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POINTWISE CONVERGENCE OF SERIES OF BESSEL FUNCTIONS

A. Benedek and R. Panzone

1. INTRODUCTION. The following result is proved in [5], Ch. 4. *Let be* $f(x) \in L^1(0,1)$, $\nu > -1/2$ *and* $x \in (0,1)$. *If the Fourier series of* f *converges to* 1 *at the point* x *then the Fourier-Bessel series of* f *converges to the same value at the same point.*
This result cannot be extended to $\nu > -1$ since, in general, a function in L^1 has not Fourier-Bessel coefficients for $\nu \in (-1,-1/2)$. However, it is possible to prove the following result which is contained in theorem 4 of this paper. *Assume* $\mu = (1/2 + \nu) \wedge 0$, $\nu > -1$, $x \in (0,1)$ *and* $f(x)x^{\mu} \in L^1(0,1)$. *Then, the expansions of* f *with respect to the trigonometric system and the Bessel system are equiconvergent.*
To prove this result, which is crucial in the proof of the main result of this paper, we prove two theorems on equiconvergence of the Fourier expansion with the expansion of the function under consideration with respect to the system of eigenfunctions of a second order differential equation. This is done following the line of proof of theorem 9.5 in Titchmarsh's book.
The main result of this paper is theorem 5 which roughly states that any system of solutions of Bessel's equation, orthogonal with respect to xdx, complete in the space L^2 defined by means of this measure, has the property that if the space L^p admits convergence in the norm then it also admits convergence almost everywhere.
All these systems are described and also a family of L^p-spaces which admit convergence in the norm.

2. AUXILIARY RESULTS AND NOTATION.

Assume $0 < x < 1$, $-1 < \nu < \infty$ and $\{u_n : n=1,2,...\}$ defined by one of the following formulae:

(1) $u_n(x) = a_n^{-\nu} J_\nu(a_n x) + K.a_n^{\nu} J_{-\nu}(a_n x)$, $K \in (-\infty,\infty)$, $K=0$ if $\nu \geq 0$

(2) $u_n(x) = -(2/\pi)\lg(ka_n)\ J_o(a_n x) + Y_o(a_n x), \quad k \in (0,\infty)$

with $\arg a_n \in [0,\pi)$. Each system $\{u_n\}$ is associated to a value of K or k. Each function u_n is a solution of the equation:

$$(3) \qquad (xy')' + (a_n^2 x - \nu^2/x)y = 0$$

with $\nu=0$ for the functions of the systems (2).
The a_n's are obtained from the homogeneous boundary condition:

$$(4) \quad ((\sin\alpha)/2 + \cos\alpha)u_n(1) + (\sin\alpha)u_n'(1) = 0 \ , \quad \alpha \in [0,\pi)$$

For a fixed α the set of solutions of (4), $\{a_n\}$, is a denumerable family. This set together with the value of K or k define the system $\{u_n\}$, i.e. each system is determined by two parameters: α and K (or k). Besides, *each of them is orthogonal with respect to the weight* x *and complete in the space* L^2 *determined by this weight*, (cf. [5], Ch. 4).
We shall denote with $L^p(w(x))$, $\infty > w(x) > 0$, $1 \leqslant p \leqslant \infty$, the real space of p-integrable functions with respect to the measure $w(x)dx$ and shall say that $L^p(w)$ *admits convergence in the norm* with respect to $\{u_n\}$ if for $f \in L^p$, the expansion of f with respect to the system converges to the function in L^p. That is, let be

$$S_N(f) = \sum_1^N c_n u_n \ , \ c_n = \int_0^1 u_n\ f\ xdx \Big/ \int_0^1 u_n^2\ xdx \text{ and } f \in L^p(w),$$

then $\|(S_n f - f)\ w\|_p \longrightarrow 0$ if $n \longrightarrow \infty$. Observe that u_n is not conjugated because it is always a *real function*.

3. POINTWISE CONVERGENCE OF EIGENFUNCTION EXPANSIONS IN L^2.

In what follows assume the results and notation of chapters 1 and 2 of [5]. Let q be continuous in $0 \leqslant x < \Delta$, $\Delta \leqslant \infty$. Consider the differential equation

$$(5) \qquad y'' + (\lambda - q)y = 0$$

and the solutions ϕ and θ such that

(6) $\phi(0,\lambda)=\sin\alpha$, $\phi'(0,\lambda)=-\cos\alpha$; $\theta(0,\lambda)=\cos\alpha$, $\theta'(0,\lambda)=\sin\alpha$.

Assume that $m(\lambda)$ is an analytic function, holomorphic in the complex plane except for simple poles at the real points $\lambda_o < \lambda_1 < \ldots < \lambda_n < \ldots$ such that $m(\bar{\lambda}) = \overline{m(\lambda)}$.

Assume that for non-real λ, $m(\lambda)$ is on the "limit circle" at λ and such that there exist two sequences $\{b_n\}$ and $\{\beta_n\}$ with the property that for any λ with $\mathrm{Im}(\lambda) > 0$ it holds:

$$m(\lambda) = \lim_{b_n \uparrow \Delta} - \frac{\theta(b_n)\ \mathrm{cotg}\ \beta_n + \theta'(b_n)}{\phi(b_n)\ \mathrm{cotg}\ \beta_n + \phi'(b_n)} = \lim_{b_n \uparrow \Delta} 1_{b_n,\beta_n}(\lambda) ,$$

$\beta_n \in [0,2\pi)$, $\beta_n \longrightarrow \beta$.

The residues of the function $\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\ \phi(x,\lambda)$ at the poles of $m(\lambda)$ are:

$$\psi_n(x)/\|\phi(.,\lambda_n)\|_2 = \phi(x,\lambda_n)/\|\phi(.,\lambda_n)\|_2^2$$

$\{\psi_n(x)\}$ is an *orthonormal* family of functions in $L^2(0,\Delta)$.

We shall prove next theorem following the pattern given by paragraphs 1-4 of chapter 9 in Titchmarsh's book.

THEOREM 1. *Assume that at* $x \in (0,\Delta)$ *the functions* $\psi_n(x)$ *verify*

$$\sum_{T^2-T<\lambda_n<T^2+T} |\psi_n(x)|^2 = O(1) \quad \text{if} \quad T \longrightarrow \infty$$

If f *is a real function,* $f \in L^2(0,\Delta)$, *and the ordinary Fourier series of* f *converges at* x *to* 1, *then the series*

$$\sum c_n\ \psi_n(x) \quad , \quad c_n = \int_0^\Delta f(y)\ \psi_n(y)\ dy ,$$

also converges to 1.

By "the ordinary Fourier series of f" we understand the expansion of the restriction of f to the finite interval $(0,b)\ni x$ with respect to the system $\{\sin 2\pi\ nx/b,\ \cos 2\pi\ nx/b\}$. Next lemma 2 asserts that if the ordinary Fourier series of f converges at x to 1 then this value is independent of b whenever $0 < x < b < \Delta$.

To prove theorem 1 we need some estimations from Titchmarsh's book. Assume $0 \leqslant \arg\lambda \leqslant \pi$, $s = \sqrt{\lambda} = \sigma + it$, $0 \leqslant \arg s \leqslant \pi/2$, $\Delta > b > b_o > 0$.

Then, (cf. [5], 1.7. (ii))

$$(7)\quad \begin{cases} \phi(x,\lambda) = \sin\alpha \cos sx + O(1)\, e^{tx}/|s| & \text{if } \sin\alpha \neq 0 \\ \phi(x,\lambda) = -(\sin sx)/s + O(1)\, e^{tx}/|s|^2 & \text{if } \sin\alpha = 0 \end{cases}$$

where the O's are uniform in x and s whenever $x \in (0,b)$ and $|s| > \sigma_o(b)$. A careful examination of the constants in the proofs of 9.2.12 and 9.2.13 in [5], proves that next formulae (8) and (9) hold.

$$(8)\quad \begin{cases} \psi(x,\lambda) = e^{isx}/is \sin\alpha + O(1)\, e^{t(x-2b)}/\sigma + O(1) e^{-xt}/|s|^2 & \text{if } t>1 \\ \psi(x,\lambda) = O(1)/\sigma t & \text{if } 0 < t \leqslant 1 \end{cases}$$

where $\sin\alpha \neq 0$, and if $\sin\alpha = 0$,

$$(9)\quad \begin{cases} \psi(x,\lambda) = e^{isx} + O(1)|s|\, e^{t(x-2b)}/\sigma + O(1)\, e^{-xt}/|s| & \text{if } t > 1 \\ \psi(x,\lambda) = O(1)|s|/\sigma t & \text{if } 0 < t \leqslant 1 \end{cases}$$

In both formulae, *the O's are uniform whenever* $x \in (0,b)$ *and* $|s| > \sigma_o(b)$. (For this it is convenient to consider formula 9.2.11 of [5] for the cases $bt > 1$ and $bt \leqslant 1$).
We need also the following result on analytic functions (cf. [5], lemma 2.11).

LEMMA 1. *Let* $F(z)$ *be holomorphic on* $-R \leqslant x \leqslant R$, $-r \leqslant y \leqslant r$ *where* $z=x+iy$. *If in this rectangle* $|F| < M/|y|$ *then* $|F(z)| \leqslant M_o.M$ *in* $-r \leqslant y \leqslant r$, $-R/2 \leqslant x \leqslant R/2$, *where* $M_o = M_o(r,R)$.

Since $\psi(x,s^2) - e^{isx}/is \sin\alpha$ is regular in the square $-1 \leqslant \sigma \leqslant 1$, $T-1 \leqslant t \leqslant T+1$ for T great enough and takes conjugate values at points symmetric with respect to the imaginary axis, we can apply the preceding lemma to obtain from (8):

$$(10)\quad \begin{cases} \psi(x,\lambda) = e^{isx}/is \sin\alpha + O(1) e^{t(x-2b)}/(\sigma \vee 1) + O(1) e^{-xt}/|s|^2 = \\ \qquad = O(1)\, e^{-xt}/|s| \quad , \text{ if } t > 1 \\ \psi(x,\lambda) = O(1)/\sigma t \quad , \text{ if } 0 < t \leqslant 1 \end{cases}$$

for $\sin\alpha \neq 0$ and from (9), when $\sin\alpha = 0$:

$$(11)\quad \begin{cases} \psi(x,\lambda) = e^{isx} + O(1)|s|\, e^{t(x-2b)}/(\sigma \vee 1) + O(1) e^{-xt}/|s| = \\ \qquad = O(1)\, e^{-xt} \quad \text{if } t > 1 \\ \psi(x,\lambda) = O(1)|s|/\sigma t \quad \text{if } 0 < t \leqslant 1 \end{cases}$$

In formulae (10) and (11), the O's have the same properties as in formulae (8) and (9).
After integration of formulae (7) we obtain:

$$(12)\quad \begin{cases} \displaystyle\int_0^x \phi(y,\lambda)\,dy = O(1)e^{tx}(1+x)/|s| & \text{if } \sin\alpha \neq 0 \\[2ex] \displaystyle\int_0^x \phi(y,\lambda)\,dy = O(1)\; e^{tx}(1+x)/|s|^2 & \text{if } \sin\alpha = 0 \end{cases}$$

where the O's have same properties as in (8) and (9).
Next, we shall see that if $x < b$ then

$$(13)\quad \begin{cases} \displaystyle\int_x^b \psi(y,\lambda)\,dy = O(1)e^{-xt+b.d(t)}/(t\wedge 1)|s|^2, & \text{for } \sin\alpha \neq 0 \\[2ex] \displaystyle\int_x^b \psi(y,\lambda)\,dy = O(1)e^{-xt+b.d(t)}/(t\wedge 1)|s| & \text{for } \sin\alpha = 0 \end{cases}$$

where the O's have the same properties as in formulae (8) and (9), and $d(t)=t$ if $0 < t \leqslant 1$, $=0$ if $t > 1$.
To prove (13), observe that $\psi(x,\lambda)$ satisfies the integral equation:

$$(14)\quad \psi(x,\lambda) = \psi(b,\lambda)\cos(s(x-b)) + \psi'(b,\lambda)\sin(s(x-b))/s + \\ + \int_b^x s^{-1}\sin(s(x-y))\; q(y)\; \psi(y,\lambda)\,dy\;.$$

After integration of (14), we obtain:

$$\int_b^x \psi(y,\lambda)dy = \psi(b,\lambda)\; s^{-1}\sin(s(x-b)) + \psi'(b,\lambda)(1-\cos(s(x-b)))s^{-2} + \\ + \int_b^x q(y)\;\psi(y,\lambda)(1-\cos(s(x-y)))\; s^{-2}\,dy\;.$$

Then

$$(15)\quad |e^{xt}\int_b^x \psi(y,\lambda)dy| \leqslant O(1)s^{-1}.\sup_{0\le y\le b}|e^{yt}\;\psi(y,\lambda)| + \\ + 2|s|^{-2}.\;|\psi'(b,\lambda)\;e^{bt}|$$

where O(1) has the same properties as in formula (8).
From (14) it follows that for b fixed and $x < b$:

$$\left|\frac{\psi'(b,\lambda)}{s}\right| e^{bt} \leqslant \frac{e^{t(b-x)}}{|\sin(s(x-b))|} \{|e^{xt}\,\psi(x,\lambda)| + |e^{bt}\,\psi(b,\lambda)\,.$$

$$.\,\cos(s(x-b))e^{t(x-b)}| + \int_x^b \left|\frac{\sin(s(x-y))}{s.e^{t(y-x)}}\,q(y)\,e^{ty}\,\psi(y,\lambda)\right| dy\} \leqslant$$

$$\leqslant \frac{M\, e^{(b-x)t}}{|\sin(s(b-x))|}\,.\ \sup_{0\le x\le b} |e^{xt}\,\psi(x,\lambda)|$$

where M is an absolut constant if $|s| > \sigma_0(b)$.
In consequence,

$$(16)\quad \left|\frac{\psi'(b,\lambda).e^{bt}}{s}\right| \leqslant M.\min_{0\le x\le b}\left|\frac{e^{(b-x)t}}{\sin(s(b-x))}\right|.\max_{0\le x\le b}|e^{xt}\,\psi(x,\lambda)| \leqslant$$

$$\leqslant M'.\max_{0\le x\le b}|e^{xt}\,\psi(x,\lambda)|$$

where M' is an absolut constant when $|s| > \sigma_0(b)$, as it can be seen using the formula: $|\sin(x+iy)|^2 = \sin^2 x + \sinh^2 y$ to estimate the minimum.

(13) follows now from (15) and (16), and the estimations (10) and (11).

QED.

It also holds (cf. 9.3.1 and 9.3.2 in [5]):

$$(17)\quad \int_b^\Delta |\psi(y,\lambda)|^2 dy = \begin{cases} O(1)/\sigma^2 t^2, & 0 < t \leqslant 1 \\ O(1)e^{-2bt}/\sigma^2 t, & t > 1 \end{cases}$$

whenever $\sin\alpha \neq 0$. In case $\sin\alpha = 0$,

$$(18)\quad \int_b^\Delta |\psi(y,\lambda)|^2 dy = \begin{cases} O(1)\,|s|^2/\sigma^2 t^2, & 0 < t \leqslant 1 \\ O(1)\,s^2 e^{-2bt}/\sigma^2 t, & t > 1 \end{cases}$$

In both formulae the O's are independent of b if $|s| > \sigma_0(b)$.

Finally, we mention a familiar result on convergence of Fourier series (cf. [6], p.242, Th. 1.3).

LEMMA 2. *Assume* $\int_0^b |f(y)|\,dy < \infty$, $\int_0^b |f_1(y)|\,dy < \infty$ *and* $f=f_1$ *in a neighbourhood of* $x \in (0,b)$. *Then, the ordinary Fourier series of* f *relative to the interval* (0,b) *converges to* 1 *if and only if*

$$\lim_{T\to\infty} \frac{1}{\pi} \int_0^b f_1(y)\,\frac{\sin(T(x-y))}{x-y}\,dy = 1$$

4. PROOF OF THEOREM 1.

Let us assume that C_1 is the circumference of radius $T^2=|\lambda|$ and that in the interval $(-\infty,-T^2]$ there is no eigenvalue of the equation. If T^2 is not an eigenvalue and

$$(19)\quad \Phi(x,\lambda) = \psi(x,\lambda) \int_0^x \phi(y,\lambda)f(y)dy + \phi(x,\lambda) \int_x^\Delta \psi(y,\lambda)f(y)dy$$

then (cf. [5], Th. 2.17):

$$\frac{1}{2\pi i} \int_{C_1} \Phi(x,\lambda)d\lambda = \sum_{\lambda_n<T^2} c_n\psi_n(x)$$

Since $\Phi(x,\bar\lambda) = \overline{\Phi(x,\lambda)}$, if C_2 is the half circle in the λ-plane image of $C = \{s=Te^{i\varphi},\ 0 \leqslant \varphi \leqslant \pi/2\}$, $s=\sqrt{\lambda}$, we have:

$$\frac{1}{2i} \int_{C_1} \Phi(x,\lambda)d\lambda = \mathrm{Im} \int_{C_2} \Phi(x,\lambda)d\lambda$$

In consequence,

$$(20)\qquad \sum_{\lambda_n<T^2} c_n\ \psi_n(x) = \frac{2}{\pi}\ \mathrm{Im} \int_C \Phi(x,s^2)\ s\,ds$$

Let ε and η be arbitrary positive numbers.
Write $f=f_1+f_2+f_3$, where f_2 is a continuously differentiable function equal to zero on (b,Δ) and in a neighbourhood $(x-\delta,x+\delta)$ of the point x, f_1 is zero on (b,Δ) and such that $\int_0^\Delta |f_1|\,dy < \eta$ and f_3 is zero in (0,b) with $\int_0^\Delta |f_3|^2 dy < \varepsilon^2$

Since $f \in L^2$, this can be done if b is chosen great enough and greater than x and δ sufficiently small.

With this decomposition $\Phi(x,\lambda) = \Phi_1(x,\lambda) + \Phi_2(x,\lambda) + \Phi_3(x,\lambda)$, where Φ_i is given by (19) with f_i instead of f.

In the following formulae (21),(22) and (23), the O's are independent of b if $|s| > \sigma_o(b)$.

$$(21)\quad \begin{cases} \Phi_1(x,\lambda) = \displaystyle\int_o^b e^{is|x-y|} f_1(y)(2is)^{-1}dy + \\ \qquad + O(1)\left(\dfrac{e^{-xt}}{|s|} + \dfrac{1}{|s|^2} + \dfrac{e^{(x-b)t}}{\sigma \vee 1}\right).\eta \quad \text{if } t > 1, \\ \qquad = O(1)\ \eta\ e^{xt}/\sigma t \quad \text{if } 0 < t \leqslant 1. \end{cases}$$

$$(22)\quad \Phi_2(x,\lambda) = e^{-\delta t}(1+x)\ O(1)/|s|^2 \quad \text{if } t > 1\ ,$$

$$= O(1)(1+x)e^{bt}/t|s|^2 \quad \text{if } 0 < t \leqslant 1.$$

$$(23)\quad \Phi_3(x,\lambda) = e^{(x-b)t}\ O(1)\ \varepsilon/\sigma \vee 1 \quad \text{if } t > 1,$$

$$= O(1)\ \varepsilon\ e^{xt}/\sigma t \quad \text{if } 0 < t \leqslant 1.$$

(21) is a direct consequence of (7) and (10) and (11). In fact, because of these formulae:

$$\Phi_1(x,\lambda) = \int_o^b \left[\frac{e^{is|x-y|}}{2is} + O(1)\left(\frac{e^{-(x+y)t}}{|s|} + \frac{e^{-|x-y|t}}{|s|^2} + \right.\right.$$

$$\left.\left. + \frac{e^{(x+y-2b)t}}{\sigma \vee 1}\right)\right] f_1(y)dy \quad \text{if } t > 1; \text{ and if } t \in (0,1],$$

$$\Phi_1(x,\lambda) = \int_o^b [O(1)\ e^{xt} f_1(y)/\sigma t]\ dy.$$

To prove (22) observe that

$$\Phi_2(x,\lambda) = \psi(x,\lambda) \int_o^{x-\delta} \phi(y,\lambda) f_2(y)dy + \phi(x,\lambda) \int_{x+\delta}^b \psi(y,\lambda) f_2(y)dy =$$

$$= -\ \psi(x,\lambda) \int_o^{x-\delta} \phi^*(y,\lambda) f_2'(y)dy + \phi(x,\lambda) \int_{x+\delta}^b \psi^*(y,\lambda) f_2'(y)dy$$

where $\phi^*(y,\lambda) = \int_o^y \phi(r,\lambda)dr$, $\psi^*(y,\lambda) = \int_y^b \psi(r,\lambda)dr$.

Applying to this formula the estimations (7),(10) (or (11)), (12) and (13), we obtain (22). Finally,

$$|\Phi_3(x,\lambda)| = |\phi(x,\lambda)\int_b^\Delta \psi(y,\lambda)f(y)dy| \leqslant |\phi(x,\lambda)|\varepsilon(\int_b^\Delta |\psi(y,\lambda)|^2 dy)^{1/2}.$$

After application of (7) and (17) (or (18)), (23) follows for $0 < t \leqslant 1$, and $\Phi_3(x,\lambda) = \varepsilon\, O(1)(\exp(x-b)t)/\sigma$ for $t > 1$. Since $\Phi_3(x,s^2)$ is regular in the square $-1 < \sigma < 1$, $T-1 < t < T+1$ (if T is great enough) and takes conjugate values at symmetric points with respect to the imaginary axis, we can apply lemma 1, obtaining so (23).

Calling $C'=C \cap \{t > 1\}$, $C''=C \cap \{t \leqslant 1\}$, we obtain from (21):

$$\int_{C'} \Phi_1(x,s^2)\ 2s\ ds = -i \int_{C'} ds \int_o^b e^{is|x-y|}\ f_1(y)\ dy +$$

$$+ \eta . O(1) \int_{C'} (\frac{e^{-xt}}{|s|} + \frac{1}{|s|^2} + \frac{e^{(x-b)t}}{\sigma V1})|s||ds| =$$

$$= -i \int_C ds \int_o^b e^{is|x-y|} f_1(y)dy + \eta . O(1)(1+x^{-1} + (b-x)^{-1} + (b-x)^{-2})$$

where $O(1)$ does not depend on b if $|s|$ is great enough. Then,

$$(24) \quad \int_{C'} \Phi_1(x,s^2)2sds = \int_o^b (e^{iT|x-y|} - e^{-T|x-y|})f_1(y)|x-y|^{-1}dy + \eta O(1)$$

where $O(1)$ is independent of b if $b > b_1 > b_o$, $|s|$ is great enough and x runs in a fixed closed interval contained in $(0,b_1)$.

Applying (22), it is obtained:

$$(25) \quad \int_{C'} \Phi_2(x,s^2)2sds = O(1) \int_{C'} \exp(-\delta t)|ds|/T = O(1)/T\delta$$

Applying (23), it follows,

$$(26) \quad \int_{C'} \Phi_3(x,s^2)2sds = O(1)\ \varepsilon \int_{C'} \frac{e^{(x-b)t}}{\sigma V1} |s||ds| = O(1)\ \varepsilon$$

In (25) and (26) the O's are as in (24). From (24), (25) and (26) it follows that:

$$(27)\quad \mathrm{Im}\int_{C'}\Phi(x,s^2)2sds = \int_0^b \frac{\sin(T|x-y|)}{|x-y|} f_1(y)dy+O(1)(\varepsilon+\eta+1/T\delta)$$

To study the analogous integral on C" we define the auxiliary function:

$$\Omega(x,s^2) = \Phi(x,s^2) - \sum_{T^2-T<\lambda_n<T^2+T} c_n\ \psi_n(x)/(s^2-\lambda_n)$$

This function is regular on the strip $(T^2-T)^{1/2} < \sigma < (T^2+T)^{1/2}$ and therefore on $T-1/4 < \sigma < T+1/4$, if T is great enough. Besides, because of (21), (22) and (23) we have for $|\mathrm{Im}\ s| = |t| \leqslant 1$,

$$(28)\quad \begin{aligned}\Omega(x,s^2) &= O(1)\{\varepsilon+\eta+T^{-1}e^{bt}+\sum|c_n||\psi_n(x)|\}/t.\sigma = \\ &= O(1)\{\varepsilon+\eta+T^{-1}e^{bt}+(\sum|c_n|^2)^{1/2}(\sum|\psi_n|^2)^{1/2}\}/t.T\end{aligned}$$

Here the sums are on the set of $\lambda_n \in (T^2-T,T^2+T)$ and O is as in formula (24).

Since $f \in L^2(0,\Delta)$, $\sum|c_n|^2$ converges. In consequence, $\sum c_n^2$ in formula (28) is o(1) for $T \to \infty$. By hypothesis, $\sum|\psi_n(x)|^2 = O(1)$. Therefore, from (28) we obtain next formula (29) where o(1) is independent of b:

$$(29)\quad \Omega(x,s^2) = O(1)(\eta+\varepsilon+o(1))/tT$$

From (29) and lemma 1 we obtain that on C":

$$\Omega(x,s^2) = O(1)(\eta+\varepsilon+o(1))/T$$

Therefore,

$$(30)\quad \int_{C''}\Omega(x,s^2)2sds = O(1)(\varepsilon+\eta) + o(1)$$

On the other hand we have: $\left|\mathrm{Im}\int_{C''}\frac{2s}{s^2-\lambda_n}ds\right| = |\mathrm{var.Im}\ \lg(s^2-\lambda_n)| < \pi$.

Then

$$(31)\quad \left|\mathrm{Im}\int_{C''}\sum\frac{c_n\ \psi_n}{s^2-\lambda_n}\,2s\ ds\right| \leqslant \pi\ \sum|c_n\ \psi_n(x)| = o(1)$$

In formula (31) the sum is on the set of $\lambda_n \in (T^2-T,T^2+T)$. From (30) and (31), it follows,

$$(32) \qquad \mathrm{Im} \int_{C''} \Phi(x,s^2)\ 2s\ ds = O(1)(\varepsilon+\eta) + o(1)$$

Taking $\varepsilon=\eta$, from (20), (27) and (32) we obtain:

$$(33) \quad \sum_{\lambda_n<T^2} c_n\psi_n(x) = \frac{1}{\pi}\int_o^b \frac{\sin(T(x-y))}{x-y} f_1(y)dy + O(1)[\varepsilon+(T\delta)^{-1}]+o(1)$$

Since $f_1(y)=f(y)$ in a neighbourhood of x, from lemma 2 it follows that the integral in formula (33) converges to 1 when $T \to \infty$.

Therefore , $\overline{\lim_{T\to\infty}} \left| \sum_{\lambda_n<T^2} c_n\ \psi_n(x) - 1\right| \leqslant C\ \varepsilon$

where C is a constant which does not depend on b neither on x, if x belongs to a fixed closed interval contained in $(0,b_1)$. Since ε is arbitrarily small, we obtain $\sum c_n\psi_n(x) = 1$.

QED.

From Carleson's theorem (cf. [3] or [4]), it immediately follows next corollary.

COROLLARY. *Under the hypothesis of theorem* 1, $\sum c_n\psi_n(x)$ *converges to* f(x) *for a.e.* x.

5. APPLICATION TO BESSEL'S EQUATION.

In this section we apply the results of the preceding section to the case of Bessel's second order differential equation:

$$(34) \qquad Y''(x) + \left(\lambda - \frac{\nu^2-1/4}{x^2}\right) Y(x) = 0, \quad 0 < x < 1 ,$$

$$Y(1)\cos\alpha + Y'(1)\sin\alpha = 0.$$

In this case $\Delta=0$. In [5], Ch.4, §8, Weyl's theory is applied to this equation. There it is obtained that the functions satisfying the equation and the boundary conditions (6) with $\alpha=0$ are:

$$(35) \quad \phi(x,\lambda) = \frac{\pi\sqrt{x}}{2}\{J_\nu(xs)Y_\nu(s) - Y_\nu(xs)J_\nu(s)\}$$

$$\theta(x,\lambda) = \frac{\pi\sqrt{x}\ s}{2}\{J_\nu(xs)Y_\nu'(s) - Y_\nu(xs)J_\nu'(s)\} + \phi(x,\lambda)/2$$

and also that for $0 < \nu < 1$, $\nu \neq 1/2$ (limit-circle case):

$$m(\lambda) = -s\,\frac{c.s^{-\nu}J_\nu'(s) + s^\nu J_{-\nu}'(s)}{c.s^{-\nu}J_\nu(s) + s^\nu J_{-\nu}(s)} - \frac{1}{2}\ , \quad -\infty < c \leqslant \infty$$

The same formula holds for $\nu=1/2$, as it can be easily verified. In this case the equation has no singularity at $x=0$.
For $\nu=0$ (limit-circle case):

$$m(\lambda) = -s\,\frac{2J_o'(s)\ \lg(sc) - \pi Y_o'(s)}{2\ J_o(s)\ \lg(sc) - \pi Y_o(s)} - \frac{1}{2}\ , \quad 0 < c \leqslant \infty$$

and for $\nu \geqslant 1$ (limit-point case):

$$m(\lambda) = -s\,\frac{J_\nu'(s)}{J_\nu(s)} - \frac{1}{2}$$

where $m(\lambda)$ is limit of $l(\lambda) = -(\theta\,\mathrm{cotg}\beta + \theta')/(\phi\,\mathrm{cotg}\beta + \phi')$. The corresponding functions in the case of $\sin\alpha \neq 0$ are, as it is easy to see:

$$(36)\quad \begin{cases} \tilde\phi(x,\lambda) = \phi\cos\alpha + \theta\sin\alpha \\ \tilde\theta(x,\lambda) = -\phi\sin\alpha + \theta\cos\alpha \end{cases}$$

and therefore, $\tilde m(\lambda)$ is the limit of the circles ($b \to 0$):

$$\tilde l(\lambda) = -(\tilde\theta(b,\lambda)\ \mathrm{cotg}\beta + \tilde\theta'(b,\lambda))/(\tilde\phi(b,\lambda)\ \mathrm{cotg}\beta + \tilde\phi'(b,\lambda)).$$

From (36) we obtain:

$$\tilde l(\lambda) = \frac{\sin\alpha - (\cos\alpha)[(\theta\,\mathrm{cotg}\beta + \theta')/(\phi\,\mathrm{cotg}\beta + \phi')]}{\cos\alpha + (\sin\alpha)[(\theta\,\mathrm{cotg}\beta + \theta')/(\phi\,\mathrm{cotg}\beta + \phi')]} =$$

$$= \frac{\sin\alpha + l(\lambda)\ \cos\alpha}{\cos\alpha - l(\lambda)\ \sin\alpha}$$

and since $m(\lambda)$ is the limit of the circles $l(\lambda)$, we have:

$$(37)\quad \tilde m(\lambda) = \frac{m(\lambda)\ \cos\alpha + \sin\alpha}{-m(\lambda)\ \sin\alpha + \cos\alpha}$$

Using (36) and (37), we obtain for any α:

(38) $\quad \widetilde{\psi}(x,\lambda) = \widetilde{\theta} + \widetilde{m}(\lambda)\widetilde{\phi} = (\theta + m.\phi)/(\cos\alpha - m.\sin\alpha)$.

Using (35) and the corresponding values of $m(\lambda)$, we have in the different cases:

(39) $\quad \widetilde{\psi}(x,\lambda) =$

$$= \frac{\sqrt{x}(cs^{-\nu}J_\nu(xs)+s^\nu J_{-\nu}(xs))}{[(\sin\alpha)/2+\cos\alpha](cs^{-\nu}J_\nu(s)+s^\nu J_{-\nu}(s))+(\sin\alpha)s(cs^{-\nu}J'_\nu(s)+s^\nu J'_{-\nu}(s))},$$

$0 < \nu < 1$, $c \in (-\infty,+\infty]$;

(40) $\quad \widetilde{\psi}(x,\lambda) =$

$$= \frac{\sqrt{x}\ [(2/\pi)J_0(xs)\lg(sc)-Y_0(xs)]}{[(\sin\alpha)/2+\cos\alpha][\frac{2}{\pi}J_0(s)\lg(sc)-Y_0(s)]+(\sin\alpha)s[\frac{2}{\pi}J'_0(s)\lg(sc)-Y'_0(s)]},$$

$\nu=0$, $c \in (0,\infty]$;

(41) $$\widetilde{\psi}(x,\lambda) = \frac{\sqrt{x}\ J_\nu(xs)}{[(\sin\alpha)/2+\cos\alpha]J_\nu(s)+sJ'_\nu(s).\sin\alpha}, \quad \nu \geqslant 1 .$$

Then, the orthogonal systems which correspond to the different functions $m(\lambda)$ are given by: $\{\psi_n(x) = \sqrt{x}\ u_n(x)/\|\sqrt{.}\ u_n(.)\|_2\}$ where u_n is defined by (1) or (2) and satisfies (4). Using the asymptotic formulae for Bessel's functions, it follows that $\psi_n(x) = O(1)$ if x is fixed and $\lambda_n \sim n^2\pi^2$. Therefore, $\sum |\psi_n|^2 = O(1)$ for $T^2-T <$ $< \lambda_n < T^2+T$, and the hypotheses of theorem 1 are verified for these cases. In consequence, we have:

THEOREM 2. *Assume* $\nu > -1$ *and* $f \in L^2$. *Then* $S_nf = \sum_{j\leq n} c_j\psi_j$ *converges to* f *a.e.*.

6. POINTWISE CONVERGENCE OF EIGENFUNCTION EXPANSIONS IN L^p.

Let be $1 \leqslant p \leqslant \infty$ and $0 < u(x) < \infty$ a.e., $0 < x < \Delta$. We have defined as $L^p(u)$ the space of "functions" f such that $uf \in L^p$. *We shall write*: $\|uf\|_p = \|f\|_{p,u}$. Let $\{\psi_n\}$ be the system defined at the beginning of paragraph 3. With the same notation of that section and the same hypotheses on $m(\lambda)$, it holds:

THEOREM 3. *Assume* $f \in L^p(u) \cap L^1$, $1 \leqslant p < \infty$, *and that the following hypotheses are satisfied*:

1) *There exist the Fourier coefficients of* f *with respect to* $\{\psi_n\}$,

$c_n = \int_0^\Delta f\psi_n dx$, *and* $\int_0^\Delta |\psi(x,\lambda)|.|f(x)|dx$ *is locally bounded in* $R^2-\{\lambda_n\}$.

2) $\sum \{c_n\psi_n;\ T^2-T < \lambda_n < T^2+T\} = o(1)$ *in a given* $x \in (0,\Delta)$ *for* $T\to\infty$.

3) *If* $1/p + 1/q = 1$, *the following equalities hold with the* O's *independent of* b *whenever* $|\lambda| > \sigma_0(b)$:

$$\left.\begin{aligned}\left(\int_b^\Delta |\psi(x,s^2)/u(x)|^q dx\right)^{1/q} &= O(1)/\sigma t \quad \text{if } 0 < t \leqslant 1\\ &= O(1)\ e^{-bt}/\sigma \quad \text{if } t \geqslant 1\end{aligned}\right\}\ \sin\alpha \neq 0\ ,$$

$$\left.\begin{aligned}&= O(1)|s|/\sigma t \quad \text{if } 0 < t \leqslant 1\\ &= O(1)|s|e^{-bt}/\sigma \quad \text{if } t \geqslant 1\end{aligned}\right\}\ \sin\alpha = 0\ .$$

(*If* $q=\infty$, *the left hand side of* 3) *should be replaced by*

$\operatorname{ess.sup}_{x\in(b,\Delta)} |\psi(x,s^2)/u(x)|$).

Then, $\sum c_j\psi_j$ *and the ordinary Fourier series of* f *(in the sence of theorem* 1) *are equiconvergent: if the last one converges to the number* 1 *then also* $\sum c_j\psi_j = 1$.

Proof. From 1) it follows that the function:

$$\Phi(x,\lambda) = \psi(x,\lambda)\int_0^x \phi(x,\lambda)f(y)dy + \phi(x,\lambda)\int_x^\Delta \psi(y,\lambda)f(y)dy$$

is, for x fixed, an analytic function in the complex plane with simple poles at the points λ_n and residues $c_n\psi_n(x)$. In fact, it is sufficient to prove that $F(\lambda) = \int_x^\Delta \psi(y,\lambda)f(y)dy$ has at λ_n a simple pole with residue $\int_x^\Delta \phi(y,\lambda_n)f(y)dy/\|\phi(.,\lambda_n)\|_2^2$. Consider $F_b(\lambda) = \int_x^b \psi(y,\lambda)f(y)dy$, $b < \Delta$. Then, from section 3, we have:

$$F_b(\lambda) = G_b(\lambda) + (\lambda-\lambda_n)^{-1}\int_x^b \phi(y,\lambda_n)f(y)dy/\int_0^b |\phi(x,\lambda_n)|^2 dx$$

where G_b is regular in a neighbourhood of λ_n. For $b \to \Delta$, it follows that

$$F(\lambda) = G(\lambda) + (\lambda-\lambda_n)^{-1} \int_x^\Delta \phi(y,\lambda_n)f(y)dy/\|\phi\|_2^2 , \quad \lambda \,\overline{\in}\, \{\lambda_j\} .$$

Here, F and G are uniform limits on compact sets. In consequence, G is regular in a neighbourhood of λ_n and F has the desired properties.

Following the proof of theorem 1, we see that (20) holds. Now, we decompose f as in that theorem, $f = f_1 + f_2 + f_3$, with the same requirements on f_1 and f_2 and the support of f_3, but requiring in this case: $\|f_3\|_{p,u} < \varepsilon$.

Then (21) and (22) hold. Using in the proof of (23) that

$$\left|\int_b^\Delta \psi(y,\lambda)f(y)dy\right| \leqslant \|f_3\|_{p,u}\left(\int_b^\Delta |\psi(y,\lambda)/u(y)|^q dy\right)^{1/q}$$

and the hypothesis 3) instead of (17) or (18), we obtain (27). (32) also holds if we take into account (cf. formulae (28) and (31)) hypothesis 2). Therefore, (33) holds and the theorem follows.

QED.

7. APPLICATION TO SERIES OF BESSEL FUNCTIONS.

In this section we apply last theorem to Bessel and Dini series. The function

$$\text{(42)} \qquad \psi(x,\lambda) = \frac{\sqrt{x}\; J_\nu(xs)}{[\cos\alpha+(\sin\alpha)/2]\; J_\nu(s)+(\sin\alpha)sJ_\nu'(s)}, \quad -1 < \nu < \infty$$

corresponds to (39) if c=0 and $-1 < \nu < 0$, also to (39) if $c=\infty$ and $0 < \nu < 1$, to (40) if $c=\infty$ and $\nu=0$ and to (41) if $\nu \geqslant 1$. The system associated to this function is:

$$\text{(43)} \qquad \psi_n(x) = \frac{\sqrt{x}\; J_\nu(xs_n)}{\|\sqrt{x}\; J_\nu(xs_n)\|_2} \sim \sqrt{xs_n}\; J_\nu(xs_n)$$

where $\{s_n\}$ is the set of zeros of the denominator in (42) and such that $0 \leqslant \arg s_n \leqslant \pi/2$.

THEOREM 4. *Assume* $\nu > -1$, $\mu = (1/2 + \nu) \wedge 0$ *and* $f \in L^1(x^\mu)(\subset L^1)$.

Then, the expansion of f *with respect to the system* (43) *is equiconvergent with the ordinary Fourier expansion in the sense already defined for* $x \in (0,1)$.

Proof. Let us verify the hypotheses of theorem 3 with p=1. Since ψ and ψ_n are $O(x^\mu)$,1) follows. (As before, for this application $\Delta=0$ and 1 instead of 0). Using the asymptotic expansions in the denominator of (42) we know that $\lambda_n \sim n^2\pi^2$. Since $\psi_n(x)=O(1)$, to prove 2) it is enough to see that $c_n=o(1)$. From (43) it follows that:

$$(44)\qquad \psi_n(x) = \begin{cases} x^\mu O(1) & \text{if } 0 < x < 1 , \\ C_n \cos(xs_n - \nu\pi/2 - \pi/4) + O(1)/xs_n & \text{if } xs_n > 1 . \end{cases}$$

Here, the O's do not depend neither on x nor on n, and C_n is bounded above independently of n. Assume that $h=s_n^{-1/2}$; applying (44) we obtain:

$$|c_n| = \left|\int_0^1 \psi_n(x)f(x)dx\right| = O(1)\left(\int_0^h x^\mu |f|dx + \left|\int_h^1 \cos(ys_n - \nu\tfrac{\pi}{2} - \tfrac{\pi}{4})f(y)dy\right| +\right.$$

$$\left. + \int_h^1 |f(y)|/ys_n dy\right) = O(1)\left(\int_0^h x^\mu |f|dx + \left|\int_0^1 \cos(xs_n - \nu\tfrac{\pi}{2} - \tfrac{\pi}{4})f(x)dx\right| +\right.$$

$$\left. + s_n^{-1/2}\int_0^1 |f(y)|dy\right) = o(1)$$

as it easily follows from the hypoteses and Riemann-Lebesgue's lemma.

Let us prove 3). Since in this case the singular extreme is 0, the norm to be bounded is calculated on (0,1-b), $b \to 1$. Because of the asymptotic formula for J_ν :

$$(45)\qquad \sqrt{xs}\, J_\nu(xs)x^{-\mu} = O(e^{tx}) \qquad \text{if } 0 < t \ , \ |xs| > 1$$

Because of the behaviour of J_ν at the origin, the same formula holds for $|xs| \leqslant 1$. Then

$$(46)\qquad \sup_{x\in(0,1-b)} \left|\frac{\psi(x,s^2)}{x^\mu}\right| = \frac{O(e^{t(1-b)})}{|A\sqrt{s}\, J_\nu(s) + B\sqrt{s^3}\, J_\nu'(s)|} \ ;$$

$A = (\sin\alpha)/2 + \cos\alpha \quad , \quad B = \sin\alpha$.

Again, because of the symptotic formulae for Bessel functions and their derivatives:

$$(47) \qquad |A\sqrt{s}\, J_\nu(s) + B\sqrt{s^3}\, J_\nu'(s)| \geqslant M(1\wedge t)e^t(|A|+|Bs|)$$

with $M > 0$, independent of s if s is great enough. From (46) and (47), 3) follows. QED.

LEMMA 3. $1 \leqslant r \leqslant p \leqslant \infty$. *If* γ *and* β *are real numbers verifying* $\beta + 1/p < \gamma + 1/r$ *then* $L^p(x^\beta) \subset L^r(x^\gamma)$.

Proof. Assume $p < \infty$. From Hölder's inequality:

$$\int_0^1 (|f|x^\gamma)^r dx = \int_0^1 (|f|x^\beta)^r x^{(\gamma-\beta)r} dx \leqslant [\int_0^1 (|f|x^\beta)^p dx]^{r/p} \,.$$

$$.\ \|x^{(\gamma-\beta)r}\|_{(p/r)^*} \,.$$

Since $(p/r)^*=p/(p-r)$, last norm is finite whenever $r < p$ and $(\gamma-\beta)rp/(p-r) > -1$, or whenever $r=p$ and $\beta < \gamma$, i.e. if the hypothesis is verified. Therefore, $\|fx^\beta\|_p C \geqslant \|fx^\gamma\|_r$. This proves in part the lemma. The case $p=\infty$ follows in the same way . QED.

COROLLARY TO THEOREM 4. *Assume that* p *and* β*, real, verify*

$$(48) \qquad 1/p < 1 + \mu - \beta \quad , \quad p > 1 \,,$$

and that $f \in L^p(x^\beta)$ *on the interval* (0,1). *Then the expansion of* f *with respect to the system* (43) *converges a.e. to* f.

Proof. $\mu \leqslant 0$ implies the existence of a number $r > 1$ such that $1/p + \beta < 1/r < 1 + \mu$. From lemma 3: $f \in L^r \subset L^1(x^\mu)$. The corollary follows from the theorem of Carleson and Hunt on convergence of ordinary Fourier series, (cf. [3] and [4]).

QED.

REMARK. We could now obtain the analogous result for the systems described in §2. To avoid calculations, which for these systems are slightly more complicated, we give an alternative proof in next paragraph.

8. CONVERGENCE A.E. OF ORTHOGONAL SERIES OF BESSEL FUNCTIONS.

The aim of this section is to prove the following result.

THEOREM 5. *Let* $\{u_n\}$ *be a system of solutions of Bessel's equation of order* ν*,* ν *any real number, satisfying a real homogeneous boundary condition at* $x=1$*, orthogonal and complete in* $L^2(\sqrt{x})$*. Then* $\nu > -1$*, and if* p*,* β *and* ν *satisfy*

$$(49) \qquad 1 < p \quad ; \quad \{(\nu + 1/2)\wedge 0\} + 3/2 - \beta > 1/p \ ,$$

then for any $f \in L^p(x^\beta)$ *its expansion with respect to the system* $\{u_n\}$ *(in the sense of paragraph* 2*) converges to* f *a.e..*

This result is a consequence of the following theorem (which is proved in [2]) and theorem 7.

THEOREM 6. *Assume* $-1 < \nu$ *and* $\{u_n\}$ *given by* (1) *or* (2)*, and* $0 < x < 1$.

a) If $1 \leqslant p \leqslant \infty$, $-\infty < \beta < \infty$ *, then* $L^p(x^\beta)$ *admits convergence in the norm with respect to the system* $\{u_n\}$ *if and only if* (49) *and* (50) *hold:*

$$(50) \qquad p < \infty \quad ; \quad 1/p > 1/2 - \beta - \{(\nu + 1/2)\wedge 0\}.$$

b) Any system of solutions of Bessel's equation of order $\nu \in (-\infty,\infty)$, $\{w_n\}$*, satisfying a real homogeneous boundary condition* (4)*, orthogonal with respect to* xdx*, complete in* $L^2(x^{1/2})$*, is equivalent to one of the systems* (1) *or* (2) *in the sense that for each* n *there exists a number* $h_n \neq 0$ *such that* $w_n=h_n u_n$.

c) For each system, $\{a_n\}$ *is real if* $n \geqslant n_o$*, where* n_o *depends on the system. Moreover,* $a_n \to \infty$ *if* $n \to \infty$ *. So, we shall suppose that* $n \geqslant n_o$ *implies* $a_n < a_{n+1}$*. If* a_n *is not real then it is purely imaginary.*

d) Let $\{u_n\}$ *be a system obtained from* (1) *with* $K\neq 0$ *and* $\nu \in (-1,0)$*. We associate to it another system* $\{\tilde{u}_n\}$*, and precisely that obtained from* (1) *with the same boundary condition and* $K=0$*. If* $\{u_n\}$ *is obtained from formula* (2) *then* $\{\tilde{u}_n\}$ *will be that obtained (with the same boundary condition) from formula* (1) *for the case* $\nu=0$ *(and therefore* $K=0$*).*

For each system and its associate there exist a number m *and a fn.* G(x) *such that if* $S_n f(\tilde{S}_n f)$ *designates the* n^{th} *partial sum of the expansion of* f *with respect to the system* $\{u_n\}$ $(\{\tilde{u}_n\})$ *then, if* (49) *and* (50) *hold then for any* $f \in L^p(x^\beta)$ *it also holds:*

$$(51) \qquad |S_n f(x) - \tilde{S}_{n+m} f(x)| \leqslant C_n \, G(x) \|f.x^\beta\|_p \ , \quad \forall n \ ,$$

where $C_n = n^{2\nu}$ *if* $-1 < \nu < 0$, $C_n = 1/\lg n$ *if* $\nu = 0$, *and* G(x) *is finite everywhere and depends only on* ν, β *and* p.

THEOREM 7. *Assume* $\nu > -1$ *and* ν, β *and* p *satifying* (49). *Then, if* $\{u_n\}$ *is one of the systems defined by* (1) *or* (2) *then* $L^p(x^\beta)$ *admits convergence a.e. with respect to it.*

Proof. Assume that the system under consideration is defined by (1) with K=0. This system after multiplication by $\sqrt{x}$, coincides except for normalization with the system $\{\psi_n\}$ of paragraph 7.
The corollary to theorem 4 implies that if $F \in L^p(x^\delta)$ and $1/p < 1 \wedge (1+\mu-\delta)$, then the expansion of F with respect to the system $\{\sqrt{x}u_n(x)\}$ converges to F a.e..
Definig f and β by $F = f\sqrt{x}$, $\beta = \delta + 1/2$, we have: $f \in L^p(x^\beta)$, and in consequence, theorem 7 for K=0 follows.
From theorem 6,d), it follows that theorem 7 holds whenever the following inequalities hold:

$$(52) \qquad 1 < p < \infty \quad ; \quad 1/2 - \mu < \beta + 1/p < 3/2 + \mu .$$

Assume $f \in L^s(x^\gamma)$, and $\gamma + 1/s \leqslant 1/2 - \mu$. In this situation, there exist p and β verifying (52) and such that $p < s$. From lemma 3, we obtain $L^p(x^\beta) \supset L^s(x^\gamma)$.

QED.

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Universidad Nacional del Sur
Bahía Blanca,Argentina.

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