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EXISTENCE OF SOLUTIONS FOR GENERALIZED CAUCHY-GOURSAT TYPE PROBLEMS FOR HYPERBOLIC EQUATIONS

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INTRODUCTION. Let X be a Banach space and R the set of real numbers. If $S \subset R^n$ is a Lebesgue measurable set we will denote by $L_q(S,X)$ the set of all Lebesgue-Bochner measurable functions with power q summable on the set S into the Banach space X. Let $a_i \in R$, $a_i > 0$ (i= 1,2) and consider the closed intervals $I_i = <0,a_i>$ for i= 1,2 .

Let the graphs of the functions $g_1:I_1 \to I_2$, $g_2:I_2 \to I_1$ represent two continuous non-decreasing curves with (0,0) as their only point in common. Denote by Δ the set of all points (x_1,x_2)in the x_1x_2-plane such that $g_1(x_1) \leqslant x_2 \leqslant a_2$ and $g_2(x_2) \leqslant x_1 \leqslant a_1$. Take $p_i \in <1,\infty>$ (i= 0,1,2), $p_o \geqslant \max(p_1,p_2)$ and let $p_3=(p_o,p_1,p_2)$. In this paper the derivatives we understand in the sense of S.Sobolev (i.e., L. Schwartz derivatives representable by a Lebesgue-Bochner locally summable function).

In the first section we define a class of functions U_{p_3}. This class is a subset of the set of continuous functions u from $I_1 \times I_2$ into X which have S. Sobolev partial derivatives u_{x_1}, u_{x_2}, $u_{x_1x_2}$. We prove that the class U_{p_3} is linearly isomorphic to the product space $W_{p_3} = L_{p_o}(\Delta,X) \times L_{p_1}(I_1,X) \times L_{p_2}(I_2,X) \times X$. Thus the class U_{p_3} inherits a Banach type structure from the product space W_{p_3}.

In the sequel we shall be concerned with the following hyperbolic equation (0.1) $u_{x_1x_2}(x_1,x_2) = f(x_1,x_2)$ a.e. on Δ, where $f: \Delta \to X$ is a bounded Bochner measurable function on Δ .

Let Y= L(X,X) denote the collection of all linear continuous mappings from X into itself. Let $V = B(\Delta,X) \times L_p(I_1,Y) \times L_\infty(I_1,Y) \times L_p(I_1,X) \times L_p(I_2,Y) \times L_\infty(I_2,Y) \times L_p(I_2,X) \times X$ where $B(\Delta,X)$ is the space of bounded Bochner measurable functions with the supremum norm from Δ into X, and $p \in <1,\infty>$. Take $(f,\alpha_o,\alpha_1,\alpha_2,\beta_o,\beta_1,\beta_2,\gamma) \in V$ and let $\overline{p} = (\infty,p,p)$. By a solution of the generalized Cauchy-Goursat boundary problem in the class $U_{\overline{p}}$ for the hyperbolic equation we mean a function $u \in U_{\bar{p}}$ satisfying equation (0.1) and the boundary conditions (0.2)

$$u_{x_1}(\cdot,g_1(\cdot)) = \alpha_o(\cdot)\,.\,u(\cdot,g_1(\cdot)) + \alpha_1(\cdot)\,.\,u_{x_2}(\cdot,g_1(\cdot)) + \alpha_2(\cdot)$$

a.e. on I_1

$$u_{x_2}(g_2(\cdot),\cdot) = \beta_o(\cdot)\,.\,u(g_2(\cdot),\cdot) + \beta_1(\cdot)\,.\,u_{x_1}(g_2(\cdot),\cdot) + \beta_2(\cdot)$$

a.e. on I_2

$$u(0,0) = \gamma$$

In the third section we establish that the generalized Cauchy-Goursat boundary problem is meaningful, i.e., all the operations appearing in the definition of the problem make sense. Also we prove the existence and uniqueness of the solutions for the initial data from the product space V in the fourth section.

The continuity of the solutions on the initial data in the sense of the topology of the normed space V is also established.

1. DEFINITION OF THE CLASS U_{p_3} .

From now on when dealing with derivatives we will specify if they are to be taken in Sobolev sense, otherwise they will be taken in the usual sense.

DEFINITION 1.1. *A function* $u: I_1 \times I_2 \to X$ *belongs to the class* U_{p_3}*, if and only if,* u *is continuous on* $I_1 \times I_2$ *and there exist* $u_1 \in L_{p_1}(I_1 \times I_2, X)$, $u_2 \in L_{p_2}(I_1 \times I_2, X)$, $u_{12} \in L_{p_o}(\Delta, X)$ *such that:*

(a) $D_1 u = u_1$, $D_2 u = u_2$, $D_{12} u = u_{12}$ *where the derivatives are taken in Sobolev sense.*

(b) *There exists a set* $A_1 \subset I_1$ *of Lebesgue measure zero such that the function* $x_2 \to u_1(x_1,x_2)$ *is continuous on* I_2 *for every fixed* $x_1 \notin A_1$; *the function* $u_1(\cdot,g_1(\cdot)) \in L_{p_1}(I_1,X)$; $u_1(x_1,g_1(x_1)) = u_1(x_1,x_2)$ *for all* $x_2 \in \langle 0,g_1(x_1)\rangle$ *at each* $x_1 \in I_1$; $u_1(x_1,c_{x_1}) = u_1(x_1,x_2)$ *for all* $x_2 \in \langle c_{x_1},a_2\rangle$ *at each* $x_1 \in \langle 0,g_2(a_2)\rangle$ *, where* $c_{x_1} = \sup\{x_2 \in I_2 : g_2(x_2) = x_1\}$.

(c) *Symmetrically, there exists a set* $A_2 \subset I_2$ *of measure zero such that the function* $x_1 \to u_2(x_1,x_2)$ *is continuous on* I_1 *for every fixed* $x_2 \notin A_2$; *the function* $u_2(g_2(\cdot),\cdot) \in L_{p_2}(I_2,X)$;

$u_2(g_2(x_2),x_2)= u_2(x_1,x_2)$ *for all* $x_1 \in \langle 0,g_2(x_2)\rangle$ *at each* $x_2 \in I_2$;
$u_2(c_{x_2},x_2)= u_2(x_1,x_2)$ *for all* $x_1 \in \langle c_{x_2},a_1\rangle$ *at each* $x_2 \in \langle 0,g_1(a_1)\rangle$
where $c_{x_2}= \sup\{x_1 \in I_1; g_1(x_1)= x_2\}$.

DEFINITION 1.2. *Let* $s \in L_q(I_1 \times I_2,X)$, $q \geq 1$. *We define the operators* $J_i (i= 1,2)$ *by the formulas*

$$J_1 s.(x_1,x_2)= \int_0^{x_1} s(t,x_2)dt$$

$$J_2 s.(x_1,x_2)= \int_0^{x_2} s(x_1,r)dr \quad \text{for all} \quad (x_1,x_2) \in I_1 \times I_2$$

LEMMA 1.1. *The operators* $J_i (i= 1,2)$ *are well defined bounded linear operators on* $L_q(I_1 \times I_2)$.

LEMMA 1.2. *The operator* T *given by the formula:*

$$T(s,\phi,\psi,\gamma)= J_2 J_1 \bar{s} + J_1\phi + J_1\psi + \gamma$$

where $\bar{s}= s$ *on* Δ *and* $\bar{s}= 0$ *on* $I_1 \times I_2 \setminus \Delta$, *is a well defined linear operator from the product* W_{p_3} *into the space* U_{p_3}.

Proof. Let $u= T(s,\phi,\psi,\gamma)$, where $(s,\phi,\psi,\gamma) \in W_{p_3}$. Clearly, u is continuous on $I_1 \times I_2$ and $D_1 u= J_2\bar{s} + \phi$, $D_2 u= J_1\bar{s} + \psi$, $D_{12}u= \bar{s}$, where the derivatives are taken in the sense of Sobolev.

Letting $u_1= J_2\bar{s} + \phi$, $u_2= J_1\bar{s} + \psi$, $u_{12}= \bar{s}$ one can prove that u satisfies all the conditions specified in the definition of U_{p_3}. Thus, T is a well defined mapping.

From the linearity of the integral and the fact that U_{p_3} is a linear space follows the linearity of the operator T.

LEMMA 1.3. *Let the set* $A \subset I_1 \times I_2$, $A_1 \subset I_1$, $B_1 \subset I_2$ *be of measure zero. Then the boundary value problem*

$$w_{12}(x_1,x_2)= 0 \quad \text{if} \quad (x_1,x_2) \notin A$$

$$w_1(x_1,g_1(x_1))= 0 \quad \text{if} \quad x_1 \notin A_1$$

$$w_2(g_2(x_2),x_2)= 0 \quad \text{if} \quad x_2 \notin B_1$$

$$w(0,0)= 0$$

has a unique solution in the class U_{p_3}, *namely*, $w \equiv 0$, *where derivatives are taken in the sense of Sobolev*.

Proof. It is evident that $w \equiv 0$ satisfy the given boundary value problem. Suppose $w \in U_{p_3}$ is a solution of the boundary value problem. Then there exists a set $B_2 \subset I_2$ of measure zero such that

$w_2(x_1,x_2)= J_1w_{12}.(x_1,x_2) + w_2(0,x_2)$ if $x_2 \notin B_2$, $x_1 \in I_1$.

The equation $J_2J_1w_{12}.(x_1,x_2)= 0$ for all $(x_1,x_2)\in I_1 \times I_2$ implies the existence of a set $B_3 \subset I_2$ of measure zero such that $J_1w_{12}.(x_1,x_2)= 0$ if $x_2 \notin B_3$, $x_1 \in I_1$.

Hence $w_2(x_1,x_2)= w_2(0,x_2)= w_2(g_2(x_2),x_2)= 0$ if $x_1 \in I_1$, $x_2 \notin B_1 \cup B_2 \cup B_3$.

Similarly we obtain sets $A_2, A_3 \subset I_1$ of measure zero such that

$$w_1(x_1,x_2)= 0 \text{ if } x_1 \notin A_1 \cup A_2 \cup A_3, \quad x_2 \in I_2 .$$

Also there exist sets $A_4 \subset I_1$, $B_4 \subset I_2$ of measure zero such that

$$w(x_1,x_2)= J_1w_1.(x_1,x_2) + w(0,x_2) \text{ if } x_2 \notin B_4, \quad x_1 \in I_1$$

$$w(x_1,x_2)= J_2w_2.(x_1,x_2) + w(x_1,0) \text{ if } x_1 \notin A_4, \quad x_2 \in I_2$$

Hence,

$$w(x_1,x_2)= w(0,x_2) \quad \text{if} \quad x_2 \notin B_4, \quad x_1 \in I_1$$

$$w(x_1,x_2)= w(x_1,0) \quad \text{if} \quad x_1 \notin A_4, \quad x_2 \in I_2$$

The last two equalities and the continuity of w imply that there exists $k \in X$ such that $w(x_1,x_2)= k$ for all $(x_1,x_2) \in I_1 \times I_2$.

So, from $w(0,0)= 0$ we have that $w \equiv 0$ on $I_1 \times I_2$.

THEOREM 1.1. *The map* T *defined in Lemma 1.2 establishes a linear isomorphism between the product* W_{p_3} *and the space* U_{p_3}. *The inverse map* F *is given by the formulas*:

$$s= u_{12} \quad \text{a.e. on } \Delta$$

$$\phi= u_1(\cdot,g_1(\cdot)) \quad \text{a.e. on } I_1$$

$$\psi= u_2(g_2(\cdot),\cdot) \quad \text{a.e. on } I_2$$

$$\gamma= u(0,0)$$

Proof. It is clear that F is a well defined linear map. Let $(s,\phi,\psi,\gamma) \in W_{p_3}$, $u= T(s,\phi,\psi,\gamma)$, and $F(u)= (\bar{s},\bar{\Phi},\bar{\Psi},\bar{\gamma})$.

By definition of the map F we have:

$$\bar{s}= u_{12} \quad \text{a.e. on } \Delta$$

$$\bar{\Phi}= u_1(\cdot,g_1(\cdot)) \quad \text{a.e. on } I_1$$

$$\bar{\Psi}= u_2(g_2(\cdot),\cdot) \quad \text{a.e. on } I_2$$

$$\bar{\gamma}= \gamma$$

But, $u_{12}= s$, $u_1= J_2s + \phi$, $u_2= J_1s + \psi$. Hence, $s= \bar{s}$ a.e. on Δ, $\phi= \bar{\Phi}$ a.e. on I_1, $\psi= \bar{\Psi}$ a.e. on I_2, $\gamma= \bar{\gamma}$, or equivalently $F \circ T= I_{W_{p_3}}$ i.e. the identity map on W_{p_3}.

Let $v \in U_{p_3}$, $F(v)= (s,\phi,\psi,\gamma)$, $u= T(s,\phi,\psi,\gamma)$.

Letting w= u - v we obtain:

$$\begin{array}{ll} w \in U_{p_3} & \\ w_{12} = 0 & \text{a.e. on } \Delta \\ w_1(\cdot, g_1(\cdot)) = 0 & \text{a.e. on } I_1 \\ w_2(g_2(\cdot), \cdot) = 0 & \text{a.e. on } I_2 \\ w(0,0) = 0 & \end{array}$$

Therefore from Lemma 1.3 it follows that $w \equiv 0$ on $I_1 \times I_2$, or equivalently $T \circ F = I_{U_{p_3}}$. This completes the proof of the theorem.

COROLLARY 1.1. *The space* U_{p_3} *is a Banach space with the norm* $|\ |$ *defined by the formula*

$$|u| = \|F(u)\| \quad \textit{for all } u \in U_{p_3}$$

The operator T *establishes a linear isomorphism and isometry between the spaces* W_{p_3} *and* U_{p_3}.

2. A CAUCHY-GOURSAT TYPE PROBLEM IN THE CLASS $U_{\overline{p}}$.

We are going to enunciate a series of hypothesis which will be used throughout the remainder of this paper.

HYPOTHESIS (A_1). The functions g_1, g_2 are continuous, strictly increasing, $g_i(0) = 0$ for i= 1,2 , and $x_2 = g_1(x_1)$, $x_1 = g_2(x_2)$ imply $x_1 = x_2 = 0$.

HYPOTHESIS (A_2). The functions g_1, g_2 satisfy hypothesis (A_1), g_i^{-1} (i= 1,2) are absolutely continuous functions on their domain of definition, and the derivatives $(g_i^{-1})'$ (i= 1,2) are essentially bounded functions.

HYPOTHESIS (A_3). The curves g_1, g_2 satisfy hypothesis (A_2) and they are absolutely continuous on their domain of definition.

HYPOTHESIS (A_4). The functions g_i(i= 1,2) are such that

$$g_i(x_i) \leqslant x_i \quad \text{for all} \quad x_i \in I_i \quad (i=1,2)$$

HYPOTHESIS (A_5). $(f, \alpha_o, \alpha_1, \alpha_2, \beta_o, \beta_1, \beta_2, \gamma) \in V$.

DEFINITION 2.1. *Under Hypothesis* (A_1) *and* (A_4) *we want to find a function* $u \in U_{\overline{p}}$ *satisfying equation* (0.1) *and the boundary conditions* (0.2) *where the derivatives are understood in the sense of Sobolev. Such a function* u, *if it exists, will be called a solution of the Cauchy-Goursat problem for equation* (0.1) *under the bounda-*

ry conditions (0.2).

3. THE CAUCHY-GOURSAT PROBLEM IS MEANINGFUL.

The following two lemmas will be needed in this section.

LEMMA 2.1. *If* $w_i \in L_\infty(I_i,Y)$, $\psi_j \in L_p(I_j,X)$ *and* g_i *satisfy Hypothesis* (A_2), *then the function* $x_i \to w_i(x_i)(\psi_j o g_i(x_i))$ *belongs to the space* $L_p(I_i,X)$, *where* $i, j \in \{1,2\}$, $i \neq j$.

LEMMA 2.2. *If* $f:\Delta \to X$ *is Bochner measurable and bounded on* Δ, $w_i \in L_\infty(I_i,Y)$ $(i=1,2)$, $g_i (i=1,2)$ *satisfy Hypothesis* (A_2), *then the function* $x_i \to w_i(x_i)(\int_0^{x_i} f(t,g_i(x_i))dt)$ *belongs to the space* $L_p(I_i,X)$.

THEOREM 2.1. *Under Hypothesis* (A_2) *and* (A_5) *the Cauchy-Goursat problem is meaningful.*

Proof. Because of Theorem 1.1 every $u \in U_{\bar{p}}$ has a representation of the form $u = J_2J_1s + J_1\phi + J_2\psi + \gamma$ where $(s,\phi,\psi,\gamma) \in W_{\bar{p}}$. For any function $G_i \in L_p(I_i,X)$ $(i=1,2)$ we are going to write $J_iG_i.(x_1,x_2) = J_iG_i.(x_i)$, for all $x_i \in I_i$.

To find $u \in U_{\bar{p}}$ satisfying the generalized Cauchy-Goursat boundary problem is equivalent to find $(s,\phi,\psi,\gamma) \in W_{\bar{p}}$ such that

$$(2.1)\quad \begin{cases} \phi(\cdot) = \alpha_0(\cdot)[J_1J_2f.(\cdot,g_1(\cdot)) + J_1\phi.(\cdot) + J_2\psi.(g_1(\cdot)) + \gamma] + \\ \qquad + \alpha_1(\cdot)[J_1f.(\cdot,g_1(\cdot)) + \psi(g_1(\cdot))] + \alpha_2(\cdot) \text{ a.e. on } I_1 \\ \psi(\cdot) = \beta_0(\cdot)[J_1J_2f.(g_2(\cdot),\cdot) + J_1\phi.(g_2(\cdot)) + J_2\psi.(\cdot) + \gamma] + \\ \qquad + \beta_1(\cdot)[J_2f.(g_2(\cdot),\cdot) + \phi(g_2(\cdot))] + \beta_2(\cdot) \text{ a.e. on } I_2 \end{cases}$$

From Lemmas 2.1 and 2.2 it follows that the equations of system (2.1) are meaningful. This completes the proof of the theorem.

3. THE OPERATORS H AND J.

DEFINITION 3.1. *Under hypothesis* (A_2) *and* (A_5) *let us define the operators* H *and* J *from the space* $L_p(I_1,X) \times L_p(I_2,X)$ *into itself by the formulas*

$$H\begin{pmatrix}\phi\\ \psi\end{pmatrix}=\begin{pmatrix}\alpha_1(x_1)(\psi(g_1(x_1))\\ \beta_1(x_2)(\phi(g_2(x_2))\end{pmatrix}\quad J\begin{pmatrix}\phi\\ \psi\end{pmatrix}=\begin{pmatrix}\alpha_o(x_1)[J_1\phi.(x_1)+J_2\psi.(g_1(x_1))]\\ \beta_o(x_2)[J_1\phi.(g_2(x_2))+J_2\psi.(x_2)]\end{pmatrix}$$

LEMMA 3.1. *The operators* H *and* J *are well defined.*

For every $\phi\in L_p(I_1,X)$, $\psi\in L_p(I_2,X)$, let $\tau=\begin{pmatrix}\phi\\ \psi\end{pmatrix}$ and

$$\tau_o=\begin{pmatrix}\alpha_o(x_1)[J_1J_2f.(x_1,g_1(x_1))+\gamma]+\alpha_1(x_1)[J_1f.(x_1,g_1(x_1))]+\alpha_2(x_1)\\ \beta_o(x_2)[J_1J_2f.(g_2(x_2),x_2)+\gamma]+\beta_1(x_2)[J_2f.(g_2(x_2),x_2)]+\beta_2(x_2)\end{pmatrix}\tag{3.1}$$

assuming that Hypothesis (A_2) and (A_5) hold.

By means of the operators H and J equation (2.1) can be written

$$\tau=J\tau+H\tau+\tau_o\tag{3.2}$$

Thus to solve the Cauchy-Goursat problem is equivalent to find a solution τ of equation (3.2).

DEFINITION 3.2. *Under Hypothesis* (A_1) *define the functions* $\lambda_i^n: I_i\to I_i$, (i= 1,2), n *a non-negative integer, by the formulas*

$$\lambda_i^o(x_i)=x_i \qquad \textit{for all } x_i\in I_i$$

$$\lambda_i^1(x_i)=\lambda_i(x_i)=g_j o g_i(x_i) \qquad \textit{for all } x_i\in I_i,\ (j=1,2,\ j\neq i)$$

$$\textit{and}\quad \lambda_i^n(x_i)=\lambda_i(\lambda_i^{n-1}(x_i)) \qquad \textit{for all } x_i\in I_i,\ n>1$$

LEMMA 3.2. *If* g_i (i= 1,2) *satisfy the Hypothesis* (A_3), *then* $(\lambda_i^n)^{-1}$ (i= 1,2) *are strictly increasing absolutely continuous functions on* $<0,\lambda_i^n(a_i)>$, *and the derivatives* $((\lambda_i^n)^{-1})'$ *are essentially bounded, where* n= 0,1,2,....

LEMMA 3.3. *If* g_i (i=1,2) *satisfy Hypothesis* (A_1), *then the sequences* λ_i^n (i= 1,2) *are non-increasing sequences converging uniformly toward zero in* I_i.

This Lemma is proven by J. Kisynski and M. Bielecki in [3].

DEFINITION 3.3. *Under Hypothesis* (A_2) *and* (A_5) *let us define the functions*:

$$\mu_1(x_1)=\alpha_1(x_1)\beta_1(g_1(x_1)) \qquad \textit{for all}\quad x_1\in I_1$$

$$\mu_{1n}(x_1) = \mu_1(x_1) \ldots \mu_1(\lambda_1^{n-1}(x_1)) \quad \textit{for all} \quad x_1 \in I_1,\ n > 1$$

$$\mu_2(x_2) = \beta_1(x_2)\alpha_1(g_2(x_2)) \quad \textit{for all} \quad x_2 \in I_2$$

$$\mu_{2n}(x_2) = \mu_2(x_2) \ldots \mu_2(\lambda_2^{n-1}(x_2)) \quad \textit{for all} \quad x_2 \in I_2,\ n > 1$$

LEMMA 3.4. *Let* $\overline{\alpha}_1 = \alpha_1/(g_1{}')^{1/p}$, $\overline{\beta}_1 = \beta_1/(g_2{}')^{1/p}$, *where* $\alpha_1 : I_1 \to Y$, $\beta_1 : I_2 \to Y$. *If* $\overline{\alpha}_1$ *and* $\overline{\beta}_1$ *are essentially bounded functions, and* g_i *satisfy Hypothesis* (A_3), *then* $\mu_{in}/((\lambda_i^n)')^{1/p}$ $(i = 1,2)$ *are essentially bounded functions on* I_i *for every natural number* n.

DEFINITION 3.4. *Let* $f \in L_p(\Delta, X)$, $p \geq 1$. *Define* $\|f\|_k =$
$$= \sup\{e^{-k(x_1+x_2)}\left(\int_0^{x_1}\int_0^{x_2} \|\overline{f}\|^p\right)^{1/p} : (x_1,x_2) \in \Delta\}$$ *where* $k > 0$ *and* $\overline{f} = f$ *on* Δ, $\overline{f} = 0$ *on* $I_1 \times I_2 \setminus \Delta$. *From now on we will write* f *instead of* $\overline{f}$.

One can prove that $(L_p(\Delta,X), \|\ \|_k)$ is a Banach space for $k > 0$. This type of norm was introduced by M.A. Bielecki in [2].

DEFINITION 3.5. *If* $(\phi,\psi) \in L_p(I_1,X) \times L_p(I_2,X)$, $p \geq 1$, *we define* $\|(\phi,\psi)\|_k = \max(\|\phi\|_k, \|\psi\|_k)$, $k > 0$. *It is known that* $L_p(I_1,X) \times L_p(I_2,X)$ *is complete under the above defined norm.*

HYPOTHESIS (A_6). The functions $g_i (i = 1,2)$ satisfy Hypothesis (A_3) and (A_4). The functions $\alpha_1 \in L_\infty(I_1,Y)$ and $\beta_1 \in L_\infty(I_2,Y)$ are such that $\overline{\alpha}_1 \in L_\infty(I_1,Y)$, $\overline{\beta}_1 \in L_\infty(I_2,Y)$, and $\lim_{A_1 \not\ni x_1 \to 0^+} \overline{\alpha}_1(x_1) = \overline{\alpha}_1(0)$, $\lim_{A_2 \not\ni x_2 \to 0^+} \overline{\beta}_1(x_2) = \overline{\beta}_1(0)$ exist in the sense of the norm of Y where
$$\overline{\alpha}_1 = \frac{\alpha_1}{(g_1')^{1/p}}, \quad \overline{\beta}_1 = \frac{\beta_1}{(g_2')^{1/p}}$$
and $A_i \subset I_i$ $(i = 1,2)$ are of Lebesgue measure zero. Finally, $\|\overline{\alpha}_1(0)\overline{\beta}_1(0)\| < 1$.

LEMMA 3.5. *Under Hypothesis* (A_6) *the operator* $A = (I-H)^{-1}$, *from* $L_p(I_1 \times X) \times L_p(I_2 \times X)$ *into itself, is bounded and linear. Moreover, there exists* M *independent of* $k > 0$ *such that* $\|A\|_k \leq M$.

Proof. It is clear that H is a well defined linear operator. We have (3.3) $\|H\|_k \leq M_1$ where $M_1 = \max(\|\alpha_1\|_\infty \|(g_1^{-1})'\|_\infty^{1/p}, \|\beta_1\|_\infty \|(g_2^{-1})'\|_\infty^{1/p})$ is independent of $k > 0$.

From the definition of the operator H it follows that

$$H^{2n}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix}\mu_{1n}(x_1)\cdot\phi(\lambda_1^n(x_1))\\ \mu_{2n}(x_2)\cdot\psi(\lambda_2^n(x_2))\end{pmatrix}$$

for all $\phi \in L_p(I_1,X)$, $\psi \in L_p(I_2,X)$, $n= 1,2,\dots$

We have the following inequality for every natural number n:

$$(3.4)\qquad \int_0^{x_1}\int_0^{x_2}\|\mu_{1n}(\xi)\cdot\phi(\lambda_1^n(\xi))\|^p d\eta d\xi \leqslant$$
$$\leqslant \int_0^{x_1}\int_0^{x_2}\|((\lambda_1^n)^{-1})')^{1/p}(t)\mu_{1n}((\lambda_1^n)^{-1}(t))\|^p\|\phi(t)\|^p d\eta dt$$

Let us note that $\lim\limits_{C\not\ni x_1\to 0^+}\mu_1(x_1)/(\lambda_1'(x_1))^{1/p}= \overline{\alpha}_1(0)\overline{\beta}_1(0)$ where $C= A_1 U g_1^{-1}(A_2)$ is of Lebesgue measure zero. Thus, given $q > 0$ such that $\|\overline{\alpha}_1(0)\overline{\beta}_1(0)\| < q^2 < 1$ there exists $\delta > 0$ such that $\|\mu_1(x_1)/(\lambda_1'(x_1))^{1/p}\| < q^2$ if $x_1 \notin C$, $x_1 \in I_1$, $0 < x_1 < \delta$. Since by Lemma 3.3 there exists n_o such that $0 \leqslant \lambda_1^{n-1}(x_1) < \delta$ for all $x_1 \in I_1$, $n \geqslant n_o$ we have

$$(3.5)\qquad \|\mu_1(\lambda_1^{n-1}(x_1))/(\lambda_1'(\lambda_1^{n-1}(x_1)))^{1/p}\| < q^2 \text{ for all } n \geqslant n_o \text{ and}$$

$x_1 \notin (\lambda_1^{n-1})^{-1}(C)$, which is a set of Lebesgue measure zero.

From (3.4), (3.5) we obtain

$$(3.6)\qquad \|\mu_{1n}(x_1)\cdot\phi(\lambda_1^n(x_1))\|_k \leqslant (\|\overline{\alpha}_1\|_\infty\|\overline{\beta}_1\|_\infty)^{n_o-1}q^{2(n-n_o+1)}$$

Similarly there exists n_1 such that

$$(3.7)\qquad \|\mu_{2n}(x_2)\cdot\psi(\lambda_2^n(x_2))\|_k \leqslant (\|\overline{\alpha}_1\|_\infty\|\overline{\beta}_1\|_\infty)^{n_1-1}q^{2(n-n_1+1)}$$

Hence $\|H^{2n}\|_k \leqslant M_2 q^{2n}$ for all $n \geqslant \max(n_o,n_1)$, where M_2 independent of k is defined in an obvious way.

Noting that $A= (I + H)\cdot B$, where $B= I + H^2 + H^4 + \dots + H^{2n} + \dots$ and using (3.3), (3.6), and (3.7) we can obtain the desired result.

DEFINITION 3.6. *Let* $C(I_i)$ $(i= 1,2)$ *denote the set of all continuous functions* $f: I_i \to X$. *For every* $f \in C(I_i)$ *define* $\|f\|_{kc}^{(i)} = \sup\{e^{-kx_i}\|f(x_i)\| : x_i \in I_i\}$.

It is known that $(C(I_i), \|\ \|_{kc}^{(i)})$ is a Banach space.

LEMMA 3.6. *The operators* T_i (i= 1,2) *from the space* $(L_p(I_i,X),\| \|_k)$ *into the space* $(C(I_i),\| \|_{kc}^{(i)})$ *defined by the formula*

$$(T_i\phi)(x_i)= \int_0^{x_i} \phi(t)\,dt \quad \textit{for all} \quad x_i \in I_i$$

are well defined bounded linear operators and $\|T_i\|_k=$ $= (kep)^{1/p}\, a_i^{(p-1)/p}$ *if* $1/pk \leqslant \min(a_1,a_2)$, $p \geqslant 1$, $k > 0$.

LEMMA 3.7. *Let* $\alpha: I_1 \to Y$, $\beta: I_2 \to Y$ *be* p-*Bochner summable functions on* I_1 *and* I_2 *respectively. Then, the operators*

$$H_i:(C(I_i),\| \|_{kc}^{(i)}) \to (L_p(I_i,X),\| \|_k) \qquad (i= 1,2)$$

defined by the formulas

$$(H_1f)(x_1)= \alpha(x_1).f(x_1) \quad \textit{for all} \quad x_1 \in I_1,\ f \in C(I_1)$$
$$(H_2g)(x_2)= \beta(x_2).g(x_2) \quad \textit{for all} \quad x_2 \in I_2,\ g \in C(I_2)$$

are well defined bounded linear operators. Moreover, for any $\varepsilon > 0$ *there exists* k_o *such that* $\|H_i\|_k \leqslant (\varepsilon/pke)^{1/p}$ *for all* $k \geqslant k_o$ (i=1,2).

Proof. It is clear that H_i (i= 1,2) are well defined and linear. We have also

$$(3.8) \qquad \left(\int_0^{x_1}\int_0^{x_2} \|\alpha(t).f(t)\|^p dr\, dt\right)^{1/p} \leqslant$$
$$\leqslant \|f\|_{kc}^{(1)} \left(\int_0^{x_1}\int_0^{x_2} \|\alpha(t)\|^p e^{kpt} dr\, dt\right)^{1/p}$$

Let k_1 be such that $\frac{1}{pk} \leqslant \min(a_1,a_2)$ for all $k \geqslant k_1$. For any $k > k_1$ we have:

$$(3.9) \qquad e^{-pk(x_1+x_2)} \int_0^{x_1}\int_0^{x_2} \|\alpha(t)\|^p e^{kpt} dr\, dt \leqslant$$
$$\leqslant (pke)^{-1} e^{-kpx_1} \int_0^{x_1} \|\alpha(t)\|^p e^{kpt} dt \ , \quad (x_1,x_2) \in I_1 \times I_2$$

Let $s \geqslant 0$ be a simple function defined on I_1 such that $\int_0^{a_1} |\,\|\alpha(t)\|^p - s(t)|\,dt < \varepsilon/2$ where $\varepsilon > 0$ is given. So,

$$(3.10) \qquad \int_0^{x_1} \|\alpha(t)\|^p e^{-kp(x_1-t)} dt < \frac{\varepsilon}{2} + \frac{\|s\|_\infty}{kp}$$

From inequalities (3.8), (3.9),(3.10) it follows that

$$\|H_1\|_k \leq (pke)^{-1/p}[\frac{\varepsilon}{2} + \frac{\|s\|_\infty}{kp}]^{1/p}$$

Similarly we obtain $\|H_2\|_k \leq (pke)^{-1/p}[\frac{\varepsilon}{2} + \frac{\|\bar{s}\|_\infty}{kp}]^{1/p}$ for some simple function $\bar{s}$ defined on I_2. Let k_2 be such that $\frac{\|s\|_\infty}{k_2 p} < \frac{\varepsilon}{2}$, $\frac{\|\bar{s}\|_\infty}{k_2 p} < \frac{\varepsilon}{2}$. Thus for all $k > k_o = \max(k_1, k_2)$ we have $\|H_i\|_k \leq$ $\leq (\varepsilon/pke)^{1/p}$ for i= 1,2.

LEMMA 3.8. *Assume* g_1, g_2 *satisfy Hypothesis* (A_1) *and* (A_4). *The operators* $T_j : (L_p(I_i, X), \| \ \|_k) \to (C(I_m), \| \ \|_{kc}^{(m)})$ *(where* $p \geq 1, i,m \in \{1,2\}$, $i \neq m$, $j = i+2$) *defined by the formulas* $(T_j\phi)(x_m) = \int_0^{g_m(x_m)} \phi(t)\,dt$, *for all* $x_m \in I_m$ *are well defined bounded linear operators. Moreover, if* $\frac{1}{kp} \leq \min(a_1, a_2)$ *then* $\|T_j\|_k \leq (kep)^{1/p} a_i^{(p-1)/p}$.

LEMMA 3.9. *Under Hypothesis* (A_1), (A_4) *and* (A_5) *the operator* J *(Definition* 3.1) *is a well defined bounded linear operator. Moreover, for any given* $\varepsilon > 0$ *there exists* k_o *such that* $\|J\|_k \leq \varepsilon$ *for all* $k \geq k_o$.

The proof follows easily from Lemmas 3.6, 3.7 and 3.8.

4. EXISTENCE THEOREMS FOR THE CAUCHY-GOURSAT PROBLEM IN THE CLASS $U_{\bar{p}}$.

Under Hypothesis (A_5) and (A_6) equation (3.2) can be written:
(4.1) $\tau = (I-H)^{-1}J\tau + (I-H)^{-1}\tau_o$. Let us define the operator F_1 by the formula: (4.2) $F_1\tau = AJ\tau + A\tau_o$. Clearly F_1 is a well defined operator from the space $L_p(I_1,X) \times L_p(I_2,X)$ into itself because of Lemmas 3.5 and 3.9.

Moreover, from equation (4.1) it follows that to find a solution of the Cauchy-Goursat problem in the class $U_{\bar{p}}$ is equivalent to find a fixed point of the operator F_1.

THEOREM 4.1. *Under Hypothesis* (A_5) *and* (A_6) *the Cauchy-Goursat problem has a unique solution in the class* $U_{\bar{p}}$.

Proof. Note that $\|F_1\tau_1 - F_1\tau_2\|_k \leq M\|J\|_k\|\tau_1 - \tau_2\|_k$ where M is as in Lemma 3.5. From Lemma 3.9 it follows that there exists k_o such that $\|F_1\tau_1 - F_1\tau_2\|_k \leq \frac{1}{2}\|\tau_1 - \tau_2\|_k$ for all $k > k_o$.

Thus for any fixed $k > k_o$ the operator F_1 has a unique fixed point because of Banach Fixed Point Theorem.

THEOREM 4.2. *Under Hypothesis* (A_1), (A_4) *and* (A_5) *the boundary value problem* $u_{12}= f$ *a.e. in* Δ; $u_1(\cdot,g_1(\cdot))= \alpha_o(\cdot).u(\cdot,g_1(\cdot)) + \alpha_2(\cdot)$ *a.e. in* I_1; $u_2(g_2(\cdot),\cdot)= \beta_o(\cdot).u(g_2(\cdot),\cdot) + \beta_2(\cdot)$ *a.e. in* I_2; $u(0,0)= \gamma$, *has a unique solution in the class* $U_{\bar{p}}$.

Proof. This theorem is proven as the preceding one considering the operator $F_2\tau= J\tau + \tau_o$ from the space $L_p(I_1,X) \times L_p(I_2,X)$ into itself, where

$$\tau_o=\begin{pmatrix}\alpha_o(x_1)(J_1J_2f.(x_1,g_1(x_1)) + \gamma) + \alpha_2(x_1)\\ \beta_o(x_2)(J_1J_2f.(g_2(x_2),x_2) + \gamma) + \beta_2(x_2)\end{pmatrix}$$

REMARK. If in the Cauchy-Goursat problem we let $\alpha_o= 0$, $\beta_o= 0$, then we cannot weaken the conditions on $g_i(i= 1,2)$ as we did in Theorem 4.2.

THEOREM 4.3. *If* $g_i(i=1,2)$ *are non-decreasing functions, and* f, α_2, β_2, γ *are as in Hypothesis* (A_5) *then the boundary value problem* $u_{12}= f$ *a.e. in* Δ, $u_1(\cdot,g_1(\cdot))= \alpha_2(\cdot)$ *a.e. in* I_1, $u_2(g_2(\cdot),\cdot)=\beta_2(\cdot)$ *a.e. in* I_2, $u(0,0)= \gamma$, *has a unique solution in the class* $U_{\bar{p}}$.

Proof. From the isomorphism of the spaces $U_{\bar{p}}$ and $W_{\bar{p}}$ it follows that the unique solution of our boundary value problem in the class $U_{\bar{p}}$ is $u= J_1J_2f + J_1\alpha_2 + J_2\beta_2 + \gamma$.

5. CONTINUOUS DEPENDENCE OF THE SOLUTION ON THE INITIAL DATA FOR THE CAUCHY-GOURSAT PROBLEM IN THE CLASS $U_{\bar{p}}$.

Throughout this section we assume that the functions $g_i(i=1,2)$ satisfy Hypothesis (A_3) and (A_4). Let V_1 be the subset of V such that the coordinates α_1 and β_1 satisfy the conditions specified in Hypothesis (A_6).

DEFINITION 5.1. *We define the operator* $S:V_1 \to U_{\bar{p}}$ *as follows* :

$S(v) = u$, $v \in V_1$, *if and only if,* u *is the unique solution of the Cauchy-Goursat problem corresponding to the initial data* v.

Note that S is a well defined operator because of Theorem 4.1. It is easy to show that V_1 is a normed space. In $U_{\bar{p}}$ consider the norm introduced in Corollary 1.1.

To prove continuous dependence of the solution on the initial data for the Cauchy-Goursat problem in the class $U_{\bar{p}}$ is equivalent to show that the operator S is continuous.

Take $v_n = (f, \alpha_{on}, \alpha_{1n}, \alpha_{2n}, \beta_{on}, \beta_{1n}, \beta_{2n}, \gamma_n)$ in V_1 and $v = (f, \alpha_o, \alpha_1, \alpha_2, \beta_o, \beta_1, \beta_2, \gamma)$ in V_1 such that $|v_n - v| \to 0$ as $n \to \infty$. Let $S(v_n) = u_n$, $S(v) = u$. We know that there exists $(s_n, \phi_n, \psi_n, \overline{\gamma}_n) \in W_{\bar{p}}$, $(s, \phi, \psi, \overline{\gamma}) \in W_{\bar{p}}$ such that $F(u_n) = (s_n, \phi_n, \psi_n, \overline{\gamma}_n) = (f_n, \phi_n, \psi_n, \gamma_n)$, $F(u) = (s, \phi, \psi, \overline{\gamma}) = (f, \phi, \psi, \gamma)$. Taking $|u_n - u| = \max(|f_n - f|_\infty, |\phi_n - \phi|_p, |\psi_n - \psi|_p, \|\gamma_n - \gamma\|)$ it is evident that the operator S is continuous if $|\phi_n - \phi|_p \to 0$, $|\psi_n - \psi|_p \to 0$ as $n \to \infty$.

For each $(f_n, \phi_n, \psi_n, \gamma_n)$, $n = 1, 2, \ldots$, we can write an equation of the form (4.1).

Letting τ_n, τ_{on} be as in equation (3.1) we have (5.1): $\tau_n = (\dot{I} - H)^{-1} J \tau_n + (\dot{I} - H)^{-1} \tau_{on}$. Similarly for (f, ϕ, ψ, γ) we have τ and τ_o such that (5.2): $\tau = (\dot{I} - H)^{-1} J \tau + (\dot{I} - H)^{-1} \tau_o$.

LEMMA 5.1. *If* $|\tau_{on} - \tau_o|_p \to 0$ *as* $n \to \infty$ *then* $|\tau_n - \tau|_p \to 0$ *as* $n \to \infty$.

Proof. From equations (5.1) and (5.2) we obtain $\tau_n - \tau = (\dot{I} - H)^{-1} J(\tau_n - \tau) + (\dot{I} - H)^{-1}(\tau_{on} - \tau_o)$.

Using the properties of the operators $(\dot{I} - H)^{-1}$ and J already established the lemma is proven.

LEMMA 5.2. $|\tau_{on} - \tau_o|_p \to 0$ *as* $n \to \infty$.

Proof. Let $\tau_{on} - \tau_o = \begin{pmatrix} \overline{\phi}_n \\ \overline{\psi}_n \end{pmatrix}$. If $v_n \to v$ as $n \to \infty$, then $|\overline{\phi}_n|_p$ and $|\overline{\psi}_n|_p$ converge toward 0 as $n \to \infty$. Hence, $|\tau_{on} - \tau_o|_p = \max(|\overline{\phi}_n|_p, |\overline{\psi}_n|_p) \to 0$ as $n \to \infty$.

THEOREM 5.1. *The operator* S *is continuous. Moreover for all* $\varepsilon > 0$

there exists $\delta > 0$ *such that if* u, $\bar{u}$ *are solutions of the Cauchy-Goursat boundary problem corresponding to initial datas* v, $\bar{v} \in V_1$ *respectively, then* $|u(x_1,x_2)-\bar{u}(x_1,x_2)| < \varepsilon$ *for all* $(x_1,x_2)\in I_1 \times I_2$ *if* $|v-\bar{v}| < \delta$.

Proof. The continuity of the operator S follows from the considerations made in this section and Lemmas 5.1 and 5.2.

Let $(s,\phi,\psi,\gamma)\in W_{\bar{p}}$, $(\bar{s},\bar{\phi},\bar{\psi},\bar{\gamma})\in W_{\bar{p}}$ be such that $F(u)= (s,\phi,\psi,\gamma)= (f,\phi,\psi,\gamma)$; $F(\bar{u})= (\bar{s},\bar{\phi},\bar{\psi},\bar{\gamma})= (\bar{f},\bar{\phi},\bar{\psi},\bar{\gamma})$. One can prove that

(5.3) $|u(x_1,x_2)-\bar{u}(x_1,x_2)| \leqslant K|u-\bar{u}|$ for all $(x_1,x_2)\in I_1 \times I_2$

where $K= a_1a_2 + a_1^{(p-1)/p} + a_2^{(p-1)/p} + 1$.

From (5.3) it follows that $\|u-\bar{u}\|_\infty \leqslant K|u-\bar{u}|$. The continuity of the operator S implies that given $\varepsilon > 0$ there exists $\delta > 0$ such that $|S(v)-S(\bar{v})|= |u-\bar{u}| < \varepsilon/K$ if $|v-\bar{v}| < \delta$. Hence $\|u-\bar{u}\|_\infty \leqslant K|u-\bar{u}| < \varepsilon$ if $|v-\bar{v}| < \delta$. This completes the proof of the theorem.

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