in G, there exists a point x such that $x A \cap B$ has positive measure ([2], p. 261). Therefore, A, B and $x A \cap B$ have the same type of homogeneity, QED.

(As it is well-known (cf. [9]) a compact group is isomorphic to a product of unit circles, $[0,1)_i$, $1 \leqq i < \gamma$, if and only if, its dual group is isomorphic to the direct sum of L_i, $1 \leqq i < \gamma$, where each L_i represents the set of integers. Therefore, a product of unit circles is characterized as the dual group of a free group. The real line with the discret topology is not a free group, and therefore its dual group provides us of an example of a non-finite compact group not a product of circles).

Lemma 1 and Maharam's theorem show that if we have a measure space (which, for the sake of brevity, we shall suppose without atoms) (X, Σ, μ), which is m-isomorphic to the spaces $\bigcup_n P_n$, $\bigcup_m Q_m$, P_n, Q_m, compact abelian groups, then the study of the relationship between the convolution operation defined by (1) and

$$f \, (*') \, g = \tau'^{-1} \, (\sum_m \, (\tau' (f) \, \chi_{Q_m} * \tau' (g) \, \chi_{Q_m})) \qquad (2)$$

is reduced to the study of the relationship between the convolutions of two compact spaces of the same homogeneity. (We have supposed that two different Q_m have different homogeneity types).

4. *Lemma 2. Two compact (non-finite) abelian groups* (1) *are m-isomorphic if and only if their duals groups are of the same power.*

Proof. By lemma 1 it is sufficient to prove that any compact non-finite commutative group G, is m-isomorphic to a product P of as many copies of the unit circle as is the power of $G^\wedge$.

(It is immediate that the power of the set of unit circles taken into consideration is the same as that of the character group $P^\wedge$). Since a m-isomorphism preserves orthonormal complete systems of functions in L^2, the power of $P^\wedge$ is the same as that of $G^\wedge$, QED.

The same argument proves also the following extension:

Two compact (commutative or not) groups are m-isomorphic if and only if their families of sets of equivalent, irreducible, unitary matrix representations have the same power, (cf. § 32 and § 33 of [9]).

5. Given a compact abelian group G, let (e_i) be its character system. For two functions f and g of $L^2(G)$ with Fourier series: $f = \sum c_i \, e_i$, $g = \sum d_i \, e_i$, we have:

$$f \, (*) \, g = \sum c_i \, d_i \, e_i \qquad (3)$$

Formula (3) permits to define an operation of convolution type with any complete orthonormal system of functions of L^2. Among

(1) For compact groups we always suppose that they have been provided with the normalized Haar measure.

these systems there are some which can be considered as the image under an m-isomorphism of the character system of certain compact groups. An orthonormal system (η_i) of this type must verify certain conditions, for example, its functions must be uniformly bounded and constitute a multiplicative group, *i. e.*, for any j and k, $|\eta_j| \leq 1$ *a. e.*, $\eta_j \cdot \eta_k = \eta_e$ *a. e.*, and so on.

The interest of these particular systems may be justified as follows. Let $f, g \in L^2(G)$ and $f = \Sigma\, c_i\, \eta_i$, $g = \Sigma\, d_i\, \eta_i$ and consider the convolution defined by (3):

$$f\ (^*) = \Sigma\, c_i\, d_i\, \eta_i$$

and that defined by (1):

$$f\ (^*)\ g = \tau^{-1}(\tau(f) * \tau\ (g)),$$

where (η_i) is the image of (e_i) under the m-isomorphism τ^{-1} from the group F onto the group G, and where the convolution in the second member of the last equality must be understood in the usual sense. It is easy to see that both definitions provide the same function (*a.e.*). In other words, with these particular systems, (3) defines a convolution which is "essential" in the sense that, except by a m-isomorphism, it is the convolution on a certain commutative compact group.

6. It arises naturally the question, what are the conditions which must satisfy an orthonormal complete system to be the m-isomorphic image of the character system of a certain group. This question receives several answers in Part I. Now we consider an example of this situation. Let G be the unit interval [0,1), with the operation of sum (mod. 1), and F the product of countable many copies of the two-element group. It is well-known that there exists a measure preserving transformation of F onto G constructed with the dyadic intervals. But this is exactly the transformation which sends the family of characters of F onto the Walsh system of the interval, (cf. [10], p. 34, Ex. 6). We leave the easy verification to the reader.

Part I

Almost everywhere multiplicative systems. 1. Let G and F be compact abelian groups and $G^\wedge$, $F^\wedge$, their dual groups. It is well-known that:

Theorem 1. G and F are isomorphic if and only if $G^\wedge$ is isomorphic to $F^\wedge$.

Our purpose in this section is to extend theorem 1 to other situations. What theorem 1 says is that if there exist an isomorphism between the character systems $G^\wedge = (\epsilon_i)$ and $F^\wedge = (\eta_i)$, then there exists an isomorphism $T\colon F \to G$, of F onto G, such that: $\eta_i(y) = \epsilon_i(Ty)$.

Theorem 2. Let $F^\sim = (\eta_i(y))$ be a complete o.n. system of functions of $L^2(F)$, which is under the multiplication a.e. (2) *an isomorphic group to $G^\wedge = (\epsilon_i(x))$. Then, there exists an m-isomorphism between F and G such that $F^\sim$ is the image of $G^\wedge$. The converse is obviously true* (3).

Proof. Consider the unitary operator U defined by the correspondence $\epsilon_i \to \eta_i$, given by hypothesis. We see next that for any $f \in L^2(G)$ and $g \in L^\infty(G)$, *it holds*

$$U(fg) = U(f) . U(g) \qquad a.e., \tag{1}$$

Since, $U(\epsilon_i\, \epsilon_j) = U(\epsilon_{ji}) = \eta_{ij} = \eta_i . \eta_j = U(\epsilon_i) . U(\epsilon_j)$, we have:

$$U\left(\left(\sum_1^N (f, \epsilon_i)\, \epsilon_i\right) \left(\sum_1^M (g, \epsilon_j)\, \epsilon_j\right)\right) = U\left(\sum_1^N (f, \epsilon_i)\, \epsilon_i\right) . U\left(\sum_1^M (g, \epsilon_j)\, \epsilon_j\right). \tag{2}$$

If $M \to \infty$, we obtain:

$$U\left(\left(\sum_1^N (f, \epsilon_i)\, \epsilon_i\right) g\right) = U\left(\sum_1^N (f, \epsilon_i)\, \epsilon_i\right) . U(g), \tag{3}$$

If in (3) we make $N \to \infty$, the first member tends in $L^2(F)$ to $U(fg)$, and the second in $L^1(F)$ to $U(f). U(g)$.

Therefore: $U(fg) = U(f)\, U(g)$, a.e. .

The theorem follows now from the next theorem 3, which is essentially Von Neumann's multiplication theorem, (cf. [3], [4] and the crossed references there mentioned).

Theorem 3. Let (X, Σ, μ) and (Y, Φ, ν) be probability spaces. If U is a unitary operator from $L^2(X)$ onto $L^2(Y)$ which verifies

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(2) Given η_1 and η_2 of $F^\sim$, $\eta_1 . \eta_2$ is equal a.e. to an element of $F^\sim$ and there exists η_3 such that $\eta_1 \eta_3 = 1$ a.e.

(3) An m-isomorphism gives a correspondence between classes of functions and - without mentioning it every time - we pick out a representative function when it is necessary.

$$U(fg) = U(f) \cdot U(g) \quad a.\,e.,$$

for any $f \in L^2(X)$, $g \in L^\infty(X)$, *then* U *is induced by an m-isomorphism.*

Proof. If χ is a characteristic function, since $U(\chi)$ is finite a. e., we have: $(U(\chi))^2 = U(\chi)$, and therefore $U(\chi)$ is a characteristic function (a. e.). Besides, χ and $U(\chi)$ define sets of the same measure, and U determines a measure preserving mapping of Σ/N_Σ into Φ/N_Φ. From:

$$U(\chi_1 + \chi_2 - 2\chi_1\chi_2) = U(\chi_1) + U(\chi_2) - 2\,U(\chi_1) \cdot U(\chi_2),$$

we see that, $U(\chi_1) = U(\chi_2)$ a. e. if and only if $\chi_1 = \chi_2$ a. e., and the mapping is one-to-one. The continuity of the operator U implies that it is a σ-isomorphism. It is also onto. In fact, it is necessary to prove that if $U(h) = \chi$, then h is a characteristic function. Let $h_n(x) = h(x)$ if $|h(x)| \leqq n$, and $= 0$ if $|h(x)| > n$. Then,

$$U(h) \cdot U(h_n) = U(hh_n) = \chi \cdot U(h_n), \text{ and}$$

$$U^{-1}(\chi \cdot U(h_n)) = h \cdot h_n .$$

Since $U(h_n) \to \chi$ and $h_n \to h$ we have: $\lim U^{-1}(\chi \cdot U(h_n)) = U^{-1}(\chi)$. Besides, hh_n tends (pointwise) to h^2. Then,

$$U^{-1}(\chi) = h = \lim U^{-1}(\chi \cdot U(h_n)) = h^2,$$

and h is a characteristic function.

This proves theorem 3 (and 2).

2. This paragraph deals with several generalizations of theorem 2. To find the right conditions to be imposed to the η-system, we make some observations. a) If T is a measure preserving transformation from F into G, (both compact abelian groups), it may happen that $T(F)$ is not a mesasurable set, however, it is a thick subset of G, i.e., $\mu*(G - T(F)) = 0$. Besides the functions $\eta_i(y) = \epsilon_i(Ty)$, $(\epsilon_i) = G^\wedge$, are measurable functions and form a multiplicative (a.e.) group isomorphic to the ϵ-system (because $T(F)$ being thick, is dense in G). Since:

$$\int \eta_i(y)\,\overline{\eta_j(y)}\,d\nu = \int \epsilon_i(Ty)\,\overline{\epsilon_j(Ty)}\,d\nu = \int \epsilon_i(x)\,\overline{\epsilon_j(x)}\,d\nu T^{-1} =$$

$$= \int \epsilon_i(x)\,\overline{\epsilon_j(x)}\,d\mu = \delta_{ij},$$

then η-system is orthonormal.

b) For any i and y, $\eta_i(y) \in \epsilon_i(G)$. And also the functions η_i are measure preserving transformations from F into the compact subgroup $\epsilon_i(G)$ of the unit circle (always with the normalized Haar measure). This follows from the next lemma.

Lemma 1. *Any character e of a compact abelian group G, is a measure preserving transformation from G onto the compact group $e(G)$.*

Proof. Cf. [2], § 63.

c) If we ask $T(F)$ to be dense on G, then the family $\epsilon_i(Ty) = \eta_i(y)$ is isomorphic to (ϵ_i) without asking T to be measure preserving.

d) If T is also continuous, the η_i are continuous functions. Now we pass to the converses of the preceding observations.

Theorem 4. Let $F^{\sim} = (\eta_i(y))$ be a system of measurable functions on the compact abelian group F isomorphic as a multiplicative (a. e.) group to the character group $G^{\wedge} = (\epsilon_i)$ of the compact abelian group G. Suppose that for any i, $\eta_i(F) \subset \epsilon_i(G)$. Then there exists a measurable transformation from F into G such that $T(F)$ generates (4) *G and for any i, $\epsilon_i(Ty) = \eta_i(y)$ a.e.y.* (The measurability of η_i and T is with respect to the Baire σ-rings. If the measurability of the η_i's is assumed to be with respect to a σ-field containing the Baire σ-ring the same result holds).

Theorem 5. If besides of the hypotesis of theorem 4 we require the η_i to be continuous functions then T is also continuous.

Theorem 6. If besides of the hypothesis of theorem 4 we re quire every function $\eta_i(y)$ to be a measure preserving mapping from F into $\epsilon_i(G)$, then T is measure preserving.

Theorem 7. If besides of the hypothesis of theorem 4 we require the system (η_i) to be orthonormal, then T is measure preserving.

Proof of theorem 4. First of all, we want to show that we can replace the system $F^{\sim} = (\eta_i)$ by another one with the same properties and which is everywhere multiplicative, e., if $\eta_i \leftrightarrow \epsilon_i$ and $\epsilon_i \epsilon_j = \epsilon_k$, then $\eta_i(x)\, \eta_j(x) = \eta_k(x)$ for every $x \in F$. We give a proof by induction. Suppose that a certain subgroup $\triangle$ of $F^{\sim}$ has been replaced by a family $\overline{\triangle}$ in such a way that: a) if $\tilde{\eta}$ $(\in \overline{\triangle})$ replaces η $(\in \triangle)$, then $\eta = \tilde{\eta}$ a. e., b) the elements of $\overline{\triangle}$ form an everywhere multiplicative group isomorphic to $\triangle$. Let α be an element of $F^{\sim}$,

(4) G is the least closed subgroup containing $T(F)$.

$\alpha \in \triangle$, an $[\alpha]$ the subgroup generated by α. If α^n does not coincide (almost everywhere) with a function of $\triangle$ whatever be $n \neq 0$, then we define $\overline{[\triangle, \alpha]} = [\overline{\triangle}, \alpha]$, where [.] indicates the subgroup generated by the set of elements contained between the brackets. If for some $n \neq 0$, α^n coincides a. e. whith a function of $\triangle$, then let m be the least positive integer with such a property. Then, $\alpha^m = \tilde{\eta}\ (\epsilon \overline{\triangle})$ a.e. Let us define $\tilde{\alpha} = \tilde{\eta}^{1\setminus m}$ for every x where $\alpha^m(x) \neq \tilde{\eta}\ (x)$, and $\tilde{\alpha} = \alpha$ where $\alpha^m\ (x) = \tilde{\eta}\ (x)$. Then, $[\overline{\triangle}, \tilde{\alpha}]$ is an everywhere multiplicative group, and $[\overline{\triangle}, \tilde{\alpha}] \sim [\triangle, \alpha]$, and we define $\overline{[\triangle, \alpha]} = [\overline{\triangle}, \tilde{\alpha}]$. It only remains to prove that $\tilde{\alpha}\ (F) \subset \epsilon\ (G)$, where ϵ is the image of α in the assumed isomorphism between $F^{\sim}$ and $G^{\wedge}$. It is obvious if $\epsilon\ (G)$ coincides with the unit circle. If not, $\epsilon\ (G)$ is the set of all k-th roots of the unity, for some k. Since $\eta\ (F) \subset \epsilon^m\ (G)$, by the inductive hyphotesis we have, $\tilde{\eta}\ (F) \subset \epsilon^m\ (G)$. From the very definition of $\tilde{\alpha}$ we get $\tilde{\alpha}\ (F) \subset \epsilon\ (G)$. Therefore, we can suppose that our system $F^{\sim}$ is an everywhere multiplicative group such that $\eta_i\ (F) \subset \epsilon_i\ (G)$.

Let $P = \Pi\, \epsilon_i\ (G)$ be the cartesian product of the image groups $\epsilon_i\ (G)$ when ϵ_i runs through $G^{\wedge}$. It is a compact abelian group. Let S be an application from F into P defined by $(Sy)_i = \eta_i(y)$. Let G' be the compact subgroup of P generated by $S(F)$. Then $S(F) \subset G'$ and $pr_i(S(y)) = \eta_i(y)$.

It is well-known that the family of proyections pr_i of P onto $\epsilon_i(G)$ is a set of generators of the free group $P^{\wedge}$. Then, the functions pr_i restricted to G' form a set of generators of the character group $G'^{\wedge}$, (cf. [9], II).

Let $c = pr_1^{a1}\ pr_2^{a2}\ \ldots\ pr_n^{an}$, ($a_i$ integers) be a character of G'. Then, $c(Sy) = \eta_1^{a1}\ \ldots\ \eta_n^{an}$ will be, by hypothesis, equal to some η-function, say η_k : $c(Sy) = \eta_k(y)$. This means that c coincides with pr_k on $S(F)$. Since G' is the least compact subgroup of P containing $S(F)$ and since c and pr_k are characters of G' which coincide on $S(F)$, we have: $c = pr_k$ on G'. Then $G'^{\wedge}$ is isomorphic to (pr_i), and therefore to (η_i) and to $(\epsilon_i) = G^{\wedge}$. By the Pontriagin duality theorem G and G' are isomorphic. Let ψ be the isomorphism, $\psi : G' \to G$, for which $pr_i \to \epsilon_i$, and T be defined by $T(y) = \psi\ (S(y))$. T is therefore an application from F into G. We have also: $pr_i/G' = \epsilon_i \circ \psi$, and $(\epsilon_i \circ \psi \circ S)\ (y) = \epsilon_i(Ty) = pr_i(Sy) = \eta_i(y)$. For any set M of the Borel field in the unit circle, $T^{-1}(\epsilon_i^{-1}(M)) = \eta_i^{-1}(M)$ belongs to the Baire σ-field of F, and since the family of sets $\{ \epsilon_i^{-1}(\mathrm{M}); \epsilon_i \in G^\wedge, M$ a Borel set $\}$ generates the σ-ring of Baire sets, we conclude that for any Baire set of G, $T^{-1}(G)$ belongs to the Baire σ-ring of F. This concludes the proof of theorem 4.

Proof of theorem 5. Since $(Sy)_i = \eta_i(y)$, S is a continuous mapping from F into G'. From $T = \psi \circ S$, we *get* the desired continuity of T.

Proof of theorem 6. This theorem reduced to theorem 7 in the following way. We have seen that $T^{-1}(\epsilon_i^{-1}(M)) = \eta_i^{-1}(M)$. From lemma 1 and the hypothesis we obtain:

$$\nu(T^{-1}(\epsilon_i^{-1}(M))) = \nu(\epsilon_i^{-1}(M)) = m_i(M \cap \epsilon_i(G)) = \\ = \mu(\epsilon_i^{-1}(M)), \tag{4}$$

where m_i is the normalized Haar measure on $\epsilon_i(G)$.

From:

$$\int \eta_i(y)\,\overline{\eta_j(y)}\,d\nu = \int \eta_k(y)\,d\nu = \int \epsilon_k(Ty)\,d\nu = \int \epsilon_k(x)\,d\nu T^{-1} = \\ = by\ (4) = \int \epsilon_k(x)\,d\mu = \int \epsilon_i(x)\,\overline{\epsilon_j(x)}\,d\mu = \delta_{ij}, \tag{5}$$

we see that the family $F^\sim$ is orthonormal. Then, the preservation of measure follows from theorem 7.

Proof of theorem 7. From (5), we get

$$\left(\sum_1^n c_i \eta_i(y), \sum_1^m d_j \eta_j(y)\right) = \sum_{i=1, j=1}^{n,m} c_i d_j \left(\epsilon_i(Ty), \epsilon_j(Ty)\right) = \\ = \sum c_i d_j \left(\epsilon_i(x), \epsilon_j(x)\right) = \left(\sum_1^n c_i \epsilon_i(x), \sum_1^m d_j \epsilon_j(x)\right) \tag{6}$$

Therefore finite linear combinations of functions of $G^\wedge$ have an image of equal norm. We want to show that $\nu(T^{-1}M) = \mu(M)$ for any Baire set M. We shall prove it for M open.

From this it follows that the preservation of measure holds for any null set. An easy L^2 approximation argument together with the last observation and (6), conclude the proof.

If M is an open Baire set, it is σ-compact, and therefore it can be constructed a sequence (g_n) of linear combinations of functions of $G^\wedge$ such that $g_n(x) \to \chi_M(x)$ everywhere and boundedly. The same happens to $h_n(y) = g_n(Ty)$ and $\chi_{T^{-1}M}(y)$. Since $\|g_n - g_m\|_2 = \|h_n - h_m\|_2$, it follows that (h_n) converges in $L^2(F)$ to a certain function which must coincide with $\chi_{T^{-1}M}(y)$. Moreover, $\|\chi_{T^{-1}M}\|_2 = \lim \| h_n \|_2 = \lim \| g_n \|_2 = \| \chi_M \|_2$.

Independece. 3. In this paragraph we want to see how the concepts of free group and independence in the sense of probability are related. We call *almost free* an abelian group G which is isomorphic to the direct product ΠZ_i of a family (Z_i), $i \in I$, of cyclic groups. We also say that a subset Γ of G is an *almost free family of generators* of G if in the isomorphism between G and ΠZ_i, the image of Γ is exactly a family of generators of the cyclic groups Z^i. Finally, we say that a set Γ of functions on a compact group G is *p-independent* if it is a set of generators of $G^\wedge$ independent in the sense of probability.

Proposition 1. Let G be a compact abelian group and $G^\wedge$ its character group. Let $\triangle$ be a subgroup of $G^\wedge$. Then Γ is a p-independent set of $\triangle$ if and only if it is an almost free family of generators of $\triangle$.

Proof. Let us see the "only if" part. First of all we observe that the results mentioned in [2], pp. 191-193 remain true, even for not necessarily real functions. We need only to prove that:

$$\prod_{j=1}^{N-1} e_{i_j}^{n_j} = e_{i_N}^{n_N} \quad \text{implies} \quad e_{i_N}^{n_N} \equiv 1 .$$

We have:

$$1 = \int \Big(\prod_{j=1}^{N-1} e_{i_j}^{n_j} \Big) e_{i_N}^{-n_N} \, d\mu = \text{(by the hypothesis of independence)} =$$

$$= \Big(\prod_{j=1}^{N-1} \int e_{i_j}^{n_j} \, d\mu \Big) \cdot \int \overline{e_{i_N}^{n_N}} \, d\mu = \Big| \int e_{i_N}^{n_N} \, d\mu \Big|^2, \text{ and therefore:}$$

$$1 = \Big| \int e_{i_N}^{n_N} \, d\mu \Big| .$$

From the last equality we obtain that $e_{i_N}^{n_N}$ must be identically one.

We pass now to the "if" part. Suppose first that $\triangle = G^\wedge$. Then G is isomorphic to a product of compact groups G_i and such that $(\Pi G_i)^\wedge = \Pi Z_i$. From this follows immediately the p-independence of Γ. If $\triangle$ is not all of $G^\wedge$, let us consider the subgroup $H = \bigcap \{ x \in G;\ g(x) = 1,\ g \in \triangle \}$. Observe now that the product measure of the (normalized) Haar measures on H and G/H is the Haar measure on G. Besides $\triangle$ is the character group of G/H.

These two observations reduce the case $\triangle \neq G$ to the case $\triangle = G$, Q.E.D..

To finish this paragraph we shall observe how these concepts are invariant under measure preserving transformations.

Proposition 2. Let T be a measurable transformation from F into G, F and G compact abelian groups. Let $(\epsilon_i(x)) = G^\wedge$ and $\eta_i(y) = \epsilon_i(Ty)$. a) *If $T(G)$ is dense in G then $\Gamma = (\epsilon_{i_s}\ (x))$* is an *almost free family of $G^\wedge$ if and only if $(\eta_{i_s}\ (y))$ is almost free* (5). b) *If T is measure preserving, then $(\epsilon_{i_s}\ (x))$ is p-independent if and only if $(\eta_{i_s}\ (y))$ is p-independent* (6).

Proof. a) follows from the definitions. b) is a consequence of

$$\nu\ \Big(\bigcap_{i=1}^{n} \eta_i^{-1}\,(M_i)\Big) = \nu\,\Big(\bigcap_{i=1}^{n} T^{-1}\,(\epsilon_i^{-1}\,(M_i))\Big) =$$

$$= \nu\ \Big(T^{-1}\Big[\bigcap_{i=1}^{n} \epsilon_i^{-1}\,(M_i)\Big]\Big) =$$

$$= \mu\ \Big(\bigcap_{i=1}^{n} \epsilon_i^{-1}\,(M_i)\Big) = \prod_{i=1}^{n} \mu\,(\epsilon_i^{-1}\,(M_i)) = \prod_{i=1}^{n} \nu\,(\eta_i^{-1}\,(M_i))\,.$$

Part II

1. Let $\{ G_i \}$ be a family of locally compact groups, pairwise disjoint and all of them commutative. We denote by G the union $\bigcup_{i \in I} G_i$ with the supremum topology, i.e., a set is open if and only if it intersects in an open set every G_i. Therefore, G is a locally compact space and every Baire (Borel) set in G is of the form $\sum_{i \in J} M_i$, $M_i \subset G_i$, where J is a denumerable subset of I and M_i is a Baire (Borel) set of G_i. For any function f on G, f_i will

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(5) In the following sense $\eta_{i_1}^{m_1} \cdots \eta_{i_n}^{m_n} = \eta^{k^l}$ implies $\eta^{k^l} \equiv 1$.

(6) In an obvious sense.

denote its restriction to the clopen subset G_i. *Evidently,* $f \in L^1(G)$ when and only when $f = \Sigma_{i \in J} f_i$, J is finite or countably infinite, $f_i \in L^1(G_i)$ *and* $\| f \|_1 = \Sigma_{i \in J} \int_{G_i} | f_i | \, d\mu_i < \infty$. Briefly, $L^1(G) = \Sigma L^1(G_i)$. We define now the convolution between two functions f, g of $L^1(G)$ *by*:

$$f * g = \Sigma f_i * g_i \quad . \tag{1}$$

From, $\| f * g \|_p = \| \Sigma f_i * g_i \|_p \leqslant \Sigma \| f_i \|_1 \; \| g_i \|_p \leqslant$ $\leqslant \| g \|_p . \Sigma \| f_i \| = \| f \|_1 . \| g \|_p$, we see that $L^1(G)$ is a Banach algebra with this operation as multiplication. Of course, convolution is commutative, associative and bilinear. To avoid long proofs and to reduce the repetitions we stick to Loomis' book for the nomenclature and references on Banach and group algebras.

Let M be a regular maximal ideal of $L^1(G)$. From the very definitions follow that: 1) the restrictions M_i to G_i of the functions of M constitute an ideal; 2) the restriction to G_i of an identity of $L^1(G)$ modulo M, is an identity of $L^1(G_i)$ mod. M_i; 3) every M_i is maximal or equal to $L^1(G_i)$, and there is one and only one different from $L^1(G_i)$. Then,

Lemma 1. The space of regular maximal ideals of $L^1(G)$ *coincides with the set theoretic union of the spaces of maximal regular ideals of the* $L^1(G_i)$.

Each maximal regular ideal is the kernel of a multiplicative linear functional, and conversely. Since, the space G is such that $(L^1(G))^* = L^\infty(G)$, (cf. [7], p. 43), any maximal regular ideal is associated to a function of $L^\infty(G)$. This function will be called a *character* of G. Let k be the index for which $M_k \neq L^1(G_k)$, and $a_M(x)$ the character associated to M. *For* any function f such that $f_k = 0$ a.e., it holds: $f(M) = \int f(x) \overline{a_M(x)} \, d\mu = 0$, since $f \in M$. Therefore, $a_M(x)$ is zero except on G_k, and obviously, coincides there with a character of G_k. Then,

Lemma 2. The characters of G define a set in one-to-one correspondence with the union of the character groups G_i; each character of G is zero on every G_i except on one of them and there coincides with a character of that group.

The topology of the family of characters $G^\wedge$ of G, is by definition the weak topology induced by the functions $\hat{f}(M) = \hat{f}(a_M) = \int f(x) \, \overline{a_M} \, (x) \, dx$, $f \in L^1(G)$. It follows easily that, *Lemma 3. The topology of $G^\wedge$ is equal to the supremum of the topologies of the spaces G_i. It coincides with the topology of the uniform convergence on compact sets of G.*

The last assertion of lemma 3 is very easy, and also the Pontrjagin's theorem. The space of maximal regular ideals of $G^\wedge$ is homeomorphic to G.

Moreover, two locally compact spaces like G, G^1 and G^2, such that there exists a homeomorphism which is a group isomorphism from every group contained in G^1 onto a group in G^2, will be called an *isomorphism* between G_1 and G_2. Therefore, *$G^{\wedge\wedge}$ is isomorphic to G.*

2. $L^1(G)$ has a symmetric involution defined by $f^{\#} = \Sigma f^{\#}_i$ where $f_i^{\#} = f_i(x^{-1})$. Obviously, $f^{\#\wedge} = f^{\wedge -}$. This involution is an isometry on $L^1(G)$.

Let $L^0(G)$ be a dense ideal in $L^1(G)$ defined as: $f \in L^0(G)$ iff $f \in L^1(G)$, f is a continuous function and $\Sigma \| f_i \|_\infty < \infty$. Let $\phi(f)$ be the positive linear functional (i.e., $\phi(f * f^{\#}) \geqslant 0$ for any $f \in L^1$) on L^0, defined by:

$$\phi(f) = \Sigma f_i(e_i),$$

where e_i is the identity of G_i.

An element p of $L^\infty(G)$ will be called *positive definite* if $\phi(p * f) = \Theta_p(f)$ is a positive functional on $L^1(G)$.

Auxiliary the orem. If $p \in L^0$ is positive definite and extendible (7) *then there exists a unique Baire measure m on $G^\wedge$ such that* $\phi(p * f) = \int_{G^\wedge} \hat{f}(a)\, \hat{p}(a)\, dm(a) = \Sigma (p_i * f_i)(e_i)$, *and* $\hat{p} \in L^1(G^\wedge, m)$.

Proof. Notice that $L^1(G)$ is semi-simple and self-adjoint. Then, the theorem follows from theorem 26. *J* of [7].

The restriction of p to G_i, p_i, verifies:

$$(p_i * f_i)(e_i) = \int_{G_i^\wedge} \hat{f}_i(a)\, \hat{p}_i(a)\, dm(a)$$

and the proof in 36. *B*, [7], shows that the restriction of m to $G_i^\wedge$ coincides with its Haar measure there.

(7) $\Theta_p(f)$ can be extended so as to remain positive when an identity is added to L_1

3. *First Bochner Theorem. The formula,*

$$p(x) = \int a(x)\, d\,\mu\,(a),$$

establishes an isomorphism between the functions $p(x) \in L^\infty(G)$ *which define positive linear functionals and the positive Baire measure* μ *such that:*

$$\int_{G_i^\wedge} d\,\mu\,(a) \leqq k < \infty, \text{ for any } i.$$

Proof. If $p \in L^\infty(G)$ defines a positive functional (that is, $(f * f^\#, p) \geqslant 0$) then p_i defines a positive functional and

$$\| p_i \|_\infty \leqq \| p \|_\infty .$$

From the Bochner theorem (36.A, [7]) it follows that: $p_i(x) = \int_{G_i^\wedge} a\,(x)\ d\,\mu_i\,(a)$, where μ_i is a Baire measure, positive and such that $\int_{G_i^\wedge} d\,\mu_i\,(a) = \| p_i \|_\infty \leqq \| p \|_\infty = k.$

Then, $p\,(x) = \int_{G^\wedge} a\,(x)\ d\,\mu\,(a)$, where $\mu\,(a)$ coincides with μ_i on G_i.

Conversely, given a $\mu\,(a)$ with the mentioned properties, $p_i\,(x) = \int_{G_i^\wedge} a\,(x)\ d\,\mu\,(a)$, is a function of $L^\infty\,(G_i)$ which defines a positive functional on $L^1\,(G_i)$. Besides, $\| p_i \|_\infty = \int_{G_i^\wedge} d\,\mu\,(a) \leqq$ $\leqq k$, and $(f * f^\#, p) = \Sigma\,(f_i * f^\#{}_i, p_i) \geqq 0$.
(As in the corollary to 36.A, [7], every p_i is essentially uniformly continuous on G_i).

Second Bochner theorem. Let $p \in L^\infty\,(G)$. p *defines a positive and extendible linear functional iff* $\int_{G^\wedge} d\,\mu\,(a) < \infty$.

Proof. It is a direct application of the Herglotz-Bochner-Weyl-Raikov theorem (cf. [7]).

Corollary. If $p(x)$ *defines an extendible positive linear functional then* $\Sigma \| p_i \|_\infty < \infty$. *If besides* $p(x) \in L^1\,(G)$, *then* $p(x) \in L^0\,(G)$.

Proof. It follows from the preceding theorem, observing that $\int_{G_i^\wedge} d\,\mu_i\,(a) = \| p_i \|_\infty$ and that p_i is esentially uniformly continuous.

Lemma 4. $p \in L^\infty\,(G)$ defines an extendible positive linear functional if and only if p defines an extendible positive definite linear functional.

Proof. If p defines an extendible positive functional, then $(f, p) =$ (cf. p.96, [7]) $= \overline{(f^{\#}, p)} = (p, f^{\#}) = (f, p^{\#})$, and therefore, $p = p^{\#}$. Since $\phi\,(p * f) = (f, p^{\#}) = (f, p)$, p is positive definite.

If p is positive definite, the functional $(f, p^{\#}) = \phi\,(p * f)$ is positive and extendible, and therefore, $p^{\#} = (p^{\#})^{\#} = p$ defines a positive functional.

For the group algebra $L^1\,(G)$ of a locally compact abelian group G, a linear positive functional is continuous if and only if it is extendible, (cf. [7], p. 126). However, for locally compact spaces of the type defined in the first paragraph, the continuity of a linear positive functional is not equivalent to its extendibility as first and second Bochner theorem show. For G a group, $p \in L^\infty\,(G)$ defines a positive functional iff it is definite positive ($\in L^0\,(G)$). An essential role is played by the extendibility, but this is not showed up because of its equivalence with continuity. This can be seen from lemma 4. If in that lemma we drop the condition on extendibility on the positive linear functional defined by $p\,(x) \in L^\infty\,(G)$, from Bochner theorems it follows that, in general, is not true that $p \in L^0(G)$. This different behaviour is a consequence of the lack of an approximate identity on $L^1\,(\bigcup_{i \in I} G_i)$ when I is not finite.

4. *Inversion theorem. If $p \in L^1\,(G) \cap L^\infty\,(G)$ and defines an extendible positive linear functional, then $\hat{p} \in L^1\,(G^\wedge)$ and*

$$p(x) = \int a(x)\,\hat{p}(a)\,d\,a\,,$$

where $d\,a$ is a certain measure on $G^\wedge$ which coincides with a Haar measure on every $G_i^\wedge$.

Proof. From lemma 4, it follows that $p\,(x)$ is positive definite. Since p_i is positive definite on G_i, p may be assumed to be continuous, and therefore $p \in L^0\,(G)$.

Then by the auxiliary theorem,

$$(f, p) = \phi\,(p * f) = \int_{G^\wedge} \hat{f}\,(a)\,\hat{p}\,(a)\,d\,a, \text{ and } \hat{p} \in L^1\,(G^\wedge).$$

On the other hand we have:

$$(p, f) = (f^{\#}, p^{\#}) = (f^{\#}, p) = \int \hat{f}^{\#} \hat{p}\, d\,\alpha = \int \hat{p}\, \overline{\hat{f}}\, d\,\alpha.$$

From the last formula we obtain:

$(p, f) = (\int \hat{p}\,(\alpha)\;\alpha\,(x)\; d\,\alpha, f)$ for any $f \in L^1$.

We shall denote by $P\ (\subset L^0)$ the family of functions of $L^1 \cap L^\infty$ which define positive definite and extendible linear functionals on $L^1\,(G)$ and by $P^\wedge$, the analogous family on $G^\wedge$. By $[P]$ we designe the subspace generated algebraically by P.

Plancherel Theorem. The Fourier transformation $f \rightarrow \hat{f}$ preserves scalar products when confined to $[P]$. Its L^2-closure is a unitary mapping from $L^2\,(G)$ onto $L^2\,(\hat{G})$.

Proof. For $p \in P$, we have $p = p^{\#}$, and therefore $\hat{p} = \overline{\hat{p}}$. Then $\phi\,(p_1 * p_2 = (p_1, p_2{}^{\#}) = (p_1, p_2)$, equals by the auxiliary theorem do $(\hat{p}_1, \overline{\hat{p}_2}) = (\hat{p}_1, \hat{p}_2)$. Then, $(p_1, p_2) = (\hat{p}_1, \hat{p}_2)$. This equality can be extended to $[P]$ and to the L^2-clousure of $[P]$, i.e., to $L^2\,(G)$. The Fourier transformation is onto because it is so for $L^2\,(G_i)$ and $L^2\,(G^\wedge{}_i)$.

We want to prove now that $[P]^\wedge = [P^\wedge]$. Given $p \in P$, let us take the positive part q of its real component. Then, $\hat{q}$ defines a positive definite functional on $L^1\,(G^\wedge)$. Besides,

$$\Sigma\, \| \hat{q}_i \|_\infty \leqq \Sigma\, \| q_i \|_1 < \infty\,.$$

From second Bochner theorem it follows that it is extendible, and from the inversion theorem, that $\hat{q} \in L^1\,(G)$. Therefore, $\hat{p} \in [P^\wedge]$.

The inclusion in the other sense follows from the inversion and Pontrjagin theorems.

5. Finally, we observe that the regularity of $L\,(G)$, the tauberian theorem, the theorem on invariant subspaces, and the condition D for $L\,(G)$, admit the same statement for G a locally compact abelian group or G a locally compact space as defined in paragraph 1. The proofs are trivial or follow the same lines as given in [7]. (Under a translate of $f\,(x)$ in $\{y_i\}$, $y_i \in G_i$, the function equal to $f\,(xy_k)$ on G_k, $k \in I$, is to be understood). The generalized Wiener tauberian theorem can be translated in almost the same way as it is very easy to verify. It has no content if every G_i is compact.

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BIBLIOGRAFIA

RENATO CACCIOPPOLI, *Opere*, en dos volúmenes, Cremonese, Roma 1963.

La Unión Matemática Italiana, con la contribución del Consiglio Nazionale delle Ricerche, encomendó a una comisión presidida por el Prof. Mauro Picone e integrada por ocho profesores, entre los cuales se cuentan quienes fueron discípulos avanzados y, posteriormente, estrechos amigos del singular matemático napolitano, la realización de esta obra en la que se ha reunido, prácticamente, la totalidad de las publicaciones que, desde la primera de 1926 (resumen de su tesis de doctorado) hasta la última de 1955 traducen el pensamiento científico de Renato Caccioppoli.

Las mismas se han distribuído en dos volúmenes siguiendo el criterio, según se aclara en el prefacio, de incluir en el primero los trabajos sobre argumentos de la teoría propiamente dicha de funciones de variable real: integración, totalización, funciones de conjunto, investigaciones vinculadas al análisis funcional, a las ecuaciones diferenciales ordinarias y en derivadas parciales, a las funciones de una o varias variables complejas y las referentes a las funciones pseudo-analíticas. También en el prefacio se incluye un útil comentario, a modo de orientación, de las publicaciones contenidas en el texto, señalándose en él, fundamentalmente, las ideas centrales, varias de ellas originales del propio Caccioppoli, que le sirvieron de guía en sus trabajos de investigación. Se incluye, por último, una lista en orden cronológico de la totalidad de las publicaciones por él realizadas.

La obra satisface una necesidad evidente. Los trabajos en ella reproducidos traducen (*) "una personalidad científica de un vigor y de una originalidad

(*) GIUSEPPE SCORZA DRAGONI, *Renato Caccioppoli*, Appendice necrol. ai Rend. dei Lincei, Fasc. III, Roma 1963. (Esta necrología contribuye en mu-