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ON WEIGHTED AVERAGES OF RANDOM VARIABLES

Hugo Aimar and Roberto Scotto

Presentado por Carlos Segovia Fernández

Abstract: In this note we study boundedness and convergence properties of the weighted averages

$$\frac{\sum^m X_i w_i}{(\sum^m w_i)^{1/p}},$$

for pairwise independent sequences of random variables and non-negative weight sequences $\{w_i\}$.

§1. INTRODUCTION

Let $\{X_i\}$ be a sequence of identically distributed pairwise independent random variables. Let $\{w_i\}$ be a sequence of non-negative real numbers such that $\sum_{i=1}^{\infty} w_i = \infty$. We shall consider boundedness and limit properties of the weighted averages:

$$A_{n,p} = \frac{S_n}{W_n^{1/p}},$$

where $S_n = \sum_{i=1}^{n} X_i w_i$, $W_n = \sum_{i=1}^{n} w_i$ and $p \geq 1$.

In [4], B. Jamison, S. Orey and W. Pruitt gave a characterization of the sequences of weights for which the weighted strong law of large numbers holds for all sequences of i.i.d.r.v. in L^1. A new and elementary proof of the non-weighted strong law of large numbers has been given by N. Etemadi in [2]. In a more recent paper, [1], C. Calderón gives a proof of the maximal inequality associated with Etemadi's theorem. In that paper, C. Calderón also consider the maximal function of the averages

$$\frac{1}{2^{k/p}} \sum_{i=1}^{2^k} X_i$$

and proves the corresponding weak type inequality for symmetrically distributed random variables.

The purpose of this note is to give necessary and sufficient conditions on the sequence of wieghts, $\{w_i\}$,in order to get weak and strong type inequalities for the maximal operator of averages, and weighted strong laws of large numbers. It is worthy of notice that the condition for $p = 1$ is the same as the condition of Jamison-Orey and Pruitt, and that we can get their result by applying Etemadi's method for pairwise independent random variables.

All the results are stated and proved in §3. In section 2 we introduce the weight sequences and prove some elementary lemmas concerning their behavior.

§2. PRELIMINARY LEMMAS

Given a subset of $\mathbb{Z}^+$ we shall denote the number of elements of E by $|E|$. If r is a real number, $[r]$ in the largest integer less than or equal to r. Let $p \geq 1$ be given.

(2.1) Definition: Let $\{w_i\}$ be a sequence of non-negative numbers such that $\sum_{i=1}^{\infty} w_i = \infty$. Let $W_n = \sum_{i=1}^{n} w_i$ and let $v_p(n) = w_n W_n^{-1/p}$. We say that $\{w_i\}$ satisfies the property Λ_p, or breifly $\{w_i\} \in \Lambda_p$, if there exists a constant C such that the inequality

$$|\{n : v_p(n) > \lambda\}| \leq \frac{C}{\lambda^p}$$

holds for every $\lambda > 0$.

(2.2) Lemma: *Let $\{w_i\} \in \Lambda_p$ and $\alpha > 1$. Then, there exists $K = K(\alpha, p)$ such that for every $k \geq K$ there is an integer m satisfyng*

(2.3) $$[\alpha^k] \leq W_m^{1/p} < [\alpha^{k+1}]$$

Proof: Assume that there exists a sequence of integers, $\{k_i\}$ such that (2.3) does not hold for any integer m. Since $\{W_n\}$ is a divergent sequence, it is possible to choose integers m_i verifying

$$W_{m_i-1}^{1/p} < [\alpha^{k_i}] < [\alpha^{k_i+1}] \leq W_{m_i}^{1/p}.$$

Thus, we get

$$w_{m_i} = W_{m_i} - W_{m_i-1} > (\alpha^{k_i+1} - 1)^p - \alpha^{k_i p}$$

and

$$w_{m_i} W_{m_i-1}^{-1} > \frac{(\alpha^{k_i+1} - 1)^p - \alpha^{k_i p}}{\alpha^{k_i p}} = \alpha^p (1 - \alpha^{-k_i-1})^p - 1.$$

Therefore, for large values of k_i, we have

$$w_{m_i} W_{m_i-1}^{-1} > (\alpha^p - 1)/2.$$

Since

$$w_{m_i} W_{m_i-1} = v_1(m_i)(1 - v_1(m_i))^{-1},$$

it follows that $v_1(m_i) > \frac{\alpha^p - 1}{\alpha^p + 1}$ for infinitely many m_i. On the other hand $v_p(n) \geq v_1(n)$. Therefore

$$|\{n : v_p(n) > \frac{\alpha^p - 1}{\alpha^p + 1}\}| = \infty,$$

and consequently $\{w_i\} \notin \Lambda_p$. #

In the sequel, K shall denote the constant in Lemma (2.2).
Now, for $\alpha > 1$ fixed, set

$$\varphi(\alpha,p;k) := min\{j : W_j^{1/p} \geq [\alpha^k]\}$$

Lemma: *Let $\{w_i\} \in \Lambda_p$. Then*
(2.4) $\varphi(\alpha,p;k)$ is a non-decreading function of k, and is one-to-one for $k \geq K$,
(2.5) if $k_i = min\{k : \varphi(\alpha,p;k) \geq i\}$, we have the inequality $W_i^{1/p} \leq \alpha^{k_i+1}$, for every $i \geq \varphi(\alpha,p;K)$.

Proof: Let $k' > k \geq K$ and let m be given by lemma (2.2), then

$$[\alpha^k] \leq W_m^{1/p} < [\alpha^{k+1}] \leq [\alpha^{k'}].$$

Therefore $\varphi(\alpha,p;k) < \varphi(\alpha,p;k')$. To prove (2.5), observe that $W_i^{1/p} \leq W_{\varphi(\alpha,p;k_i)}^{1/p}$, and that $W_{\varphi(\alpha,p;k_i)}^{1/p} < [\alpha^{k_i+1}] \leq \alpha^{k_i+1}$. #

If $(H,\mathcal{H},\nu)$ is a σ-finite measure space and f is a measurable function defined on H, such that

$$\nu\{x : |f(x)| > \lambda\} \leq \frac{C}{\lambda^p}$$

for $p < 2$, we have

$$\begin{aligned}\int_{\{x:|f(x)|\leq r\}} |f(x)|^2\, d\nu &\leq 2\int_0^r \lambda\nu\{x : |f(x)| > \lambda\}\, d\lambda \\ &\leq 2C\int_0^r \lambda^{1-p}\, d\lambda = \frac{2C}{2-p}r^{2-p}.\end{aligned}$$

In particular, if $H = \mathbb{Z}^+$ and ν is the counting measure we have the following lemma.

(2.6) Lemma: *Let $\{w_i\} \in \Lambda_p$ and $p < 2$, then*

$$\sum_{\{n:v_p(n)\leq r\}} v_p(n)^2 \leq \frac{C}{2-p}r^{2-p}.$$

(2.7) Lemma: *Let $\{w_i\}$ be a sequence of positive real numbers such that $\{w_i\} \notin \Lambda_p$. Then there exist an unbounded sequence of integers $\{m_j\}$ and a decreasing sequence of positive real numbers $\{\lambda_i\}$, such that*

(2.8) $$\lim_{i\to\infty} \lambda_i^p |\{j : v_p(j) > \lambda_i m_j\}| = \infty$$

Proof: Let E_λ be the set $\{n : v_p(n) > \lambda\}$. From the fact that $\{w_i\} \notin \Lambda_p$ we have that $\lambda^p|E_\lambda|$ is an unbounded function of λ close to zero. Assume that $|E_\lambda| < \infty$ for every λ. Choose γ_1 such that $\gamma_1^p|E_{\gamma_1}| \geq 1$. Assume that γ_{i-1} has been chosen. Since for $0 < \gamma < \gamma_{i-1}$ we have the equality

$$\gamma^p|E_\gamma| = \gamma^p|E_\gamma - E_{\gamma_{i-1}}| + \gamma^p|E_{\gamma_{i-1}}|,$$

then there exists γ_i such that

$$\gamma_i^p|E_{\gamma_i} - E_{\gamma_{i-1}}| \geq i^{p+1}.$$

Once the sequence $\{\gamma_i\}$ is constructed, define $\lambda_i = \frac{\gamma_i}{i}$ and $m_j = i$ for $j \in E_{\gamma_i} - E_{\gamma_i-1}$. Then

$$\begin{aligned}\lambda_i^p|\{j : v_p(j) > \lambda_i m_j\}| &\geq (\frac{\gamma_i}{i})^p|\{j : \gamma_{i-1} \geq v_p(j) > \gamma_i\}|\\ &= (\frac{\gamma_i}{i})^p|E_{\gamma_i} - E_{\gamma_{i-1}}| \geq i\end{aligned}$$

Let us now suppose that there exists $\lambda_o > 0$ such that $|E_{\lambda_o}| = \infty$. Write $E_{\lambda_o} = \{n(i)\}$ with $n(i) < n(i+1)$ and take $\lambda_i = \frac{1}{i}$ and $m_j = \log i$ for $n(i-1) < j \leq n(i)$.#

§3. THE RESULTS

Let $\{w_i\}$ be a sequence of weights and X_i be a sequence of random variables on the probability space $(\Omega, \mathcal{F}, P)$. We define the maximal function

$$\sigma^*(y) := \sup_{n\geq 1} \frac{\sum_{i=1}^n |X_i(y)|w_i}{\sum_{i=1}^n w_i}$$

The main result concerning the behavior of σ^* in relation to the sequence of weights, es the following

(3.1) Theorem: *Given a sequence of weights $\{w_i\}$, there exists a constant C such that the inequality*

$$P(\sigma^* > \lambda) \leq \frac{C}{\lambda}\varepsilon(|X_1|) \tag{3.2}$$

holds for every $\lambda > 0$ and every sequence of integrable, pairwise independent i.d.r.v., if and only if $\{w_i\} \in \Lambda_1$.

Proof: Let us show first the "only if" part. Let $\{w_i\} \notin \Lambda_1$. By lemma (2.7) there exist two sequences $\{\lambda_i\}$ and $\{m_i\}$ satisfying (2.8). We are going to construct a sequence of identically distributed, pairwise independent random variables such that the associated weighted maximal function is infinity on a set of probability one. Let $v_i = v_1(i)$. Since the following inequalities

$$|X_n|v_n \leq \frac{|S_n|}{W_n} + \frac{|S_{n-1}|}{W_{n-1}} \leq 2\sigma^* \tag{3.3}$$

hold, we have

$$limsup\{|X_j|v_j \geq m_j\} \subset \{\sigma^* = \infty\}.$$

Therefore given an integrable random variable X such that $\sum_{j=1}^{\infty} P(|X| \geq m_j v_j^{-1}) = \infty$, via the product theorem and Borel-Cantelli lemma, it is possible to find random variables X_j, with distributions identically that of X, pairwise independent and verifying $P(limsup\{|X_j|v_j \geq m_j\}) = 1$. Let us produce such an X. Let

$$b_i = \lambda_i|\{j : v_j > \lambda_i m_j\}|.$$

Since b_i is unbounded, we can find a sequence $\{a_i\}$ of non-negative real numbers such that $\sum a_i = 1$ and $\sum a_i b_i = \infty$. Let X be a random variable which takes the value λ_i^{-1} with probability $\lambda_i a_i (i \in \mathbb{Z}^+)$ and is zero otherwise. Then, clearly, $X \in L^1$ and

$$\sum_{j=1}^{\infty} P(X \geq m_j v_j^{-1}) \geq \sum_{j=1}^{\infty} \sum_{\{i:\lambda_i^{-1} > m_j v_j^{-1}\}} \lambda_i a_i$$
$$= \sum_{i=1}^{\infty} \lambda_i a_i |\{j : v_j/m_j > \lambda_i\}| = \sum_{i=1}^{\infty} a_i b_i = \infty$$

To prove sufficiency let us suppose that $\{w_i\} \in \Lambda_1$. Observe that it is enough to show the inequality

$$P(\sigma^* > 1) \leq C\varepsilon(X_1),$$

for $\varepsilon(X_1) < \frac{1}{12}$ and non-negative X_i's. Let μ be the distribution of X_i. Following C. Calderón, [1], let us introduce the exceptional set

$$E = \cup_{i=1}^{\infty} E_i, \ \textit{where} \ E_i := \{v_i X_i > 1\}$$

and let $Y_i := X_i \chi_{\Omega - E_i}$. Using the fact that $\{w_i\} \in \Lambda_1$ we obtain $P(E) \leq C\varepsilon(X_1)$. In fact

$$\begin{aligned} P(E) \leq \sum_{i=1}^{\infty} P(E_i) &\leq \sum_{i=1}^{\infty} \sum_{j=[v_i^{-1}]}^{\infty} \int_j^{j+1} d\mu \\ &\leq \sum_{j=0}^{\infty} (j+1) \int_j^{j+1} d\mu \frac{1}{j+1} |\{i : v_i > \frac{1}{j+1}\}| \\ &\leq C\varepsilon(X_1). \end{aligned}$$

We need now to get an estimate of σ^* on $\Omega - E$. Let $y \notin E$ and $\varphi(k) := \varphi(2,1;k)$, then by (2.4), for every $n > N = \varphi(K)$ there is an integer $k > K$ such that $\varphi(k-1) \leq n < \varphi(k)$ and $2^{k-1} \leq W_{\varphi(k-1)} < 2^k$. Set $S'_n = \sum_{i=1}^{n} Y_i w_i$, then

$$\begin{aligned} W_n^{-1} S_n(y) &\leq 4 W_{\varphi(k)}^{-1} S'_{\varphi(k)}(y) \\ &\leq 4[W_{\varphi(k)}^{-1} |S'_{\varphi(k)}(y) - \varepsilon(S'_{\varphi(k)})| + \varepsilon(X_1)]. \end{aligned}$$

Consequently we have the following estimate for the maximal function on $\Omega - E$

$$\sigma^*(y) \leq \sum_{n=1}^{N} W_n^{-1} S_n + 4 \sup_{k \geq K} W_{\varphi(k)}^{-1} |S'_{\varphi(k)}(y) - \varepsilon(S'_{\varphi(k)})| + 4\varepsilon(X_1).$$

Hence

$$\begin{aligned} \{\sigma^* > 1\} \cup (\Omega - E) \subset & \{\sum_{n=1}^{N} W_n^{-1} S_n > \frac{1}{3}\} \\ & \cup \{\sup_{k > K} W_{\varphi(k)}^{-1} |S'_{\varphi(k)}(y) - \varepsilon(S'_{\varphi(k)})| > \frac{1}{12}\} \\ & =: A_1 \cup A_2 \end{aligned}$$

Since it is clear that $P(A_1) \leq 3N\varepsilon(X_1)$, it remains only to prove a similar estimate for $P(A_2)$. By Chebishev's inequality we have

$$\begin{aligned} P(A_2) &\leq C \sum_{k=K}^{\infty} W_{\varphi(k)}^{-2} \varepsilon(|S'_{\varphi(k)} - \varepsilon(S'_{\varphi(k)})|^2) \\ &\leq C \sum_{k=K}^{\infty} 2^{-2k} \sum_{i=1}^{\varphi(k)} \varepsilon(Y_i^2) w_i^2 \\ &\leq \sum_{i=1}^{\infty} \varepsilon(Y_i^2) w_i^2 \sum_{\{k \geq K : \varphi(k) \geq i\}} 2^{-2k} \\ &\leq C \left\{ \sum_{i=1}^{N} \varepsilon(Y_i^2)(w_i 2^{-K})^2 + \sum_{i=N+1}^{\infty} \varepsilon(Y_i^2)(w_i 2^{-k_i})^2 \right\}, \end{aligned}$$

where k_i is that of (2.5). Then

$$\begin{aligned} P(A_2) &\leq C\sum_{i=1}^{\infty} v_i^2 \varepsilon(Y_i^2) \\ &\leq C\sum_{i=1}^{\infty} v_i^2 \sum_{j=0}^{[v_i^{-1}]} \int_j^{j+1} x^2\, d\mu \\ &\leq C\sum_{j=0}^{\infty} \int_j^{j+1} x^2\, d\mu \sum_{\{i: v_i \leq \frac{1}{j+1}\}} v_i^2. \end{aligned}$$

By Lemma (2.6) we have that

$$P(A_2) \leq C\sum_{j=0}^{\infty} (j+1)^{-1} \int_j^{j+1} x^2\, d\mu \leq C\varepsilon(X_1),$$

which finishes the proof of the theorem. #

(3.4) Corollary *If $\{w_i\} \in \Lambda_1, 1 < r < \infty$ and $\{X_i\}$ is a sequence of L^r, pairwise independent i.d.r.v., then there exists C depending only on r such that*

$$\varepsilon(\sigma^{*r}) \leq C\varepsilon(|X_1|^r).$$

This corollary follows from the theorem by interpolation.

Applying Etemadi's method in the weighted situation we get the theorem of Jamison-Orey and Pruitt in the pairwise independent case:

(3.5) Theorem: *Let $\{w_i\}$ be a sequence of weights. Then, the weighted strong law of large numbers holds for every integrable, pairwise independent sequence of i.d.r.v. if and only if $\{w_i\} \in \Lambda_1$.*

Proof: Let $\{w_i\} \in \Lambda_1$. We can assume that X_i's are non-negative random variables. Define Y_i as in theorem (3.1) and $\varphi(\alpha, k) := \varphi(\alpha, 1; k)$, for $\alpha > 1$. The same computation as in theorem (3.1) proves the estimate

$$\sum_{k=1}^{\infty} P\left(\left|\frac{S'_{\varphi(\alpha,k)} - \varepsilon(S'_{\varphi(\alpha,k)})}{W_{\varphi(\alpha,k)}}\right| > \varepsilon\right) \leq C(\varepsilon, \alpha)\varepsilon(X_1).$$

Since, on the other hand $W^{-1}_{\varphi(\alpha,k)}\varepsilon(S'_{\varphi(\alpha,k)K})$ converges to $\varepsilon(X_1)$, by Borel-Cantelli lemma we have

$$\lim_{k\to\infty} \frac{S'_{\varphi(\alpha,k)}}{W_{\varphi(\alpha,k)}} = \varepsilon(X_1) \qquad a.e..$$

The sequences $\{X_i\}$ and $\{Y_i\}$ are equivalent, i.e. $\sum_{i=1}^{\infty} P(X_i \neq Y_i) < \infty$, then we have the almost everywhere convergence of the weighted averages $W_{\varphi(\alpha,k)}^{-1} S_{\varphi(\alpha,k)}$ to $\varepsilon(X_1)$. Now, for every $n \geq \varphi(\alpha, K)$ there exists k such that $\varphi(\alpha,k) \leq n < \varphi(\alpha, k+1)$ and $[\alpha^k] \leq W_{\varphi(\alpha,k)} < [\alpha^{k+1}]$. Then for those valuis of n we have

$$\left(\frac{1}{\alpha^2} - \frac{1}{\alpha^{k+2}}\right) \frac{S_{\varphi(\alpha,k)}}{W_{\varphi(\alpha,k)}} \leq \frac{S_n}{W_n} \leq \left(\frac{\alpha^2}{1-\alpha^{-k}}\right) \frac{S_{\varphi(\alpha,k+1)}}{W_{\varphi(\alpha,k+1)}},$$

hence

$$\frac{1}{\alpha^2}\varepsilon(X_1) \leq liminf \frac{S_n}{W_n} \leq limsup \frac{S_n}{W_n} \leq \alpha^2 \varepsilon(X_1)$$

almost everywhere, for every $\alpha > 1$. Then the strong law of large numbers holds. Let now $\{w_i\}$ be a weight sequence such that $\{w_i\} \notin \Lambda_i$. The first inequality in (3.3) shows that

$$limsup\{|X_j|v_j \geq m_j\} \subset limsup\{w_j^{-1}|S_j| \geq \frac{m_j}{2}\}.$$

Then the same example as in theorem (3.1) shows that $\{w_i\} \in \Lambda_1$ is also necessary. #

Let us now consider the p^{th} weighted maximal function

$$\sigma_p^*(y) = \sup_{n \geq 1} \frac{|S_n(y)|}{W_n^{1/p}},$$

where $1 < p < 2, S_n(y) = \sum_{i=1}^{n} X_i(y) w_i$ and $W_n = \sum_{i=1}^{n} w_i$. This maximal function is the weighted extension of that of C. Calderón and corresponds to the strong law for the averages $\frac{S_n}{W_n^{1/p}}$ whose non weighted analogues can be found in [5], page 255.

In order to prove the weak type (p,p) of σ_P^* in the independent case we shall use a Kolmogorov type inequality, more precisely the following extension, proved by J. Hájek and A. Rényi in [3]:

If $\xi_1, \xi_2, \xi_3, \ldots, \xi_k, \ldots$ is an L^2 sequence of mutually independent random variables, each with mean value zero, and C_k is a non-increasing sequence of positive numbers, then the inequality

$$P\{\max_{1 \leq k \leq n} C_k|\xi_1 + \xi_2 + \ldots + \xi_k| \geq \varepsilon\} \leq \frac{1}{\varepsilon^2} \sum_{k=1}^{n} C_k^2 \sigma^2(X_k)$$

holds for every $\varepsilon > 0$ and any positive integer n.

(3.6) Theorem: *Given a sequence of weights $\{w_i\}$, there exists a constant C_p such that the inequality*

$$P(\sigma_p^* > \lambda) \leq \frac{C_p}{\lambda^p} \varepsilon(|X_1|^p) \tag{3.7}$$

holds for every $\lambda > 0$ and every sequence of L^p, independent symmetric i.d.r.v., if an only if $\{w_i\} \in \Lambda_p$.

Proof: Since all the lemmas in section 2 hold for $1 \leq p < 2$, the proof of the "only if" part follows the same pattern as in theorem (3.1).

If $\{w_i\} \in \Lambda_p$, as before, it is enough to prove the following inequality

$$P(\sigma_p^* > 1) \leq C_p \varepsilon(|X_1|^p).$$

Let us define $E_i := \{v_p(i)|X_i| > 1\}$; $E = \cup_{i=1}^{\infty} E_i$ and $Y_i = X_i \chi_{\Omega - E_i}$. Since the X_i's are symmetrically distributed, $\varepsilon(Y_i) = \varepsilon(X_i) = 0$ for every i. The estimate $P(E) \leq C\varepsilon(|X_1|^p)$ can be obtained as in theorem (3.1) taking into account that $\{w_i\} \in \Lambda_p$. Let

$$S_n'^* := \sup_{1 \leq k \leq n} W_k^{-1/p} |S_k'|$$

and

$$S'^* := \sup_{k \leq 1} W_k^{-1/p} |S_k'|,$$

where S_k' is the weighted k^{th} sum of the Y_i's. Applying Hájek-Rényi inequality to $\xi_k = Y_k w_k$ and $C_k = W_k^{-1/p}$, we obtain

$$P(S_n'^* > 1) \leq \sum_{i=1}^{n} v_p^2(i) \varepsilon(Y_i^2).$$

Then

$$P(\{\sigma_p^* > 1\} \cap \{\Omega - E\}) = P(S'^* > 1) \leq \sum_{i=1}^{\infty} v_p^2(i) \varepsilon(Y_i^2) \leq C_p \varepsilon(|X_1|^p),$$

the last inequality follows from the fact that $\{w_i\} \in \Lambda_p$ as in theorem (3.1). #

(3.8) **Remark:** The corresponding strong law

$$\frac{S_n}{W_n^{1/p}} \longrightarrow 0 \quad a.e.$$

holds if $\{w_i\} \in \Lambda_p$.

In the pairwise independent case we can consider the following weighted analogue of the diadic maximal studied by C. Calderón:

$$\sigma_{\alpha,p}^* = \sup_{k \geq 1} \frac{|S_{\varphi(\alpha,p;k)}|}{W_{\varphi(\alpha,p;k)}^{1/p}}$$

Etemadi's method used in theorem (3.1) allows us to obtain the following result:

(3.9) Theorem: *Let $1 \leq p < 2$, $\{w_i\} \in \Lambda_p$ and $\{X_i\}$ be a sequence of symmetrically and identically distributed pairwise independent L^p random variables, then*

$$P(\sigma_{\alpha,p}^* > \lambda) \leq \frac{C_{p,\alpha}}{\lambda^p} \varepsilon(|X_1|^p). \qquad \#$$

REFERENCES

[1] Calderón, C.: "*On N. Etemadi's proof of the strong law of the large numbers*". Preprint.

[2] Etemadi, N.: "*An elementary proof of the strong law of large numbers*". Z. Wahrscheinlichkeitstheorie verw. Gebiete 55, (1981), p.p. 119-122.

[3] Hájek, J. and Rényi, A.: "*Generalization of an inequality of Kolmogorov*". Acta Mathematica Academiae Scientiarum Hungaricae VI. (1955). p.p. 281-283.

[4] Jamison, B., Orey, S., and Pruitt, W. : "*Convergence of weighted averages of independent random variables*". Z. Wahrscheinlichkeitstheorie 4. (1965). p.p. 40-44.

[5] Loéve, M.: "*Probability Theory I*". Springer Verlag. New Yord. 1977.

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