REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 48, Número 2, 2007, Páginas 47–65

THE LADDER CONSTRUCTION OF PRÜFER MODULES

CLAUS MICHAEL RINGEL

Dedicated to María Inés Platzeck on the occasion of her 60th birthday

Abstract. Let R be a ring (associative, with 1). A non-zero module M is said to be a Prüfer module provided there exists a surjective, locally nilpotent endomorphism with kernel of finite length. The aim of this note is to construct Prüfer modules starting from a pair of module homomorphisms $w,v:U_0\to U_1$, where w is injective and its cokernel is of finite length. For $R=\mathbb{Z}$ the ring of integers, one can construct in this way the ordinary Prüfer groups considered in abelian group theory. Our interest lies in the case that R is an artin algebra.

1. The construction.

Let R be a ring (associative, with 1). The modules to be considered will usually be left R-modules. Our main interest will be the case where R is an artin algebra, however the basic construction should be of interest for any ring R. In fact, the standard examples of what we call Prüfer modules are the Prüfer groups in abelian group theory, thus $\mathbb{Z}$-modules. Here is the definition of a Prüfer module: it is a non-zero module P which has a surjective, locally nilpotent endomorphism ϕ with kernel of finite length. If H is the kernel of ϕ, we often will write $P = H[\infty]$, and we will denote the kernel of ϕ^t by $H[t]$. Observe the slight ambiguity: given a Prüfer module P, not only ϕ but also all non-trivial powers of ϕ and maybe many other endomorphisms will have the required properties (surjectivity, local nilpotency, finite length kernel).

The content of the paper is as follows. In the first section we show that any pair of module homomorphisms $w, v : U_0 \to U_1$, where w is injective with non-zero cokernel of finite length, gives rise to a Prüfer module. Section 2 provides some examples and section 3 outlines the relationship between Prüfer modules and various sorts of self-extensions of finite length modules. The final sections 4 and 5 deal with degenerations in the sense of Riedtmann-Zwara: we will show that this degeneration theory is intimately connected to the existence of Prüfer modules with some splitting property, and we will exhibit an extension of a recent result by Bautista and Pérez. Our interest in the questions considered here was stimulated by a series of lectures by Sverre Smalø [S] at the Mar del Plata conference, March 2006, and we are indebted to him as well as to M.C.R.Butler and G.Zwara for helpful comments.

For the relevance of Prüfer modules when dealing with artin algebras of infinite representation type, we refer to a forthcoming paper [R5]. The appendix to section 3.3 provides some indications in this direction.

1.1. The basic frame. A pair of exact sequences

$$0 \to U_0 \xrightarrow{w_0} U_1 \to H \to 0 \quad \text{and} \quad 0 \to K \to U_0 \xrightarrow{v_0} U_1 \to Q \to 0$$

yields a module U_2 and a pair of exact sequences

$$0 \to U_1 \xrightarrow{w_1} U_2 \to H \to 0 \quad \text{and} \quad 0 \to K \to U_1 \xrightarrow{v_1} U_2 \to Q \to 0$$

by forming the induced exact sequence of $0 \to U_0 \xrightarrow{w_0} U_1 \to H \to 0$ using the map v_0:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & U_0 & \xrightarrow{w_0} & U_1 & \longrightarrow & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle v_0} & & \downarrow{\scriptstyle v_1} & & \| & & \\
0 & \longrightarrow & U_1 & \xrightarrow{w_1} & U_2 & \longrightarrow & H & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & Q & = & Q & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Recall that a commutative square

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y_1 \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\
Y_2 & \xrightarrow[f']{} & Z
\end{array}$$

is said to be *exact* provided it is both a pushout and a pullback, thus if and only if the sequence

$$0 \to X \xrightarrow{\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]} Y_1 \oplus Y_2 \xrightarrow{[\,g' \; -f'\,]} Z \to 0$$

is exact. Note that our basic setting provides an exact square

$$\begin{array}{ccc}
U_0 & \xrightarrow{w_0} & U_1 \\
{\scriptstyle v_0}\downarrow & & \downarrow{\scriptstyle v_1} \\
U_1 & \xrightarrow[w_1]{} & U_2
\end{array}$$

Next, we will use that the composition of exact squares is exact:

(E1) *The composition of two exact squares*

$$\begin{array}{ccccc} X & \longrightarrow & Y_1 & \longrightarrow & Z_1 \\ \downarrow & & \downarrow & & \downarrow \\ Y_2 & \longrightarrow & Z_2 & \longrightarrow & A \end{array}$$

yields an exact square

$$\begin{array}{ccc} X & \longrightarrow & Z_1 \\ \downarrow & & \downarrow \\ Y_2 & \longrightarrow & A \end{array}$$

1.2. The ladder. Using induction, we obtain in this way modules U_i and pairs of exact sequences

$$0 \to U_i \xrightarrow{w_i} U_{i+1} \to H \to 0 \quad \text{and} \quad 0 \to K \to U_i \xrightarrow{v_i} U_{i+1} \to Q \to 0$$

for all $i \geq 0$.

We may combine the pushout diagrams constructed inductively and obtain the following ladder of commutative squares:

$$\begin{array}{ccccccccc} U_0 & \xrightarrow{w_0} & U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & \cdots \\ {\scriptstyle v_0}\downarrow & & {\scriptstyle v_1}\downarrow & & {\scriptstyle v_2}\downarrow & & {\scriptstyle v_3}\downarrow & & \\ U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & U_4 & \xrightarrow{w_4} & \cdots \end{array}$$

We form the inductive limit $U_\infty = \bigcup_i U_i$ (along the maps w_i).

Since all the squares commute, the maps v_i induce a map $U_\infty \to U_\infty$ which we denote by v_∞:

$$\begin{array}{ccccccccc} U_0 & \xrightarrow{w_0} & U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & \cdots \\ {\scriptstyle v_0}\downarrow & & {\scriptstyle v_1}\downarrow & & {\scriptstyle v_2}\downarrow & & {\scriptstyle v_3}\downarrow & & \\ U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & U_4 & \xrightarrow{w_4} & \cdots \end{array} \qquad \begin{array}{c} \bigcup_i U_i = U_\infty \\ \downarrow {\scriptstyle v_\infty} \\ \bigcup_i U_i = U_\infty \end{array}$$

We also may consider the factor modules U_∞/U_0 and U_∞/U_1. The map $v_\infty : U_\infty \to U_\infty$ maps U_0 into U_1, thus it induces a map

$$\overline{v} : U_\infty/U_0 \longrightarrow U_\infty/U_1.$$

Claim. *The map $\overline{v}$ is an isomorphism.* Namely, the commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_{i-1} & \xrightarrow{w_{i-1}} & U_i & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle v_{i-1}} & & \downarrow{\scriptstyle v_i} & & \| & & \\ 0 & \longrightarrow & U_i & \xrightarrow{w_i} & U_{i+1} & \longrightarrow & H & \longrightarrow & 0 \end{array}$$

can be rewritten as

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_{i-1} & \xrightarrow{w_{i-1}} & U_i & \longrightarrow & U_i/U_{i-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle v_{i-1}} & & \downarrow{\scriptstyle v_i} & & \downarrow{\scriptstyle \overline{v}_i} & & \\ 0 & \longrightarrow & U_i & \xrightarrow{w_i} & U_{i+1} & \longrightarrow & U_{i+1}/U_i & \longrightarrow & 0 \end{array}$$

with an isomorphism $\overline{v}_i : U_i/U_{i-1} \to U_{i+1}/U_i$. The map $\overline{v}$ is a map from a filtered module with factors U_i/U_{i-1} (where $i \geq 1$) to a filtered module with factors U_{i+1}/U_i (again with $i \geq 1$), and the maps $\overline{v_i}$ are just those induced on the factors.

It follows: The composition of maps

$$U_\infty/U_0 \xrightarrow{p} U_\infty/U_1 \xrightarrow{\overline{v}^{-1}} U_\infty/U_0$$

with p the projection map is an epimorphism ϕ with kernel U_1/U_0. It is easy to see that ϕ is locally nilpotent, namely we have $\phi^t(U_t/U_0) = 0$ for all t.

Summary. (a) *The maps v_i yield a map*

$$v_\infty : U_\infty \to U_\infty$$

with kernel K and cokernel Q.

(b) *This map v_∞ induces an isomorphism $\overline{v} : U_\infty/U_0 \to U_\infty/U_1$. Composing the inverse of this isomorphism with the canonical projection p, we obtain an endomorphism $\phi = (\overline{v})^{-1} \circ p$*

$$U_\infty/U_0 \xrightarrow{p} U_\infty/U_1 \xrightarrow{\overline{v}^{-1}} U_\infty/U_0.$$

If the cokernel H of w_0 is non-zero and of finite length, then U_∞/U_0 is a Prüfer module with respect to ϕ, with basis H; in this case, we call U_∞/U_0 (or better the pair $(U_\infty/U_0, \phi)$) the *Prüfer module defined by the pair* (w_0, v_0) or by the ladder U_i. Prüfer modules which are obtained in this way will be said to be *of ladder type.*

If necessary, we will use the following notation: $U_i(w_0, v_0) = U_i$, for all $i \in \mathbb{N} \cup \{\infty\}$ and $P(w_0, v_0) = U_\infty/U_0$ for the Prüfer module. Since $P(w_0, v_0)$ is a Prüfer module with basis the cokernel H of w_0, we will sometimes write $H[n] = U_n/U_0$ or even $H[n; w_0, v_0]$.

Remark: Using a terminology introduced for string algebras [R3], we also could say: U_∞ is *expanding,* U_∞/U_0 is *contracting.*

Lemma. *Assume that $P = P(w, v)$ with $w, v : U_0 \to U_1$. Then P is generated by U_1.*

Proof: For $i \geq 2$, the module U_i is a factor module of $U_{i-1} \oplus U_{i-1}$, thus by induction, U_i is generated by U_1.

1.3. The chessboard. Assume now that both maps $w_0, v_0 : U_0 \to U_1$ are monomorphisms. Then we get the following arrangement of commutative squares:

$$\begin{array}{ccccccccc}
U_0 & \xrightarrow{w_0} & U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & \cdots \\
\downarrow{\scriptstyle v_0} & & \downarrow{\scriptstyle v_1} & & \downarrow{\scriptstyle v_2} & & \downarrow{\scriptstyle v_3} & & \\
U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & \cdots & & \\
\downarrow{\scriptstyle v_1} & & \downarrow{\scriptstyle v_2} & & \downarrow{\scriptstyle v_3} & & & & \\
U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & \cdots & & & & \\
\downarrow{\scriptstyle v_2} & & \downarrow{\scriptstyle v_3} & & & & & & \\
U_3 & \xrightarrow{w_3} & \cdots & & & & & & \\
\downarrow{\scriptstyle v_3} & & & & & & & & \\
\cdots & & & & & & & &
\end{array}$$

Note that there are both horizontally as well as vertically ladders: the horizontal ladders yield $U_\infty(w_0, v_0)$ (and its endomorphism v_∞); the vertical ladders yield $U_\infty(v_0, w_0)$ (and its endomorphism w_∞).

2. Examples.

(1) The classical example: Let $R = \mathbb{Z}$ be the ring of integers, and $U_0 = U_1 = \mathbb{Z}$ its regular representation. Module homomorphisms $\mathbb{Z} \to \mathbb{Z}$ are given by the multiplication with some integer n, thus we denote such a map just by n. Let $w_0 = 2$ and $v_0 = n$. *If n is odd, then $P(2, n)$ is the ordinary Prüfer group for the prime* 2, and $U_\infty(2, n) = \mathbb{Z}[\frac{1}{2}]$ (the subring of $\mathbb{Q}$ generated by $\frac{1}{2}$). If n is even, then $P(2, n)$ is an elementary abelian 2-group.

(2) Let $R = K(2)$ be the Kronecker algebra over some field k. Let U_0 be simple projective, U_1 indecomposable projective of length 3 and $w_0 : U_0 \to U_1$ a non-zero map with cokernel H (one of the indecomposable modules of length 2). The module $P(w_0, v_0)$ is the Prüfer module for H if and only if $v_0 \notin kw_0$, otherwise it is a direct sum of copies of H.

(3) Trivial cases: First, let w be a split monomorphism. Then the Prüfer module with respect to any map $v : U_0 \to U_1$ is just the countable sum of copies of H. Second, let $w : U_0 \to U_1$ be an arbitrary monomorphism, let $\beta : U_1 \to U_1$ be an endomorphism. Then $P(w, \beta w)$ is the countable sum of copies of H.

(4) Assume that there exists a split monomorphism $v : U_0 \to U_1$, say $U_1 = U_0 \oplus X$ and $v = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] : U_0 \to U_1$. Then

$$0 \to U_0 \xrightarrow{w} U_0 \oplus X \to H \to 0$$

is a Riedtmann-Zwara sequence as discussed in section 4, thus H is a degeneration of X.

Remark: *Not all Prüfer modules are of ladder type.* Consider the generalized Kronecker algebra Λ with countably many arrows $\alpha_0, \alpha_1, \dots$ starting at the vertex a and ending in the vertex b. Define a representation $P = (P_a, P_b, \alpha_i)_i$ as follows: Let $P_a = P_b$ be a vector space with a countable basis $e_0, e_1, \dots$ and let $\alpha_i : P_a \to P_b$ be defined by $\alpha_i(e_j) = e_{j-i}$ provided $j \geq i$ and $\alpha_i(e_j) = 0$ otherwise. Let ϕ_a, ϕ_b be the endomorphism of P_a of P_b, respectively, which sends e_0 to 0 and e_i to e_{i-1} for $i \geq 1$. Then P is a Prüfer module (with respect to ϕ, but also with respect to any power of ϕ). Obviously, P is a faithful Λ-module. Assume that $P = P(w, v)$ for some maps $w, v : U_0 \to U_1$ with U_0, U_1 of finite length. Then P is generated by U_1, according to Lemma 1.2. However U_1 is of finite length and no finite length Λ-module is faithful.

3. Ladder extensions.

3.1. The definition. Let H be a non-zero module of finite length. A self-extension $0 \to H \to H[2] \to H \to 0$ is said to be a *ladder extension* provided there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_0 & \longrightarrow & U_1 & \xrightarrow{q} & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\ 0 & \longrightarrow & H & \longrightarrow & H[2] & \longrightarrow & H & \longrightarrow & 0 \end{array}$$

such that f factors through q, say $f = qv$ for some $v : U_0 \to U_1$.

This means that we have a commutative diagram with exact rows of the following kind (here $f = qv_0$):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_0 & \xrightarrow{w_0} & U_1 & \xrightarrow{q} & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle v_0} & & \downarrow{\scriptstyle v_1} & & \| & & \\ 0 & \longrightarrow & U_1 & \xrightarrow{w_1} & U_2 & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle q} & & \downarrow & & \| & & \\ 0 & \longrightarrow & H & \longrightarrow & H[2] & \longrightarrow & H & \longrightarrow & 0. \end{array}$$

Thus, in order to construct all the ladder extensions of H, we may start with an arbitrary epimorphism $q : U_1 \to H$, form its kernel w_0 and consider any homomorphism $v_0 : U_0 \to U_1$.

According to section 1 we know: *Ladder extensions build up to form Prüfer modules.*

Lemma. *Let k be a commutative ring and Λ a k-algebra. Then $H[2; w_0, v_0] = H[2; w_0, v_0 + \mu w_0]$ for any $\mu \in k$.*

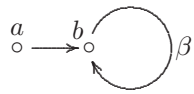
Proof: We deal with the exact sequence induced by qv_0 or $q(v_0 + \mu w_0)$, respectively. But $q(v_0 + \mu w_0) = qv_0 + q\mu w_0 = qv_0$, since $qw_0 = 0$.

Also, any central automorphism λ of U_0 yields isomorphic extensions $H[2; w_0, v_0]$ and $H[2; w_0, \lambda v_0]$. This shows that *the extension* $H[2; w_0, v_0]$ *only depends on the* k*-subspace* $\langle w_0, v_0 \rangle$.

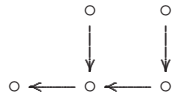
Remark. Not all self-extensions are ladder extensions. For example: *A non-zero self-extension of a simple module* S *over an artinian ring is never a ladder extension!*

Proof: Construct the corresponding ladder, thus the corresponding Prüfer module $S[\infty]$. The module $S[n]$ would be a (serial) module of Loewy length n, with n arbitrary. But the Loewy length of any module over the artinian ring R is bounded by the Loewy length of ${}_RR$, thus $S[\infty]$ cannot exist.

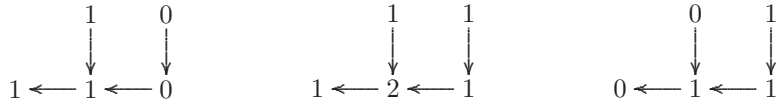
Example. Here is a further example of a self-extension which is not a ladder extension. Consider the following quiver Q



with one loop β at the vertex b, and one arrow from a to b. We consider the representations of Q with the relation $\beta^3 = 0$. The universal covering $\widetilde{Q}$ of Q has many D_5 subquivers Q' of the form



and we consider some representations of Q'; we present here the corresponding dimension vectors.



There is an obvious exact sequence

$$0 \to H \to H' \to H'' \to 0.$$

Under the covering functor, the representations H and H'' are identified, thus we obtain a self-extension. One easily checks that this self-extension is not a ladder extension.

Proposition. *Let* H *be an indecomposable module with Auslander-Reiten translate isomorphic to* H*. Assume that there is a simple submodule* S *of* H *with* $\mathrm{Ext}^1(S, S) = 0$*. Then the Auslander-Reiten sequence ending (and starting) in* H *is a ladder extension.*

Proof. Let $0 \to H \to H' \to H \to 0$ be the Auslander-Reiten sequence. Denote by $u : S \to H$ the inclusion map. Since the map $H \to H/S$ factors through

$H \to H'$, there is a commutative diagram with exact rows of the following form:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \xrightarrow{w} & U & \xrightarrow{q} & H & \longrightarrow & 0 \\ & & {\scriptstyle u}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & H & \longrightarrow & H' & \longrightarrow & H & \longrightarrow & 0 \end{array}$$

Now form the induced exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & U' & \longrightarrow & S & \longrightarrow & 0 \\ & & \| & & \downarrow & & {\scriptstyle u}\downarrow & & \\ 0 & \longrightarrow & S & \xrightarrow{w} & U & \xrightarrow{q} & H & \longrightarrow & 0 \end{array}$$

Since $\mathrm{Ext}^1(S,S) = 0$, the induced sequence splits, thus we obtain a map $v : S \to U$ with $qv = u$. It follows that $H' = H[2; w, v]$.

We do not know whether one can delete the assumption about the existence of S.

3.2. Standard self-extensions.

Let H be an R-module, say with an exact sequence $0 \to \Omega H \xrightarrow{u} PH \xrightarrow{p} H \to 0$, where PH denotes a projective cover of H. We know that

$$\mathrm{Ext}^1(H,H) = \underline{\mathrm{Hom}}(\Omega H, H) = \mathrm{Hom}(\Omega H, H)/\operatorname{Im}(\mathrm{Hom}(u,H)).$$

Note that

$$\operatorname{Im}(\mathrm{Hom}(u,H)) \subseteq \operatorname{Im}(\mathrm{Hom}(\Omega H, p)) \subseteq \mathrm{Hom}(\Omega H, H).$$

(Proof: $\mathrm{Hom}(u,H) : \mathrm{Hom}(PH,H) \to \mathrm{Hom}(\Omega H, H)$, thus take $\phi : PH \to H$ and form ϕu. Since $p : PH \to H$ is surjective and PH is projective, there is $\phi' : PH \to PH$ with $\phi = p\phi'$. Thus $\phi u = p\phi' u$ is in the image of $\mathrm{Hom}(\Omega H, p)$.)

Thus we can consider

$$\mathrm{Ext}^1(H,H)_s := \operatorname{Im}(\mathrm{Hom}(\Omega H, p))/\operatorname{Im}(\mathrm{Hom}(u,H))$$

as a subgroup of $\mathrm{Hom}(\Omega H, H)/\operatorname{Im}(\mathrm{Hom}(u,H)) = \mathrm{Ext}^1(H,H)$. We call the elements of $\mathrm{Ext}^1(H,H)_s$ the *standard* self-extensions.

Proposition. *Standard self-extensions are ladder extensions.*

Proof. Here is the usual diagram in which way a map $f : \Omega H \to H$ yields a self-extension of H

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega H & \xrightarrow{u} & PH & \xrightarrow{p} & H & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & H & \longrightarrow & H[2] & \longrightarrow & H & \longrightarrow & 0 \end{array}$$

The standard extensions are those where the map f factors through p, say $f = pw'$ with $w' : \Omega H \to PH$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega H & \xrightarrow{u} & PH & \xrightarrow{p} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle w'} & & \downarrow{\scriptstyle w_1'} & & \| & & \\
0 & \longrightarrow & PH & \xrightarrow{u_1} & U_2 & \longrightarrow & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p} & & \downarrow & & \| & & \\
0 & \longrightarrow & H & \longrightarrow & H[2] & \longrightarrow & H & \longrightarrow & 0
\end{array}$$

3.3. Modules of projective dimension 1.

Proposition. *If the projective dimension of H is at most 1, then any self-extension of H is standard, thus a ladder extension.*

Proof: Consider a module H with a projective presentation $0 \to P' \to P \xrightarrow{p} H \to 0$. Any self-extension of H is given by a diagram of the following kind:

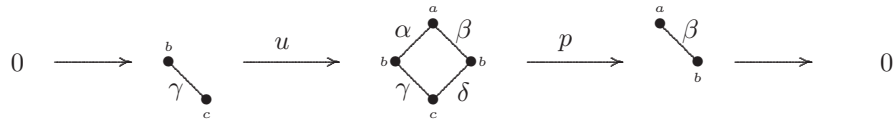
$$\begin{array}{ccccccccc}
0 & \longrightarrow & P' & \xrightarrow{u} & P & \xrightarrow{p} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\
0 & \longrightarrow & H & \longrightarrow & H[2] & \longrightarrow & H & \longrightarrow & 0
\end{array}$$

Since P' is projective and $p : P \to H$ surjective, there is a map $f' : P' \to P$ such that $f = pf'$. The self-extension is given just by $H[2] = H[2; u, f']$.

Corollary. *If R is a hereditary ring, any self-extension is standard, thus a ladder extension.*

Example of a ladder extension which is not standard. Consider the quiver Q

such that $\delta\alpha = 0 = \gamma\beta = \gamma\alpha - \delta\beta$. Consider the indecomposable length 2 module $H = (\beta : a \to b)$ annihilated by α. Then the kernel ΩH of $PH \to H$ is $\Omega H = (\gamma : b \to c)$. We may visualize this as follows:



There is a ladder extension of H, given by the non-trivial map $f : \Omega H \to H$, but this map does not factor through PH, since $\mathrm{Hom}(\Omega H, PH)$ is one-dimensional,

generated by u. Note that $\overline{f} : \Omega H/K \to H$ factors through $\overline{p} : PH/u(K) \to H$, where $K = S(c)$ is the kernel of f.

Appendix. Here, we want to indicate that the Corollary can be used in order to obtain a conceptual proof of the second Brauer-Thrall conjecture for hereditary artin algebras.

Assume that there is no generic module. We show: *Any indecomposable module is a brick without self-extensions.* Assume that there is an indecomposable module M which is not a brick or which does have self-extensions. If M is not a brick, then the brick paper [R2] shows that there are bricks M' with self-extensions. Thus, we see that there always is a brick H with self-extensions. Take any non-zero self-extension of H. According to 3.2, such a self-extension is standard, thus a ladder extension, thus we obtain a corresponding Prüfer module $H[\infty]$. The process of simplification [R1] shows that all the modules $H[n]$ are indecomposable. Thus $H[\infty]$ is not of finite type and therefore there exists a generic module [R5].

But if any indecomposable module is a brick without self-extensions, the quadratic form is weakly positive. Ovsienko asserts that then there are only finitely many positive roots, thus the algebra is of bounded representation type and therefore of finite representation type.

3.4. Warning. *A Prüfer module $H[\infty]$ is not necessarily determined by $H[2]$, even if it is of ladder type.*

As an example take the generalized Kronecker quiver with vertices a, b and three arrows $\alpha, \beta, \gamma : a \to b$. and let H be the two-dimensional indecomposable representation annihilated by α and β. Consider a projective cover $q : PH \to H$, let ΩH be its kernel, say with inclusion map $w : \Omega H \to PH$.

$$0 \to \Omega H \xrightarrow{w} PH \xrightarrow{q} H \to 0 \qquad (*)$$

The ladders to be considered are given by the various maps $f : \Omega H \to PH$ such that the image of f is not contained in ΩH (otherwise, the induced self-extension of H will split). In order to specify a self-extension $H[2]$ of H, we require that $H[2]$ is annihilated say by γ.

We will consider several copies of PH. If $e_i \in (PH)_a$ is a generator, let us denote $e_{i1} = \alpha(e_i),\ e_{i2} = \beta(e_i),\ e_{i3} = \gamma(e_i)$, thus, e_{i1}, e_{i2}, e_{i3} is a basis of $(PH)_b$.

We start with PH generated by e_1 and consider the exact sequence $(*)$ as displayed above. We see that e_{12}, e_{13} is a basis of ΩH.

Now, let us consider two maps $f, g : \Omega H \to PH$, here we denote the generator of PH by e_0. The first map f is given by $f(e_{12}) = e_{01}$ and $f(e_{13}) = 0$. The second map $g : \Omega H \to PH$ is defined by $g(e_{12}) = e_{01}$ and $g(e_{13}) = e_{02}$.

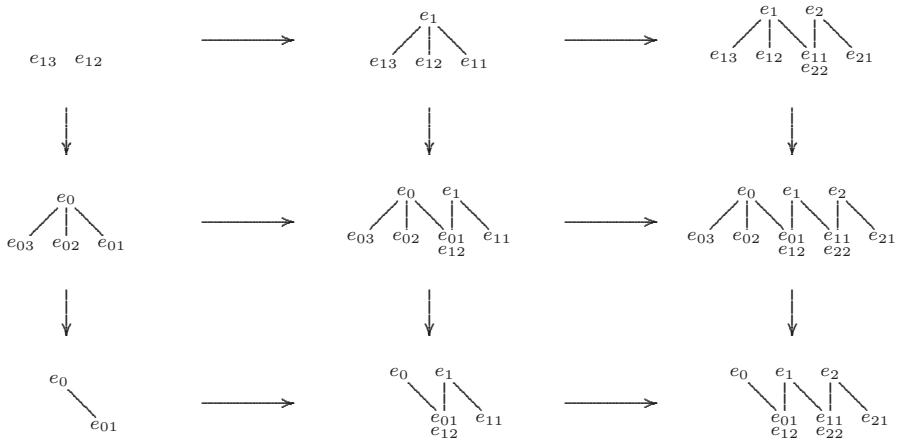
Note that $qf = qg$, thus $H[2; w, f] = H[2; w, g]$ and actually this is precisely the self-extension of H annihilated by γ.

An easy calculation shows that $H[3; w, f]$ (and even $H[\infty; w, f]$) is annihilated by γ, whereas $H[3; w, g]$ is faithful. The following displays may be helpful; always,

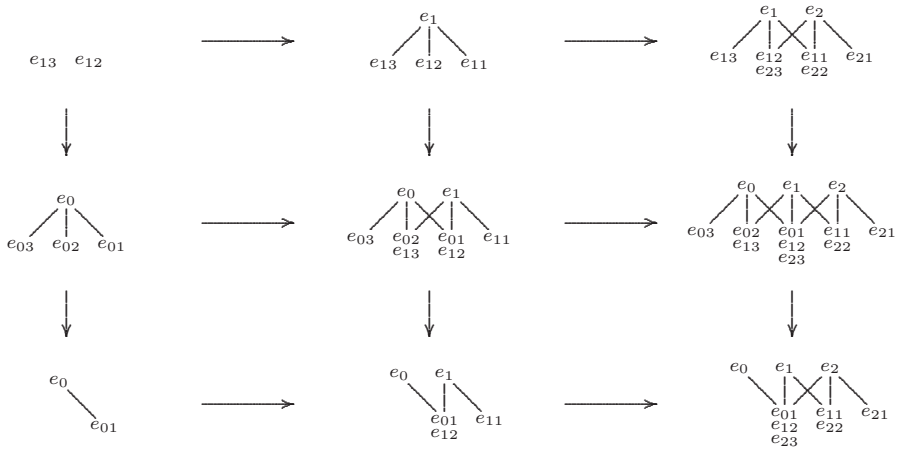
we exhibit the modules:

$$\begin{array}{ccccc} U_0 = \Omega H & \xrightarrow{w_0} & U_1 = PH & \xrightarrow{w_1} & U_2 \\ \downarrow{\scriptstyle v_0} & & \downarrow{\scriptstyle v_1} & & \downarrow{\scriptstyle v_2} \\ U_1 = PH & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 \\ \downarrow{\scriptstyle q} & & \downarrow & & \downarrow \\ H & \longrightarrow & H[2] & \longrightarrow & H[3] \end{array}$$

First the display for the homomorphism f.



Now the corresponding display for the homomorphism g.



4. Degenerations.

Definition: Let X, Y be finite length modules. Call Y a *degeneration* of X provided there is an exact sequence of the form $0 \to U \to X \oplus U \to Y \to 0$ with U

of finite length. (such a sequence will be called a *Riedtmann-Zwara sequence*). The map $U \to U$ is called a corresponding *steering* map. (Note that in case we deal with modules over a finite dimensional k-algebra and k is an algebraically closed field, then this notion of degeneration coincides with the usual one, as Zwara [Z2] has shown.)

The proof of the following result is essentially due to Zwara, he used this argument in order to show that Y is a degeneration of X if and only if there is an exact sequence $0 \to Y \to X \oplus V \to V \to 0$ (a co-Riedtmann-Zwara sequence) with V of finite length.

Proposition. *Let X, Y be Λ-modules of finite length. The following conditions are equivalent:*

(1) *Y is a degeneration of X.*
(2) *There is a Prüfer module $Y[\infty]$ and some natural number t_0 such that $Y[t+1] \simeq Y[t] \oplus X$ for all $t \geq t_0$.*
(3) *There is a Prüfer module $Y[\infty]$ and some natural number t_0 such that $Y[t_0+1] \simeq Y[t_0] \oplus X$.*

Here is the recipe how to obtain a Prüfer module $Y[\infty]$ starting from a degeneration: If Y is a degeneration of X, say with steering module U, then there exists a monomorphism $\mu : U \to U \oplus X$ with cokernel Y. The Prüfer module $Y[\infty]$ we are looking for is

$$Y[\infty] = P(\mu, \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]).$$

Proof of the implication (3) $\implies$ (1). Assume that there is a Prüfer module $Y[\infty]$ such that $Y[t+1] \simeq Y[t] \oplus X$. We get the following two exact sequences

$$0 \to Y[t] \to Y[t+1] \to Y[1] \to 0,$$
$$0 \to Y[1] \to Y[t+1] \to Y[t] \to 0, \tag{1}$$

in the first, the map $Y[t+1] \to Y[1]$ is given by applying ψ^t, in the second the map $Y[t+1] \to Y[t]$ is given by applying ψ. In both sequences, we can replace $Y[t+1]$ by $Y[t] \oplus X$. Thus we obtain as first sequence a new Riedtmann-Zwara sequence, and as second sequence a dual Riedtmann-Zwara sequence:

$$0 \to Y[t] \to Y[t] \oplus X \to Y \to 0,$$
$$0 \to Y \to Y[t] \oplus X \to Y[t] \to 0, \tag{2}$$

note that both use the same steering module, namely $Y[t]$. Thus:

Remark. We see: The module Y is a degeneration of X if and only if there exists a module V and an exact sequence $0 \to Y \to V \oplus X \to V \to 0$.

Proof of the proposition. We need further properties of exact squares:

(E2) *For any map $a : U \to V$, and any module X, the following diagram is exact:*

$$\begin{array}{ccc} U & \xrightarrow{a} & V \\ {\scriptstyle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\downarrow & & \downarrow{\scriptstyle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \\ U \oplus X & \xrightarrow[a\oplus 1_X]{} & V \oplus X. \end{array}$$

(E3) *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y_1 \\ {\scriptstyle 0}\downarrow & & \downarrow \\ Y_2 & \xrightarrow[f']{} & Z \end{array}$$

be exact. Then f' is split mono.

(E4) *Assume that we have the following exact square*

$$\begin{array}{ccc} U & \xrightarrow{a} & V \\ {\scriptstyle b}\downarrow & & {\scriptstyle b'}\downarrow \\ W & \xrightarrow[a']{} & X \end{array}$$

and that b is a split monomorphism, then the sequence

$$0 \to U \xrightarrow{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]} V \oplus W \xrightarrow{[b'\ a']} X \to 0$$

splits.

Proofs. (E2) is obvious. (E3): Since $\left[\begin{smallmatrix}f\\0\end{smallmatrix}\right]$ is injective, $f : X \to Y_1$ is injective. Let Q be the cokernel of f. We obtain the map f' by forming the induced exact sequence of $0 \to X \xrightarrow{f} Y_1 \to Q \to 0$, using the zero map $X \to Y_1$. But such an induced exact sequence splits. (E4) Assume that $pb = 1_U$. Then $[0\ p]\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right] = 1_U$.

There is the following lemma (again, see Zwara [Z1]):

Lemma (Existence of nilpotent steering maps.) *If there is an exact sequence $0 \to U \to X \oplus U \to Y \to 0$, then there is an exact sequence $0 \to U' \to X \oplus U' \to Y \to 0$ such that the map $U' \to U'$ is nilpotent.*

Proof: We can decompose $U = U_1 \oplus U_2 = U_1' \oplus U_2'$ such that the given map $f : U \to U$ maps U_1 into U_1', U_2 into U_2' and such that the induced map $f_1 : U_1 \to U_1'$ belongs to the radical of the category, whereas the induced map $f_2 : U_2 \to U_2'$ is an isomorphism. We obtain the following pair of exact squares

$$\begin{array}{ccccc} U_1 & \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & U_1 \oplus U_2 & \longrightarrow & X \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_1\oplus f_2}\downarrow & & \downarrow \\ U_1' & \xrightarrow[\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]]{} & U_1' \oplus U_2' & \longrightarrow & Y \end{array}$$

(the left square is exact according to (E2)). The composition of the squares is the desired exact square (note that U_1' is isomorphic to U_1).

Assume that a monomorphism $w = \begin{bmatrix}\phi\\ g\end{bmatrix} : U \to U \oplus X$ with cokernel Y and $\phi^t = 0$ is given. Consider also the canonical embedding $v = \begin{bmatrix}1\\ 0\end{bmatrix} : U \to U \oplus X$ and form the ladder $U_i(w, v)$ for this pair of monomorphisms w, v. The modules $Y[i] = U_i(w, v)/U_0(w, v)$ are just the modules we are looking for: As we know, there is a Prüfer module $(Y[\infty], \psi)$ with $Y[i]$ being the kernel of ψ^i.

We construct the maps w_n, v_n explicitly as follows:

$$w_n = \begin{bmatrix}\phi & \\ g & \\ & 1_{X^n}\end{bmatrix} = \begin{bmatrix}w & \\ & 1_{X^n}\end{bmatrix} : U \oplus X^n \to (U \oplus X) \oplus X^n$$

and

$$v_n = \begin{bmatrix}1_{U\oplus X_n}\\ 0\end{bmatrix} : U \oplus X^n \to U \oplus X^n \oplus X,$$

using the recipe (E2). Thus we obtain the following sequence of exact squares:

$$\begin{array}{ccccccccc}
U & \xrightarrow{\left[\begin{smallmatrix}\phi\\ g\end{smallmatrix}\right]} & U\oplus X & \xrightarrow{\left[\begin{smallmatrix}\phi & \\ g & \\ & 1\end{smallmatrix}\right]} & U\oplus X\oplus X & \xrightarrow{\left[\begin{smallmatrix}\phi & & \\ g & & \\ & 1 & \\ & & 1\end{smallmatrix}\right]} & U\oplus X\oplus X\oplus X & \longrightarrow \\
\downarrow{\scriptstyle \left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right]} & & \downarrow{\scriptstyle \left[\begin{smallmatrix}1 & \\ 0 & 1\end{smallmatrix}\right]} & & \downarrow{\scriptstyle \left[\begin{smallmatrix}1 & & \\ & 1 & \\ 0 & 0 & 1\end{smallmatrix}\right]} & & \downarrow{\scriptstyle \left[\begin{smallmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ 0 & 0 & 0 & 1\end{smallmatrix}\right]} & \\
U\oplus X & \xrightarrow[\left[\begin{smallmatrix}\phi & \\ g & \\ & 1\end{smallmatrix}\right]]{} & U\oplus X\oplus X & \xrightarrow[\left[\begin{smallmatrix}\phi & & \\ g & & \\ & 1 & \\ & & 1\end{smallmatrix}\right]]{} & U\oplus X\oplus X\oplus X & \xrightarrow[\left[\begin{smallmatrix}\phi & & & \\ g & & & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{smallmatrix}\right]]{} & U\oplus X\oplus X\oplus X\oplus X & \longrightarrow
\end{array}$$

In particular, we have $U_n = U_n(w, v) = U \oplus X^n$.

Note that the composition $w_{n-1} \cdots w_0 : U \to U \oplus X^n$ is of the form $\begin{bmatrix}\phi^n\\ g_n\end{bmatrix}$ for some $g_n : U \to X^n$.

We also have the following sequence of exact squares:

$$\begin{array}{ccccccccc}
U = U_0 & \xrightarrow{w_0} & U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y[1] & \xrightarrow{s_1} & Y[2] & \xrightarrow{s_2} & Y[3] & \xrightarrow{s_3} & \cdots
\end{array}$$

where the vertical maps are of the form

$$U_n = U \oplus X^n \xrightarrow{[h_n\ q_n]} Y[n].$$

The composition of these exact squares yields an exact square

$$\begin{array}{ccc} U & \xrightarrow{w_{n-1}\cdots w_0} & U \oplus X^n \\ \big\downarrow & & \big\downarrow{\scriptstyle [h_n\ q_n]} \\ 0 & \longrightarrow & Y[n] \end{array}$$

Here we may insert the following observation: This sequence shows that *the module* $Y[n]$ *is a degeneration of the module* X^n.

Since the composition $w_{n-1}\cdots w_0 : U \to U \oplus X^n$ is of the form $\begin{bmatrix}\phi^n \\ g_n\end{bmatrix}$, and $\phi^t = 0$, it follows that h_t is a split monomorphism, see (E3).

Also, we can consider the following two exact squares, with $w = \begin{bmatrix}\phi \\ g\end{bmatrix} : U \to V = U \oplus X$ (the upper square is exact, according to (E2)):

$$\begin{array}{ccc} U & \xrightarrow{w} & V \\ {\scriptstyle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\big\downarrow & & \big\downarrow{\scriptstyle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \\ U \oplus X^t & \xrightarrow{\left[\begin{smallmatrix}w & \\ & 1\end{smallmatrix}\right]} & V \oplus X^t \\ {\scriptstyle [h_t\ q_t]}\big\downarrow & & \big\downarrow{\scriptstyle [h_{t+1}\ q_{t+1}]} \\ Y[t] & \longrightarrow & Y[t+1] \end{array}$$

The vertical composition on the left is h_t, thus, as we have shown, a split monomorphism. This shows that the exact sequence corresponding to the composed square splits (E4): This yields

$$U \oplus Y[t+1] \simeq Y[t] \oplus V = Y[t] \oplus U \oplus X.$$

Cancelation of U gives the desired isomorphism:

$$Y[t+1] \simeq Y[t] \oplus X.$$

Remark to the proof. Given the Riedtmann-Zwara sequence

$$0 \to U \xrightarrow{\left[\begin{smallmatrix}\phi\\ g\end{smallmatrix}\right]} U \oplus X \to Y \to 0,$$

we have considered the following pair of monomorphisms

$$w = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], w' = \left[\begin{smallmatrix}\phi\\ g\end{smallmatrix}\right] : U \to U \oplus X.$$

The corresponding Prüfer modules are $X^{(\infty)}$ and $Y[\infty]$, respectively. And $U_n(w, w') = U \oplus X^n$. As we know, we can assume that ϕ is nilpotent. Then all the linear combinations

$$w + \lambda w' = \left[\begin{smallmatrix}1+\lambda\phi\\ g\end{smallmatrix}\right]$$

with $\lambda \in k$ are also split monomorphisms (with retraction $[\,\eta\ 0\,]$, where $\eta = (1 + \lambda\phi)^{-1}$).

Corollary. *Assume that Y is a degeneration of X. Then there exists a Prüfer module $Y[\infty]$ such that $Y[\infty]$ is isomorphic to $Y[t] \oplus X^{(\infty)}$ for some natural number t.*

5. Application: The theorem of Bautista-Pérez.

Here we assume that we deal with an artin algebra Λ, and all the modules are Λ-modules of finite length.

Proposition. *Let W be a module with $\mathrm{Ext}^1(W,W) = 0$ and assume there is given an exact sequence $0 \to U \to V \to W \to 0$. Then the cokernel of any monomorphism $U \to V$ is a degeneration of W.*

Corollary (Bautista-Pérez). *Let U, V be modules, and let W and W' be cokernels of monomorphisms $U \to V$. Assume that both $\mathrm{Ext}^1(W,W) = 0$ and $\mathrm{Ext}^1(W',W') = 0$. Then the modules W and W' are isomorphic.*

Both assertions are well-known in case k is an algebraically closed field: in this case, the conclusion of the proposition just asserts that W' is a degeneration of W in the sense of algebraic geometry. The main point here is to deal with the general case when Λ is an arbitrary artin algebra. The corollary stated above (under the additional assumptions that V is projective and that $w(U), w'(U)$ are contained in the radical of V) is due to Bautista and Pérez [BP] and this result was presented by Smalø with a new proof [S] at Mar del Plata.

We need the following well-known lemma.

Lemma. *Let W be a module with $\mathrm{Ext}^1(W,W) = 0$. Let $U_0 \subset U_1 \subset U_2 \subset \cdots$ be a sequence of inclusions of modules with $U_i/U_{i-1} = W$ for all $i \geq 1$. Then there is a natural number n_0 such that $U_n \subset U_{n+1}$ is a split monomorphism for all $n \geq n_0$.*

Let us use it in order to finish the proof of the proposition. Let $U_0 = U, U_1 = V$, and $w_0 : U_0 \to V_0$ the given monomorphism with cokernel W. Let $v_0 : U_0 \to U_1$ be an additional monomorphism, say with cokernel W'. Thus we are in the setting of section 1. We apply the Lemma to the chain of inclusions

$$U_0 \xrightarrow{w_0} U_1 \xrightarrow{w_1} U_2 \xrightarrow{w_2} \cdots$$

and see that there is n such that $w_n : U_n \to U_{n+1}$ splits. This shows that U_{n+1} is isomorphic to $U_n \oplus W$. But we also have the exact sequence

$$0 \to U_n \xrightarrow{v_n} U_{n+1} \to W' \to 0.$$

Replacing U_{n+1} by $U_n \oplus W$, we see that we get an exact sequence of the form

$$0 \to U_n \xrightarrow{v_n} U_n \oplus W \to W' \to 0$$

(a Riedtmann-Zwara sequence), as asserted.

Proof of the Corollary. It is well-known that the existence of exact sequences

$$0 \to X \to X \oplus W \to W' \to 0 \quad \text{and} \quad 0 \to Y \to Y \oplus W' \to W \to 0$$

implies that the modules W and W' are isomorphic [Z1]. But in our case we just have to change one line in the proof of the proposition in order to get the required isomorphism. Thus, assume that both $\mathrm{Ext}^1(W,W) = 0$ and $\mathrm{Ext}^1(W',W') = 0$. Choose n such that **both** the inclusion maps

$$w_n : U_n \to U_{n+1} \quad \text{and} \quad v_n : U_n \to U_{n+1}$$

split. Then U_{n+1} is isomorphic both to $U_n \oplus W$ and to $U_n \oplus W'$, thus it follows from the Krull-Remak-Schmidt theorem that W and W' are isomorphic.

Remark. *Assume that $w, w' : U \to V$ are monomorphisms with cokernels W and W', respectively, and that $\mathrm{Ext}^1(W,W) = 0$ and $\mathrm{Ext}^1(W',W') = 0$. Then w splits if and only if w' splits.*

Proof: According to the corollary, we can assume $W = W'$. Assume that w splits, thus V is isomorphic to $U \oplus W$. Look at the exact sequence $0 \to U \xrightarrow{w'} V \to W \to 0$. If it does not split, then $\dim \mathrm{End}(V) < \dim \mathrm{End}(U \oplus W)$, but V is isomorphic to $U \oplus W$.

As we have mentioned, the lemma is well-known; an equivalent assertion was used for example by Roiter in his proof of the first Brauer-Thrall conjecture, a corresponding proof can be found in [R4]. We include here a slightly different proof:

Applying the functor $\mathrm{Hom}(W,-)$ to the short exact sequence $0 \to U_{i-1} \xrightarrow{w_{i-1}} U_i \to W \to 0$, we obtain the exact sequence

$$\mathrm{Ext}^1(W,U_{i-1}) \to \mathrm{Ext}^1(W,U_i) \to \mathrm{Ext}^1(W,W).$$

Since the latter term is zero, we see that we have a sequence of surjective maps

$$\mathrm{Ext}^1(W,U_0) \to \mathrm{Ext}^1(W,U_1) \to \cdots \to \mathrm{Ext}^1(W,U_i) \to \cdots,$$

being induced by the inclusion maps $U_0 \to U_1 \to \cdots \to U_i \to \cdots$. The maps between the Ext-groups are k-linear. Since $\mathrm{Ext}^1(W,U_0)$ is a k-module of finite length, the sequence of surjective maps must stabilize: there is some n_0 such that the inclusion $U_n \to U_{n+1}$ induces an isomorphism

$$\mathrm{Ext}^1(W,U_n) \to \mathrm{Ext}^1(W,U_{n+1})$$

for all $n \geq n_0$. Now we consider also some Hom-terms: the exactness of

$$\mathrm{Hom}(W,U_{n+1}) \to \mathrm{Hom}(W,W) \to \mathrm{Ext}^1(W,U_n) \to \mathrm{Ext}^1(W,U_{n+1})$$

shows that the connecting homomorphism is zero, and thus that the map $\mathrm{Hom}(W,U_{n+1}) \to \mathrm{Hom}(W,W)$ (induced by the projection map $p : U_{n+1} \to W$) is surjective. But this means that there is a map $h \in \mathrm{Hom}(W,U_{n+1})$ with $ph = 1_W$, thus $p : U_{n+1} \to W$ is a split epimorphism and therefore the inclusion map $U_n \to U_{n+1}$ is a split monomorphism.

Remark. In general, there is no actual bound on the number n_0. However, in case of dealing with the chain of inclusions

$$U_0 \xrightarrow{w_0} U_1 \xrightarrow{w_1} U_2 \xrightarrow{w_n} \cdots$$

such a bound exists, namely the length of $\mathrm{Ext}^1(W, U_0)$ as a k-module, or, even better, the length of $\mathrm{Ext}^1(W, U_0)$ as an E-module, where $E = \mathrm{End}(W)$.

Proof: Look at the surjective maps

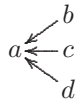
$$\mathrm{Ext}^1(W, U_0) \to \mathrm{Ext}^1(W, U_1) \to \cdots \to \mathrm{Ext}^1(W, U_i) \to \cdots,$$

being induced by the maps $U_n \xrightarrow{w_n} U_{n+1}$ (and these maps are not only k-linear, but even E-linear). Assume that $\mathrm{Ext}^1(W, U_n) \to \mathrm{Ext}^1(W, U_{n+1})$ is bijective, for some n. As we have seen above, this implies that the sequence

$$0 \to U_n \xrightarrow{w_n} U_{n+1} \to W \to 0 \qquad (*)$$

splits. Now the map w_{n+1} is obtained from $(*)$ as the induced exact sequence using the map w'_n. With $(*)$ also any induced exact sequence will split. Thus w_{n+1} is a split monomorphism (and $\mathrm{Ext}^1(W, U_{n+1}) \to \mathrm{Ext}^1(W, U_{n+2})$ will be bijective, again). Thus, as soon as we get a bijection $\mathrm{Ext}^1(W, U_n) \to \mathrm{Ext}^1(W, U_{n+1})$ for some n, then also all the following maps $\mathrm{Ext}^1(W, U_m) \to \mathrm{Ext}^1(W, U_{m+1})$ with $m > n$ are bijective.

Example. Consider the D_4-quiver with subspace orientation:



and let Λ be its path algebra over some field k. We denote the indecomposable Λ-modules by the corresponding dimension vectors. Let

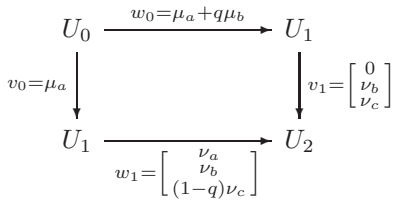
$$U_0 = \begin{smallmatrix} & & 0 \\ 1 & & 0 \\ & & 0 \end{smallmatrix}, \quad U_1 = \begin{smallmatrix} & & 1 \\ 2 & & 1 \\ & & 1 \end{smallmatrix}, \quad W = \begin{smallmatrix} & & 1 \\ 1 & & 1 \\ & & 1 \end{smallmatrix}, \quad W' = \begin{smallmatrix} & & 0 \\ 1 & & 1 \\ & & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & & 1 \\ 0 & & 0 \\ & & 0 \end{smallmatrix}.$$

Note that a map $w_0 : U_0 \to U_1$ with cokernel W exists only in case the base-field k has at least 3 elements; of course, there is always a map $w'_0 : U_0 \to U_1$ with cokernel W'.

We have $\dim \mathrm{Ext}^1(W, U_0) = 2$, and it turns out that the module U_2 is the following:

$$U_2 = \begin{smallmatrix} & & 0 \\ 1 & & 1 \\ & & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & & 1 \\ 1 & & 0 \\ & & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & & 1 \\ 1 & & 1 \\ & & 0 \end{smallmatrix}.$$

The pushout diagram involving the modules U_0, U_1 (twice) and U_2 is constructed as follows: denote by μ_a, μ_b, μ_c monomorphisms $U_0 \to U_1$ which factor through the indecomposable projective modules $P(a), P(b), P(c)$, respectively. We can assume that $\mu_c = -\mu_a - \mu_b$, so that a mesh relation is satisfied. Denote the 3 summands of U_2 by M_a, M_b, M_c, with non-zero maps $\nu_a : U_1 \to M_a$, $\nu_b : U_1 \to M_b$, $\nu_c : U_1 \to M_c$, such that $\nu_a\mu_a = 0$, $\nu_b\mu_b = 0$, $\nu_c\mu_c = 0$. There is the following commutative square, for any $q \in k$, we are interested when $q \notin \{0, 1\}$:



(the only calculation which has to be done concerns the third entries: $\nu_c(\mu_a + q\mu_b) = (1-q)\nu_c\mu_a$). Note that w_1 (as well as w_1') does not split.

But now we deal with a module U_2 such that $\mathrm{Ext}^1(W, U_2) = 0$. This implies that U_3 is isomorphic to $U_2 \oplus W$. Thus the next pushout construction yields an exact sequence of the form

$$0 \to U_2 \to U_2 \oplus W \to W' \to 0.$$

Acknowledgement. The author is indebted to Dieter Vossieck for a careful reading of the final version of the paper.

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Claus Michael Ringel
Fakultät für Mathematik, Universität Bielefeld,
POBox 100 131,
D-33 501 Bielefeld
ringel@math.uni-bielefeld.de

Recibido: 5 de marzo de 2007
Aceptado: 16 de abril de 2007