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DERIVED CATEGORIES AND THEIR APPLICATIONS

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ABSTRACT. In this notes we start with the basic definitions of derived categories, derived functors, tilting complexes and stable equivalences of Morita type. Our aim is to show via several examples that this is the best framework to do homological algebra, We also exhibit their usefulness for getting new proofs of well known results. Finally we consider the Morita invariance of Hochschild cohomology and other derived functors.

1. Introduction

Derived categories were invented by A. Grothendieck and his school in the early sixties. The volume [Ve77] reproduces some notes of his pupil J.L.Verdier, dating from 1963, which are the original source on derived categories (see also [Ve96]).

Let $\mathcal{A}$ be an abelian category and $\mathcal{C}(\mathcal{A})$ the category of complexes in $\mathcal{A}$, that is, objects are sequences of maps

$$\cdots \to X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \to \cdots$$

with $d^n \circ d^{n-1} = 0$ for all $n \in \mathbb{Z}$, and a morphism $f : X \to Y$ between complexes is a family of morphisms $f^n : X^n \to Y^n$ in $\mathcal{A}$ such that $d^n_Y \circ f^n = f^{n+1} \circ d^n_X$. The cohomology groups of the complex X are by definition

$$H^n(X) = \operatorname{Ker} d^n / \operatorname{Im} d^{n-1}.$$

A morphism $f : X \to Y$ induces a group morphism $H^n(f) : H^n(X) \to H^n(Y)$ in each degree. We say that f is a <u>quasi-isomorphism</u> if $H^n(f)$ is an isomorphism for all $n \in \mathbb{Z}$.

The <u>derived category</u> $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ is obtained from $\mathcal{C}(\mathcal{A})$ by formally inverting all quasi-isomorphisms.

Recall that the definition of derived functors in homological algebra is as follows. Assume that $\mathcal{A}$ has enough projectives and let $F : \mathcal{A} \to Ab$ be a contravariant left exact functor. Then the right derived functor $R^nF : \mathcal{A} \to Ab$ is defined in the following way. Let M be an object in $\mathcal{A}$ and let

$$\cdots \to P_2 \to P_1 \to P_0 \to 0$$

be a projective resolution of M, that is,

$$\begin{array}{ccccccccc} \cdots & \to & P_2 & \to & P_1 & \to & P_0 & \to & 0 & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots & \to & 0 & \to & 0 & \to & M & \to & 0 & \cdots \end{array}$$

is a quasi-isomorphism. Then $R^nF(M)$ is the group homology in degree n of the complex

$$0 \to F(P_0) \to F(P_1) \to F(P_2) \to \cdots .$$

These are well defined functors since the projective resolution of an object M is unique up to quasi-isomorphisms. Moreover, for any short exact sequence $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$ there is a long exact sequence

$$\cdots \to R^nF(M_3) \xrightarrow{R^nF(g)} R^nF(M_2) \xrightarrow{R^nF(f)} R^nF(M_1) \xrightarrow{\delta^n} R^{n+1}F(M_3) \to \cdots$$

where δ is the so called connecting morphism.

In fact, when we are doing homological algebra, we are not dealing with objects and cohomology groups but with complexes up to quasi-isomorphisms and their cohomology groups. Hence it is natural to work in $\mathcal{D}(\mathcal{A})$ instead of $\mathcal{A}$ or $\mathcal{C}(\mathcal{A})$.

Any object X in $\mathcal{A}$ can be identified in $\mathcal{D}(\mathcal{A})$ by the complex concentrated in degree zero, which will be denoted by $X[0]$. Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence in $\mathcal{A}$. Then $0 \to X[0] \xrightarrow{f} Y[0] \xrightarrow{g} Z[0] \to 0$ is a short exact sequence in $\mathcal{C}(\mathcal{A})$. Since

$$\begin{array}{ccccccccccc} \cdots & \to & 0 & \to & X & \xrightarrow{f} & Y & \to & 0 & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow g & & \downarrow & & \\ \cdots & \to & 0 & \to & 0 & \to & Z & \to & 0 & \to & \cdots \end{array}$$

is a quasi-isomorphism, we can replace $Z[0]$ by the complex appearing in the first line of the previous picture, hence the short exact sequence $0 \to X[0] \xrightarrow{f} Y[0] \xrightarrow{g} Z[0] \to 0$ can be identified in $\mathcal{D}(\mathcal{A})$ by the sequence of complexes

$$\begin{array}{ccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & 0 & \to & 0 & \to & X & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow f & & \downarrow \\ 0 & \to & X & \xrightarrow{f} & Y & = & Y & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

which is not an exact sequence.

In this new category, the concept of "short exact sequences" (which are determined by two morphisms f, g) will be replaced by that of "distinguished triangles"(determined by three morphisms f, g, h). If $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is a short exact sequence in $\mathcal{A}$, then

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & 0 & \to & 0 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & 0 & \to & X & = & X \\
\downarrow & & \downarrow & & \downarrow f & & \downarrow \\
X & \xrightarrow{f} & Y & = & Y & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & 0 & \to & 0 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
 & & & & & & \\
X[0] & \to & Y[0] & \to & Z[0] & \to & X[1]
\end{array}$$

is a triangle in $\mathcal{D}(\mathcal{A})$.

We will consider "cohomological functors" defined in $\mathcal{D}(\mathcal{A})$, that applied to distinguished triangles will give long exact sequences with morphisms induced by f, g, h. That is, the extra morphism we are considering when defining triangles is a morphism of complexes that induces the corresponding connecting morphism.

So, when considering the derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$, we replace objects in $\mathcal{A}$ by complexes, and invert quasi-isomorphisms between complexes. As we shall see, $\mathcal{D}(\mathcal{A})$ is not abelian if $\mathcal{A}$ is not semisimple. The abelian structure of $\mathcal{A}$, and of $\mathcal{C}(\mathcal{A})$, has to be replaced by a triangulated structure.

Given an algebra A over a commutative ring k, for simplicity, we will write $\mathcal{D}(A)$ for the derived category $\mathcal{D}(\mathrm{Mod}-A)$. Given two k-algebras A and B the natural question is: when are $\mathcal{D}(A)$ and $\mathcal{D}(B)$ equivalent categories? (triangle equivalent?). Of course, if A and B are Morita equivalent (that is, Mod-A and Mod-B are k-linearly equivalent) then $\mathcal{D}(A)$ and $\mathcal{D}(B)$ are equivalent. We will see that there other equivalences and we shall present some examples.

2. Triangulated categories

Let $\mathcal{T}$ be an additive category with an additive automorphism $T : \mathcal{T} \to \mathcal{T}$ called the translation functor. We will write $X[n]$ and $f[n]$ for $T^n(X)$ and $T^n(f)$ respectively. A triangle in $\mathcal{T}$ is a diagram of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

that will be denoted as (X, Y, Z, f, g, h), and a morphism of triangles is a triple

$$(\alpha, \beta, \gamma) : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$$

forming a commutative diagram in $\mathcal{T}$

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \alpha[1] \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1] \end{array}$$

A morphism of triangles is said to be an isomorphism if α, β, γ are isomorphisms in $\mathcal{T}$.

DEFINITION 2.1. *A structure of triangulated category on $\mathcal{T}$ is given by a translation functor T and a class of distinguished triangles verifying the following four axioms:*

TR1. a) $(X, X, 0, id_X, 0, 0)$ *is a distinguished triangle, for any object X;*
b) *Every triangle isomorphic to a distinguished one is distinguished;*
c) *Every morphism $X \xrightarrow{f} Y$ can be embedded in a distinguished triangle (X, Y, Z, f, g, h).*

TR2. *(Rotation) A triangle (X, Y, Z, f, g, h) is distinguished if and only if $(Y, Z, X[1], g, h, -f[1])$ is distinguished.*

TR3. *(Morphisms) Every commutative diagram*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \downarrow \alpha & & \downarrow \beta & & & & \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1] \end{array}$$

whose rows are distinguished triangles can be completed to a morphism of triangles by a morphism $Z \xrightarrow{\gamma} Z'$.

TR4. *(The octahedral axiom) Given $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ morphisms in $\mathcal{A}$, and distinguished triangles*

$$(X, Y, X', f, f', s), \quad (X, Z, Y', g \circ f, h, r), \quad (Y, Z, Z', g, g', t),$$

there exist morphisms $X' \xrightarrow{u} Y'$, $Y' \xrightarrow{v} Z'$ such that

$$(X', Y', Z', u, v, f'[1] \circ t)$$

is a distinguished triangle and

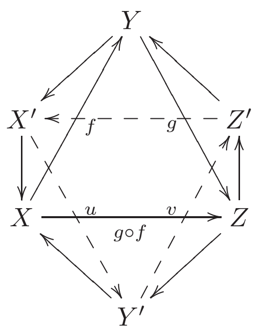
$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{f'} & X' & \xrightarrow{s} & X[1] \\ \| & & \downarrow g & & \downarrow u & & \| \\ X & \xrightarrow{g \circ f} & Z & \xrightarrow{h} & Y' & \xrightarrow{r} & X[1] \\ \downarrow f & & \| & & \downarrow v & & \downarrow f[1] \\ Y & \xrightarrow{g} & Z & \xrightarrow{g'} & Z' & \xrightarrow{t} & Y[1] \\ \downarrow f' & & \downarrow h & & \| & & \downarrow f'[1] \\ X' & \overset{u}{\dashrightarrow} & Y' & \overset{v}{\dashrightarrow} & Z' & \xrightarrow{w} & X'[1] \end{array}$$

is a commutative diagram.

Observe that all the rows in the previous diagram are distinguished triangles, and morphisms between rows determined the following morphisms of triangles

$$\begin{array}{ccccc}
(X, Y, X', f, f', s) & \overset{(id_X, g, u)}{\longrightarrow} & (X, Z, Y', g \circ f, h, r) & \overset{(f, id_Z, v)}{\longrightarrow} & (Y, Z, Z', g, g', t) \\
(Y, Z, Z', g, g', t) & \overset{(f', h, id_{Z'})}{\longrightarrow} & (X', Y', Z', u, v, w) & \overset{(s, tv, w)}{\longrightarrow} & (X[1], Y[1], X'[1], f[1], f'[1], s[1])
\end{array}$$

The name of last axiom comes from the fact that it can be viewed as a picture of a octahedron, where four faces are distinguished triangles, and all the other faces are commutative.



DEFINITION 2.2. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}$ an abelian category. An additive functor $H : \mathcal{T} \to \mathcal{A}$ is said to be a* cohomological functor *if, for any distinguished triangle (X, Y, Z, f, g, h) in $\mathcal{T}$, we get an exact sequence in $\mathcal{A}$*

$$H^i(X) \overset{H^i(f)}{\to} H^i(Y) \overset{H^i(g)}{\to} H^i(Z)$$

where H^i denotes $H \circ T^i = H(-[i])$.

REMARK 2.3. *By TR2, if (X, Y, Z, f, g, h) is a distinguished triangle then $(Y, Z, X[1], g, h, -f[1])$ is also distinguished. Then, if H is a cohomological functor,*

$$H^i(Y) \overset{H^i(g)}{\to} H^i(Z) \overset{H^i(h)}{\to} H^{i+1}(X)$$

is also exact. So we get a long exact sequence

$$\cdots \to H^i(X) \overset{H^i(f)}{\to} H^i(Y) \overset{H^i(g)}{\to} H^i(Z) \overset{H^i(h)}{\to} H^{i+1}(X) \to \cdots$$

LEMMA 2.4. *Let (X, Y, Z, f, g, h) be a distinguished triangle in $\mathcal{T}$. Then*

i) *$g \circ f = 0$;*
ii) *If $u : U \to Y$ is such that $g \circ u = 0$, then there exists $v : U \to X$ such that $u = f \circ v$;*
iii) *If $s : Y \to W$ is such that $s \circ f = 0$, then there exists $t : Z \to W$ such that $s = t \circ g$.*

Proof. i) From TR1 we know that $(X, X, 0, id_X, 0, 0)$ is a distinguished triangle, and from TR3 we know that there exists a morphism of triangles

$$\begin{array}{ccccccc} X & \stackrel{id_X}{\rightarrow} & X & \rightarrow & 0 & \rightarrow & X[1] \\ \downarrow id_X & & \downarrow f & & \downarrow s & & \downarrow id_X[1] \\ X & \stackrel{f}{\rightarrow} & Y & \stackrel{g}{\rightarrow} & Z & \stackrel{h}{\rightarrow} & X[1] \end{array}$$

and hence $g \circ f = 0$.

ii) From TR2 and TR3 we know that the diagram

$$\begin{array}{ccccccc} U & \stackrel{id_U}{\rightarrow} & U & \rightarrow & 0 & \rightarrow & U[1] \\ & & \downarrow u & & \downarrow & & \\ X & \stackrel{f}{\rightarrow} & Y & \stackrel{g}{\rightarrow} & Z & \stackrel{h}{\rightarrow} & X[1] \end{array}$$

can be completed to a morphism of triangles. Hence, there exists $U \stackrel{v}{\rightarrow} X$ such that $f \circ v = u$.

iii) It follows as (ii).

□

REMARK 2.5. *Using TR2 we also get that* $h \circ g = 0$ *and* $f[1] \circ h = 0$.

COROLLARY 2.6. *If* U *is an object in a triangulated category* $\mathcal{T}$, *then* $\operatorname{Hom}_{\mathcal{T}}(U, -) : \mathcal{T} \rightarrow Ab$ *and* $\operatorname{Hom}_{\mathcal{T}}(-, U) : \mathcal{T}^{op} \rightarrow Ab$ *are cohomological functors.*

COROLLARY 2.7. *Any distinguished triangle is determined, up to isomorphisms, by one of its morphisms.*

Proof. From TR2, it suffices to prove that the distinguished triangles (X, Y, Z, f, g, h) and (X, Y, Z', f, g', h') are isomorphic. By TR3, there exists a morphism of triangles

$$\begin{array}{ccccccc} X & \stackrel{f}{\rightarrow} & Y & \stackrel{g}{\rightarrow} & Z & \stackrel{h}{\rightarrow} & X[1] \\ \downarrow id_X & & \downarrow id_Y & & \downarrow t & & \downarrow id_X[1] \\ X & \stackrel{f}{\rightarrow} & Y & \stackrel{g'}{\rightarrow} & Z' & \stackrel{h'}{\rightarrow} & X'[1] \end{array}$$

If we apply the cohomological functors $\operatorname{Hom}_{\mathcal{T}}(-, Z)$ and $\operatorname{Hom}_{\mathcal{T}}(Z', -)$, by the 5-lemma we get that

$$t^* : \operatorname{Hom}_{\mathcal{T}}(Z', Z) \rightarrow \operatorname{Hom}_{\mathcal{T}}(Z, Z), \qquad t_* : \operatorname{Hom}_{\mathcal{T}}(Z', Z) \rightarrow \operatorname{Hom}_{\mathcal{T}}(Z', Z')$$

are isomorphisms. Then, t has left and right inverse, and therefore, t is an isomorphism. □

PROPOSITION 2.8. *For any distinguished triangle* (X, Y, Z, f, g, h), *the following conditions are equivalent:*

(i) f *is a monomorphism;*
(ii) $h = 0$;
(iii) *there exists* $Z \stackrel{s}{\rightarrow} Y$ *such that* $g \circ s = id_Z$;
(iv) g *is an epimorphism;*
(v) *there exists* $Y \stackrel{t}{\rightarrow} X$ *such that* $t \circ f = id_X$.

Proof. $(iii) \Rightarrow (iv)$ and $(v) \Rightarrow (i)$ are immediate.

$(i) \Rightarrow (ii)$ By $TR2$ and Lemma 2.4(i) we have that $f[1] \circ h = 0$. Since f is a monomorphism, $h = 0$.

$(ii) \Rightarrow (iii)$ If $h = 0$, by Corollary 2.6,

$$\mathrm{Hom}_{\mathcal{T}}(Z, Y) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{T}}(Z, Z) \xrightarrow{0} \mathrm{Hom}_{\mathcal{T}}(Z, X[1])$$

is exact, so there exists $s : Z \to Y$ such that $g \circ s = id_Z$.

$(iv) \Rightarrow (v)$ By Lemma 2.4(i) we know that $h \circ g = 0$. Since g is an epimorphism we have that $h = 0$, and by Corollary 2.6,

$$\mathrm{Hom}_{\mathcal{T}}(Y, X) \xrightarrow{g^*} \mathrm{Hom}_{\mathcal{T}}(X, X) \xrightarrow{0} \mathrm{Hom}_{\mathcal{T}}(Z[-1], X)$$

is exact, so there exists $t : Y \to X$ such that $t \circ f = id_X$. □

COROLLARY 2.9. *In any triangulated category any monomorphism splits and any epimorphism splits. Moreover, f is an isomorphism if and only if f is a monomorphism and an epimorphism.*

3. Distinguished triangles in $\mathcal{K}(\mathcal{A})$

Let $\mathcal{A}$ be an abelian category, and let $\mathcal{C}(\mathcal{A})$ be the category of complexes in $\mathcal{A}$. The homotopy category $\mathcal{K}(\mathcal{A})$ is defined as follows: the objects of $\mathcal{K}(\mathcal{A})$ are the objects in $\mathcal{C}(\mathcal{A})$, and morphisms in $\mathcal{K}(\mathcal{A})$ are the homotopy equivalence classes of morphisms in $\mathcal{C}(\mathcal{A})$. That is, $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X, Y) = \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X, Y)/\simeq$, where $f \simeq g$ if there exists $h^n : X^n \to Y^{n-1}$ with $d_Y^{n-1} \circ h^n + h^{n+1} \circ d_X^n = f^n - g^n$. So, the morphisms homotopic to zero in $\mathcal{C}(\mathcal{A})$ become the zero morphisms in $\mathcal{K}(\mathcal{A})$ and the homotopic equivalences become isomorphisms.

It can be checked that $\mathcal{K}(\mathcal{A})$ is well defined as a category, and moreover, it is an additive category, and the quotient $\mathcal{C}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$ is an additive functor.

Observe that $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X, Y)$ can also be defined as the quotient of $\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X, Y)$ by the subgroup

$$Ht(X, Y) = \{X \xrightarrow{f} Y : f \text{ is homotopic to } 0\}$$

since Ht is an ideal in $\mathcal{C}(\mathcal{A})$, that is, $g \circ f \in Ht$ if f or g belongs to Ht.

Let $T : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{A})$ be the automorphism defined in the following way: for any complex X in $\mathcal{C}(\mathcal{A})$, $T(X)^n = X^{n+1}$ and $T(d)^n = -d^{n+1}$; for any morphism $X \xrightarrow{f} Y$ in $\mathcal{C}(\mathcal{A})$, $T(f)^n = f^{n+1}$. The functor T is additive and it is an automorphism. Since $T(f)$ is a morphism homotopic to zero if and only if f is a morphism homotopic to zero, it induces an additive automorphism $T : \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$. We denote $X[n] = T^n(X)$ and $d[n] = T^n(d)$, for any $n \in \mathbb{Z}$.

We will see that $\mathcal{K}(\mathcal{A})$ is a triangulated category with translation functor T.

LEMMA 3.1. *The cohomology functors $H^n : \mathcal{C}(\mathcal{A}) \to \mathcal{A}$ induce well defined functors $H^n : \mathcal{K}(\mathcal{A}) \to \mathcal{A}$.*

Proof. If $f, g : X \to Y$ are homotopic, then $d_Y^{n-1} \circ h^n + h^{n+1} \circ d_X^n = f^n - g^n$, so $H^n(f) = H^n(g)$. $\square$

DEFINITION 3.2. *Let $X \xrightarrow{f} Y$ be a morphism in $\mathcal{C}(\mathcal{A})$.*

(i) *The* <u>mapping cone</u> *of f is the complex* $\mathrm{cone}(f)$ *such that* $\mathrm{cone}(f)^n = X^{n+1} \oplus Y^n$*, and the differential $d_f^n : X^{n+1} \oplus Y^n \to X^{n+2} \oplus Y^{n+1}$ is given by*

$$d_f^n = \begin{pmatrix} -d_X^{n+1} & 0 \\ f & d_Y^n \end{pmatrix}.$$

(ii) *The* <u>mapping cylinder</u> *of f is the complex* $\mathrm{cyl}(f)$ *such that* $\mathrm{cyl}(f)^n = X^n \oplus X^{n+1} \oplus Y^n$*, and the differential $d^n : X^n \oplus X^{n+1} \oplus Y^n \to X^{n+1} \oplus X^{n+2} \oplus Y^{n+1}$ is given by*

$$d^n = \begin{pmatrix} d_X^n & id & 0 \\ 0 & -d_X^{n+1} & 0 \\ 0 & f & d_Y^n \end{pmatrix}.$$

PROPOSITION 3.3. *Any short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathcal{C}(\mathcal{A})$ fits into a commutative diagram*

$$\begin{array}{ccccccccc} & & 0 & \to & Y & \xrightarrow{h} & \mathrm{cone}(f) & \xrightarrow{\delta} & X[1] & \to 0 \\ & & & & \downarrow \alpha & & \| & & & \\ 0 & \to & X & \xrightarrow{\nu} & \mathrm{cyl}(f) & \xrightarrow{\pi} & \mathrm{cone}(f) & \to & 0 & \\ & & \| & & \downarrow \beta & & \downarrow \gamma & & & \\ 0 & \to & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \to & 0 & \end{array}$$

defined in degree n by the following diagram

$$\begin{tikzcd} & Y^n \arrow[r, "\binom{0}{id}"] \arrow[d, "\left(\begin{smallmatrix} 0 \\ 0 \\ id \end{smallmatrix}\right)"] & X^{n+1} \oplus Y^n \arrow[r, "(id\ 0)"] \arrow[d, equal] & X^{n+1} \\ X^n \arrow[r, "\left(\begin{smallmatrix} -id \\ 0 \\ 0 \end{smallmatrix}\right)"] \arrow[d, equal] & X^n \oplus X^{n+1} \oplus Y^n \arrow[r, "\left(\begin{smallmatrix} 0 & id & 0 \\ 0 & 0 & id \end{smallmatrix}\right)"] \arrow[d, "(-f^n\ 0\ id)"] & X^{n+1} \oplus Y^n \arrow[d, "(0\ g^n)"] & \\ X^n \arrow[r, "f^n"] & Y^n \arrow[r, "g^n"] & Z^n & \end{tikzcd}$$

with exact rows, α, β homotopy inverse equivalences and γ a quasi-isomorphism.

Proof. A direct computation shows that all the maps are morphisms of complexes. It is clear that $\beta \circ \alpha = id$. On the other hand,

$$(id - \alpha\circ\beta)(x, x', y) = (x, x', y) - (0, 0, -f(x)+y) = (x, x', f(x)) = (h\circ d + d\circ h)(x, x', y),$$

where $h(x, x', y) = (0, x, 0)$. Then α and β are homotopy equivalences, so they are quasi-isomorphisms. Now, since β is a quasi-isomorphism, the 5-lemma applied to the long exact sequences obtained from the second and the third row, implies that γ is a quasi-isomorphism. □

DEFINITION 3.4.

(i) *A short exact sequence* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *in* $\mathcal{C}(\mathcal{A})$ *semi-splits if* $0 \to X^n \xrightarrow{f^n} Y^n \xrightarrow{g^n} Z^n \to 0$ *splits, for all* n.

(ii) *A triangle in* $\mathcal{K}(\mathcal{A})$ *is a distinguished triangle if it is isomorphic, in* $\mathcal{K}(\mathcal{A})$, *to one of the form* $X \xrightarrow{f} Y \xrightarrow{h} \operatorname{cone}(f) \xrightarrow{\delta} X[1]$.

THEOREM 3.5. [Ve96, II.1.3.2] *The category* $\mathcal{K}(\mathcal{A})$ *is a triangulated category with translation functor* T.

Let $H^n : \mathcal{C}(\mathcal{A}) \to \mathcal{A}$ be the functor defined by $H^n(X) = \operatorname{Ker} d^n / \operatorname{Im} d^{n-1}$. For any morphism f homotopic to zero, $H^n(f) = 0$. Then H^n factors uniquely through $\mathcal{K}(\mathcal{A})$.

The following are immediate consequences of Proposition 3.3.

PROPOSITION 3.6. *Any short exact sequence in* $\mathcal{C}(\mathcal{A})$ *is quasi-isomorphic to a semi-split short exact sequence.*

PROPOSITION 3.7. *Any distinguished triangle in* $K(\mathcal{A})$ *is quasi-isomorphic to one induced by a semi-split short exact sequence in* $C(\mathcal{A})$.

PROPOSITION 3.8. *The functor* $H^n : \mathcal{K}(\mathcal{A}) \to \mathcal{A}$ *is a cohomological functor, for any* $n \in \mathbb{Z}$.

Proof. Let $X \xrightarrow{f} Y \to Z \to X[1]$ be a distinguished triangle in $\mathcal{K}(\mathcal{A})$. By definition, it is isomorphic to

$$X \xrightarrow{f} Y \xrightarrow{h} \operatorname{cone}(f) \xrightarrow{\delta} X[1]$$

Since $0 \to Y \xrightarrow{h} \operatorname{cone}(f) \xrightarrow{\delta} X[1] \to 0$ is a short exact sequence in $\mathcal{C}(\mathcal{A})$, it induces an exact sequence

$$H^n(Y) \xrightarrow{H^n(h)} H^n(\operatorname{cone}(f)) \xrightarrow{H^n(\delta)} H^{n+1}(X)$$

□

A simple computation shows that the connecting morphism $H^{n+1}(X) \to H^{n+1}(Y)$ associated to the exact sequence appearing in the previous proof is $H^{n+1}(f)$: recall that the connecting morphism is defined by chasing the following diagram

$$\begin{tikzcd}
 & & X^{n+1}\oplus Y^n \arrow[r, "(\,id\ 0\,)"] \arrow[d, "d_f^n=\begin{pmatrix} -d_X^{n+1} & 0 \\ f & d_Y^n \end{pmatrix}"] & X^{n+1} \arrow[r] & 0 \\
0 \arrow[r] & Y^{n+1} \arrow[r, "h^{n+1}=\begin{pmatrix} 0 \\ id \end{pmatrix}"] & X^{n+2}\oplus Y^{n+1} & &
\end{tikzcd}$$

so, if $x \in X^{n+1}$ is such that $d_X^{n+1}(x) = 0$, then $x = \delta^n(x, 0)$. Finally,

$$d_f^n(x, 0) = (-d_X^{n+1}(x), f(x)) = (0, f(x)) = h^{n+1}(f(x)).$$

REMARK 3.9. *Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a semi-split short exact sequence in $\mathcal{C}(\mathcal{A})$, that is, $Y^n \simeq X^n \oplus Z^n$. Then Y is isomorphic to the complex*

$$\cdots \to X^{n-1} \oplus Z^{n-1} \xrightarrow{\begin{pmatrix} d_X^{n-1} & h^{n-1} \\ 0 & d_Z^{n-1} \end{pmatrix}} X^n \oplus Z^n \xrightarrow{\begin{pmatrix} d_X^{n} & h^{n} \\ 0 & d_Z^{n} \end{pmatrix}} X^{n+1} \oplus Z^{n+1} \to \cdots$$

where $h : Z \to X[1]$ is a morphism of complexes and

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a distinguished triangle in $\mathcal{K}(\mathcal{A})$. Moreover, the connecting morphism δ in the long exact sequence

$$\cdots \to H^n(X) \xrightarrow{H^n(f)} H^n(Y) \xrightarrow{H^n(g)} H^n(Z) \xrightarrow{\delta^n} H^{n+1}(X) \to \cdots$$

is just $H^n(h)$.

4. Derived categories

As we said in the introduction, the derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ is obtained from $\mathcal{C}(\mathcal{A})$ by formally inverting all quasi-isomorphisms. So $\mathcal{D}(\mathcal{A})$ is the localization of $\mathcal{C}(\mathcal{A})$ with respect to the class of quasi-isomorphisms in $\mathcal{C}(\mathcal{A})$: there exists a functor $Q : \mathcal{C}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ sending quasi-isomorphisms to isomorphisms which is universal for such property, that is, for any functor $F : \mathcal{C}(\mathcal{A}) \to \mathcal{B}$ sending quasi-isomorphisms to isomorphisms, there exists a functor $G : \mathcal{D}(\mathcal{A}) \to \mathcal{B}$ making the following diagram commutative

$$\begin{tikzcd}
\mathcal{C}(\mathcal{A}) \arrow[r, "Q"] \arrow[d, "F"'] & \mathcal{D}(\mathcal{A}) \arrow[dl, "G"] \\
\mathcal{B} &
\end{tikzcd}$$

The category $\mathcal{D}(\mathcal{A})$ can be obtained just by adding formally inverses for all quasi-isomorphisms. But in this case, morphisms in $\mathcal{D}(\mathcal{A})$ are just formal expressions of the form

$$f_1 \circ s_1^{-1} \circ f_2 \circ s_2^{-1} \circ \cdots \circ f_n \circ s_n^{-1},$$

where all f_i are morphism in $\mathcal{C}(\mathcal{A})$ and s_i are quasi-isomorphisms en $\mathcal{C}(\mathcal{A})$. To work with these kind of expressions, we would like to find a "common denominator". It was Verdier's observation that we can obtain a better description of the morphisms by a "calculus of fractions" if we construct $\mathcal{D}(\mathcal{A})$ in several steps:

$$\mathcal{C}(\mathcal{A}) \to \mathcal{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$$

that is, we consider complexes and their morphisms, then complexes and homotopy classes of morphisms of complexes, and finally, we invert all quasi-isomorphisms in $\mathcal{K}(\mathcal{A})$. Then $\mathcal{D}(\mathcal{A}) = \mathcal{K}(\mathcal{A})[S^{-1}]$ is the localization of $\mathcal{K}(\mathcal{A})$ with respect to the class S of all quasi-isomorphisms.

DEFINITION 4.1. *A class S of maps in a category $\mathcal{C}$ is said to be a* <u>*multiplicative system*</u> *if it satisfies the following conditions:*

S1. *For any object X in $\mathcal{C}$, $id_X \in S$. If s, t are composable maps in S, then $s \circ t \in S$.*

S2. *Given $X \xrightarrow{s} Y$, $X \xrightarrow{f} Z$ and $W \xrightarrow{g} Y$ with $s \in S$, there exist morphisms in $\mathcal{C}$ completing the following diagrams in a commutative way*

$$\begin{array}{ccc} X & \xrightarrow{s} & Y \\ \downarrow f & & \downarrow f' \\ Z & \xrightarrow{t} & X' \end{array} \qquad\qquad \begin{array}{ccc} Y' & \xrightarrow{u} & W \\ \downarrow g' & & \downarrow g \\ X & \xrightarrow{s} & Y \end{array}$$

with $t, u \in S$, that is, $t^{-1} \circ f' = f \circ s^{-1}$ and $s^{-1} \circ g = g' \circ u^{-1}$.

S3. *Given $f, g : X \to Y$, there exists $W \xrightarrow{s} X$ with $f \circ s = g \circ s$ if and only if there exists $Y \xrightarrow{t} Z$ with $t \circ f = t \circ g$, $s, t \in S$.*

The multiplicative system S is said to be <u>*saturated*</u> *if it satisfies:*

S4. *A morphism $X \xrightarrow{s} Y$ belongs to S if and only if there exist morphisms $Y \xrightarrow{f} Z$ and $W \xrightarrow{g} X$ such that $f \circ s \in S$ and $s \circ g \in S$.*

If $\mathcal{C}$ is a triangulated category with translation functor T, the multiplicative system S is said to be <u>*compatible with the triangulation*</u> *if it satisfies:*

S5. *A morphism s belongs to S if and only if $T(s)$ belongs to S.*

S6. *Given a morphism of triangles $(\alpha, \beta, \gamma) : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$, if α and β belong to S, then γ belongs to S.*

If S is a multiplicative system, the morphisms of the localization of $\mathcal{C}$ with respect to S can be described by a "calculus of fractions".

If S is saturated and $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ is the localization functor, then s belongs to S if and only if $Q(s)$ is an isomorphism.

Finally, if S is compatible with the triangulation, $\mathcal{C}[S^{-1}]$ is a triangulated category, with distinguished triangles isomorphic to images of distinguished triangles in $\mathcal{C}$.

We refer to [Ve96, II.2] for more details.

PROPOSITION 4.2. *Let S be the class of quasi-isomorphisms in $\mathcal{K}(\mathcal{A})$. Then S is a saturated multiplicative system in $\mathcal{K}(\mathcal{A})$, compatible with the triangulation.*

Proof. See for instance [Ha66, I.4]. □

REMARK 4.3.

(1) *The class $\tilde{S}$ of quasi-isomorphism in $\mathcal{C}(\mathcal{A})$ is not a multiplicative system.*
(2) *The localization of $\mathcal{C}(\mathcal{A})$ with respect to $\tilde{S}$ is isomorphic to the localization of $\mathcal{K}(\mathcal{A})$ with respect to S.*

Now we can describe $\mathcal{D}(\mathcal{A})$. The objects are those of $\mathcal{K}(\mathcal{A})$. The morphisms $X \to Y$ in $\mathcal{D}(\mathcal{A})$ are equivalence classes of pairs

$$s^{-1} \circ f = (X \xrightarrow{f} U \xleftarrow{s} Y)$$

with $s \in S$. The equivalence relation is defined as follows:

$$(X \xrightarrow{f} U \xleftarrow{s} Y) \sim (X \xrightarrow{g} V \xleftarrow{t} Y)$$

if there exists a commutative diagram

$$\begin{tikzcd}
& U \arrow[d, "\alpha"] & \\
X \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & W & Y \arrow[ul, "s"'] \arrow[l, "u"] \arrow[dl, "t"] \\
& V \arrow[u, "\beta"] &
\end{tikzcd}$$

with $u \in S$, that is, $s^{-1} \circ f \sim t^{-1} \circ g$ if and only if $(\alpha \circ s)^{-1} \circ (\alpha \circ f) = u^{-1} \circ h = (\beta \circ t)^{-1} \circ (\beta \circ g)$. The composition of morphisms in $\mathcal{D}(\mathcal{A})$ can be visualized by the diagram

$$\begin{tikzcd}
& & W & & \\
& U \arrow[ur, "h"] & & V \arrow[ul, "u"'] & \\
X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \arrow[ur, "g"] & & Z \arrow[ul, "t"']
\end{tikzcd}$$

that is,

$$(t^{-1} \circ g) \circ (s^{-1} \circ f) = (u \circ t)^{-1} \circ (h \circ f),$$

where the existence of u and h comes from S2.

Finally, the functor $Q : \mathcal{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ is defined as the identity in objects and sends a morphism $X \xrightarrow{f} Y$ to the equivalence class of the pair $id_Y^{-1} \circ f = (X \xrightarrow{f} Y \xleftarrow{id_Y} Y)$.

Since S is saturated, a morphism $X \xrightarrow{f} Y$ in $\mathcal{K}(\mathcal{A})$ is a quasi-isomorphism if and only if $Q(f)$ is an isomorphism.

REMARK 4.4.

(i) *A complex X in $\mathcal{K}(\mathcal{A})$ is quasi-isomorphic to zero if and only if $Q(X) = 0$ in $\mathcal{D}(\mathcal{A})$.*

(ii) *Let $X \xrightarrow{f} Y$ be a morphism in $\mathcal{C}(\mathcal{A})$. Then $Q(f) = 0$ in $\mathcal{D}(\mathcal{A})$ if and only if there exists a quasi-isomorphism $Y \xrightarrow{s} Z$ such that $s \circ f$ is homotopic to zero in $\mathcal{C}(\mathcal{A})$. Moreover, if f is a monomorphism (epimorphism) in $\mathcal{K}(\mathcal{A})$, then $Q(f)$ is so.*

(iii) *The cohomological functor $H^n : \mathcal{K}(\mathcal{A}) \to \mathcal{A}$ sends quasi-isomorphisms to isomorphisms, so it factors through $\mathcal{D}(\mathcal{A})$, inducing a cohomological functor $H^n : \mathcal{D}(\mathcal{A}) \to \mathcal{A}$, for any $n \in \mathbb{Z}$.*

The automorphism $T : \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$ sends quasi-isomorphisms to quasi-isomorphisms, so it induces an automorphism $T : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$.

THEOREM 4.5. *The category $\mathcal{D}(\mathcal{A})$ is a triangulated category with translation functor T and $Q : \mathcal{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ is an additive functor of triangulated categories.*

Proof. The class of quasi-isomorphisms in $\mathcal{K}(\mathcal{A})$ is a multiplicative system compatible with the triangulation, so the localization is triangulated and the distinguished triangles are those isomorphic to images by Q of distinguished triangles in $\mathcal{K}(\mathcal{A})$. Clearly Q commutes with the translation functor T and sends distinguished triangles to distinguished triangles. □

Since $\mathcal{A} \to \mathcal{C}(\mathcal{A})$ is fully faithful, we identify objects and morphisms in $\mathcal{A}$ with objects and morphisms in $\mathcal{C}(\mathcal{A})$ concentrated in degree zero.

PROPOSITION 4.6. *The composition $\mathcal{A} \to \mathcal{C}(\mathcal{A}) \to \mathcal{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ is a fully faithful functor.*

Proof. Denote $q : \mathcal{A} \to \mathcal{D}(\mathcal{A})$ the composition functor. For any object X in $\mathcal{A}$, $q(X)$ is the complex concentrated in degree zero, and for any morphism $X \xrightarrow{f} Y$, $q(f)$ is the equivalence class of the pair $(X \xrightarrow{f} Y \xleftarrow{id_Y} Y)$. Observe that the composition of q with the cohomological functor $H^0 : \mathcal{D}(\mathcal{A}) \to \mathcal{A}$ is the identity of $\mathcal{A}$.
The functor q is faithful: if $q(f) = 0$, there exists a quasi-isomorphism s such that $s \circ f$ is homotopic to zero; then $H^0(s) \circ H^0(f) = 0$, but $H^0(s)$ is an isomorphism, and hence $f = H^0(f) = 0$.
The functor q is full: let $(q(X) \xrightarrow{f} Z \xleftarrow{s} q(Y))$ be a representative of the equivalence class of a morphism in $\mathcal{D}(\mathcal{A})$ from $q(X)$ to $q(Y)$. Since s is a quasi-isomorphism, the complex Z has cohomology Y in degree zero, and zero otherwise. Then, the morphism of complexes

$$\begin{array}{ccccccccccc}
Z & & \cdots & \to & Z^{-1} & \xrightarrow{d^{-1}} & Z^0 & \xrightarrow{d^0} & Z^1 & \xrightarrow{d^1} & Z^2 & \to & \cdots \\
\downarrow u & & & & \downarrow & & \downarrow u_0 & & \| & & \| & & \\
U & & \cdots & \to & 0 & \to & Z^0/\operatorname{Im} d^{-1} & \xrightarrow{h^0} & Z^1 & \xrightarrow{d^1} & Z^2 & \to & \cdots
\end{array}$$

is a quasi-isomorphism and $Y \xrightarrow{(u \circ s)^0} Z^0/\operatorname{Im} d^{-1}$ is the kernel of $Z^0/\operatorname{Im} d^{-1} \xrightarrow{h^0} Z^1$. Now, $h^0 \circ u^0 \circ f = d^0 \circ f = 0$, so there exists a unique morphism $X \xrightarrow{g} Y$ in $\mathcal{A}$ such that $u^0 \circ s^0 \circ g = u^0 \circ f$. Finally the commutative diagram

$$\begin{tikzcd}
& Z \arrow[d, "u"] & \\
X \arrow[ur, "f"] \arrow[r, "u\circ f"] \arrow[dr, "g"'] & U & Y \arrow[ul, "s"'] \arrow[l, "u\circ s"] \arrow[dl, "id_Y"] \\
& Y \arrow[u, "u\circ s"] &
\end{tikzcd}$$

says that $s^{-1} \circ f$ and $id_Y^{-1} \circ g$ are equivalent, so q is full. $\square$

From now on, we will identify $\mathcal{A}$ with the full subcategory $q(\mathcal{A})$. We already know that $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y) \simeq \mathrm{Hom}_{\mathcal{A}}(X, Y)$ for any pair of objects X, Y in $\mathcal{A}$. Now we want to describe $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X[m], Y[n])$ for any $m, n \in \mathbb{Z}$.

It is clear that $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X[m], Y[n]) \simeq \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[n-m])$, so we only have to study $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[n])$ for $n \neq 0$.

Following Yoneda, let $\mathrm{Ext}^n_{\mathcal{A}}(X, Y)$ be the set of isomorphism classes of exact sequences

$$\psi : \qquad 0 \to Y \to Z^{-n} \to \cdots \to Z^{-1} \to X \to 0$$

in $\mathcal{A}$. It is clear that we can define a map $\mathrm{Ext}^n_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[n])$ by sending ψ to the equivalence class of the morphism $(X \xrightarrow{f} Z \xleftarrow{s} Y[n])$ given by

$$\begin{array}{ccccccccccccccc}
X & & \cdots & \to & 0 & \to & 0 & \to & \cdots & \to & 0 & \to & X & \to & 0 & \to & \cdots \\
\downarrow f & & & & \downarrow & & \downarrow & & & & \downarrow & & \| & & \downarrow & & \\
Z & & \cdots & \to & 0 & \to & Z^{-n} & \to & \cdots & \to & Z^{-1} & \to & X & \to & 0 & \to & \cdots \\
\uparrow s & & & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow & & \\
Y[n] & & \cdots & \to & 0 & \to & Y & \to & \cdots & \to & 0 & \to & 0 & \to & 0 & \to & \cdots
\end{array}$$

with s a quasi-isomorphism.

On the other hand, let $(X \xrightarrow{f} Z \xleftarrow{s} Y[n])$ be a representative of a map from X to $Y[n]$. Since s is a quasi-isomorphism, the complex Z has cohomology zero except in degree $-n$. Consider the following quasi-isomorphism

$$\begin{array}{cccccccccc}
Z & & \cdots & \to & Z^{-n-1} & \xrightarrow{d^{-n-1}} & Z^{-n} & \to & Z^{-n+1} & \to & \cdots \\
\downarrow u & & & & \downarrow & & \downarrow & & \| & & \\
U & & \cdots & \to & 0 & \to & Z^{-n}/\operatorname{Im} d^{-n-1} & \to & Z^{-n+1} & \to & \cdots
\end{array}$$

Observe that if $n < 0$, then $u \circ f = 0$, so $s^{-1} \circ f \sim (u \circ s)^{-1} \circ (u \circ f) \sim 0$. Hence $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[n]) = 0$ for any $n < 0$.

If $n > 0$, consider the quasi-isomorphism $W \xrightarrow{v} U$ given by

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, equal] & Z^{-n}/\operatorname{Im} d^{-n-1} \arrow[r] \arrow[d, equal] & Z^{-n+1} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & Z^{-1} \arrow[r] \arrow[d, equal] & \operatorname{Im} d^{-1} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & Z^{-n}/\operatorname{Im} d^{-n-1} \arrow[r] & Z^{-n+1} \arrow[r] & \cdots \arrow[r] & Z^{-1} \arrow[r] & Z^{0} \arrow[r] & Z^{1} \arrow[r] & \cdots
\end{tikzcd}$$

and observe that there exists a commutative diagram

$$\begin{tikzcd}
 & Z \arrow[d, "u"] & \\
X \arrow[ur, "f"] \arrow[r, "u\circ f"] \arrow[dr, "g"'] & U & Y[n] \arrow[ul, "s"'] \arrow[l, "u\circ s"] \arrow[dl, "t"] \\
 & W \arrow[u, "v"] &
\end{tikzcd}$$

where t, g are given by

$$\begin{tikzcd}
X \arrow[d, "g"] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
W & \cdots \arrow[r] & 0 \arrow[r] & Z^{-n}/\operatorname{Im} d^{-n-1} \arrow[r] & \cdots \arrow[r] & Z^{-1} \arrow[r] & \operatorname{Im} d^{-1} \arrow[r] & 0 \arrow[r] & \cdots \\
Y[n] \arrow[u, "t"] & \cdots \arrow[r] & 0 \arrow[r] \arrow[u] & Y \arrow[r] \arrow[u] & \cdots \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

Finally the morphism $s^{-1} \circ f \sim t^{-1} \circ g$ from X to $Y[n]$ can be associated with the exact sequence appearing in the first row of the diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Z^{-n}/\operatorname{Im} d^{-n-1} \arrow[r] \arrow[d, equal] & Z^{-n+1} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & Z^{-2} \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y \arrow[r] & Z^{-n}/\operatorname{Im} d^{-n-1} \arrow[r] & Z^{-n+1} \arrow[r] & \cdots \arrow[r] & Z^{-2} \arrow[r] & Z^{-1} \arrow[r] & \operatorname{Im} d^{-1} \arrow[r] & 0
\end{tikzcd}$$

This shows that there is a close connection between $\operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[n])$ and $\operatorname{Ext}^n_{\mathcal{A}}(X, Y)$. In fact, the following theorem holds.

THEOREM 4.7. *Let X, Y be objects in $\mathcal{A}$. Then*

(i) $\operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(X[m], Y[n]) = \operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[n-m])$ *for all* $n, m \in \mathbb{Z}$;
(ii) $\operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y) = \operatorname{Hom}_{\mathcal{A}}(X, Y)$;
(iii) $\operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[n]) = \operatorname{Ext}^n_{\mathcal{A}}(X, Y)$ *for all* $n > 0$;
(iv) $\operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[n]) = 0$ *for all* $n < 0$.

An abelian category $\mathcal{A}$ is said to be <u>semisimple</u> if any short exact sequence in $\mathcal{A}$ splits.

THEOREM 4.8. *The derived category $\mathcal{D}(\mathcal{A})$ is abelian if and only if $\mathcal{A}$ is semisimple.*

Proof. If $\mathcal{A}$ is semisimple then $\mathcal{D}(\mathcal{A})$ is equivalent to the abelian semisimple category $\prod_{\mathbb{Z}} \mathcal{A}$, as we shall see in the second example of the following section.
Assume that $\mathcal{D}(\mathcal{A})$ is abelian. We know from Proposition 2.8 that any monomorphism splits, and that any epimorphism splits. Let $X \xrightarrow{f} Y$ be a morphism in $\mathcal{D}(\mathcal{A})$. Then f is equal to the composition $X \xrightarrow{\pi} \operatorname{Im} f \xrightarrow{\iota} Y$. Since π is an epimorphism and ι is a monomorphism, they split. Hence there exist $\operatorname{Im} f \xrightarrow{s} X$ and $Y \xrightarrow{t} \operatorname{Im} f$ such that if $v = s \circ t$ then $f = f \circ v \circ f$. Assume that $\mathcal{A}$ is not semisimple and let $U \xrightarrow{g} V$ be a non-split monomorphism. Then there exists a morphism $q(V) \xrightarrow{v} q(U)$ in $\mathcal{D}(\mathcal{A})$ such that $q(g) = q(g) \circ v \circ q(g)$. Now, q is fully faithfull, so there exists a morphism $V \xrightarrow{u} U$ in $\mathcal{A}$ such that $g = g \circ u \circ g$. But g is a monomorphism, so $id_U = u \circ g$, a contradiction. ☐

5. Examples

5.1. **Hereditary categories.** Let $\mathcal{A}$ be an hereditary category, that is, $\operatorname{Ext}^2_{\mathcal{A}}(-,-) = 0$. For instance, the category of abelian groups is hereditary. In this case we can easily describe objects, morphisms and triangles in $\mathcal{D}(\mathcal{A})$.

We start with the description of the objects of $\mathcal{D}(\mathcal{A})$. Let $X = (X^n, d^n)$ be a complex in $\mathcal{C}(\mathcal{A})$. The vanishing of $\operatorname{Ext}^2_{\mathcal{A}}(-,-)$ implies that $\operatorname{Ext}^1_{\mathcal{A}}(Z,-)$ is right exact for any Z in $\mathcal{A}$. Let $X^{n-1} \xrightarrow{d^{n-1}} X^n$ and consider the short exact sequences

$$0 \to \operatorname{Ker} d^{n-1} \to X^{n-1} \to \operatorname{Im} d^{n-1} \to 0$$

$$0 \to \operatorname{Im} d^{n-1} \to \operatorname{Ker} d^n \to H^n(X) \to 0$$

Since

$$\operatorname{Ext}^1_{\mathcal{A}}(H^n(X), X^{n-1}) \to \operatorname{Ext}^1_{\mathcal{A}}(H^n(X), \operatorname{Im} d^{n-1}) \to 0$$

is exact, there exists a morphism between short exact sequences

$$\begin{array}{ccccccccc}
0 & \to & X^{n-1} & \xrightarrow{h^n} & E^n & \xrightarrow{f^n} & H^n(X) & \to & 0 \\
 & & \downarrow & & \downarrow s^n & & \| & & \\
0 & \to & \operatorname{Im} d^{n-1} & \to & \operatorname{Ker} d^n & \to & H^n(X) & \to & 0
\end{array}$$

which induces the following quasi-isomorphisms

$$\begin{array}{ccccccc}
\to & H^{n-1}(X) & \xrightarrow{0} & H^n(X) & \xrightarrow{0} & H^{n+1}(X) & \to \\
 & \uparrow & & \uparrow \begin{pmatrix} f^n & 0 \end{pmatrix} & & \uparrow & \\
\to & E^{n-1} \oplus X^{n-1} & \xrightarrow{\begin{pmatrix} 0 & h^n \\ 0 & 0 \end{pmatrix}} & E^n \oplus X^n & \to & E^{n+1} \oplus X^{n+1} & \to \\
 & \downarrow & & \downarrow \begin{pmatrix} \tilde{s}^n & id \end{pmatrix} & & \downarrow & \\
\to & X^{n-1} & \xrightarrow{d^{n-1}} & X^n & \xrightarrow{d^n} & X^{n+1} & \to
\end{array}$$

where $\tilde{s}^n$ is the composition $E^n \xrightarrow{s^n} \operatorname{Ker} d^n \to X^n$. So (X^n, d^n) is quasi-isomorphic to $(H^n(X), 0) = \prod_n H^n(X)[-n]$.

Let X, Y be objects in $\mathcal{A}$. Then

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X[n], Y[m]) \simeq \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[m-n]) \simeq \mathrm{Ext}^{m-n}_{\mathcal{A}}(X, Y)$$

which is zero except for $m-n \in \{0, 1\}$. Finally, $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X[n], Y[n]) \simeq \mathrm{Hom}_{\mathcal{A}}(X, Y)$ and $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X[n], Y[n+1]) \simeq \mathrm{Ext}^1_{\mathcal{A}}(X, Y)$.

Concerning triangles, let $X[n] \xrightarrow{f} Y[n]$ be a morphism in $\mathcal{D}(\mathcal{A})$. The previous computation applied to the complex

$$\mathrm{cone}(f) : \cdots \to 0 \to X \xrightarrow{f} Y \to 0 \to \cdots$$

says that it is quasi-isomorphic to the complex

$$\cdots \to 0 \to \mathrm{Ker}\, f \xrightarrow{0} \mathrm{Coker}\, f \to 0 \to \cdots$$

Hence any distinguished triangle is isomorphic to one of the form

$$X[n] \xrightarrow{f} Y[n] \to \mathrm{Ker}\, f[n+1] \oplus \mathrm{Coker}\, f[n] \to X[n+1]$$

In particular, if $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is a short exact sequence in $\mathcal{A}$, then

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\delta} X[1]$$

is a distinguished triangle in $\mathcal{D}(\mathcal{A})$, and $\delta \in \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(Z, X[1])$ is the morphism associated to the given short exact sequence.

5.2. **Semisimple categories.** Let $\mathcal{A}$ be a semisimple category. In this case, $\mathcal{A}$ is hereditary, so the conclusions in the previous example hold. But now $\mathrm{Ext}^1_{\mathcal{A}}(-,-) = 0$, so $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X[n], Y[n]) \simeq \mathrm{Hom}_{\mathcal{A}}(X, Y)$ and $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X[n], Y[m]) = 0$ for all $m \neq n$. Hence $\mathcal{D}(\mathcal{A})$ is equivalent to $\prod_{\mathbb{Z}} \mathcal{A}$.

5.3. $\mathcal{A} = \mathrm{mod}\, k(A_2)$**.** Let $\mathcal{A}$ be the full subcategory of finitely generated left modules over the path algebra associated to the quiver $A_2 : 1 \to 2$. In this case we only have three non-isomorphic indecomposable modules: the simple projective module S_2, the projective module of length two P_1, and the simple module S_1. Moreover, if $X, Y \in \{S_1, S_2, P_1\}$, then $\mathrm{Hom}_{\mathcal{A}}(X, Y) = k$ if $(X, Y) = (S_2, P_1)$ or (P_1, S_1) and it is zero otherwise, and $\mathrm{Ext}^1_{\mathcal{A}}(X, Y) = k$ if $(X, Y) = (S_1, S_2)$ and it is zero otherwise. Then $\mathcal{D}(\mathcal{A})$ has the following picture

$$\begin{tikzcd}
 & & S_1[-1] \arrow[dr] & & P_1[0] \arrow[dr] & & S_2[1] \arrow[dr] & & S_1[1] & \cdots \\
\cdots & {} \arrow[ur] & & S_2[0] \arrow[ur] & & S_1[0] \arrow[ur] & & P_1[1] \arrow[ur] & &
\end{tikzcd}$$

where the composition of any two consecutive arrows is zero. Observe that this category has neither monomorphisms nor epimorphisms. All distinguished triangles can be visualized in the picture as the diagrams of three consecutive arrows.

6. Morita theory in $\mathcal{D}(A)$

Given an algebra A over a commutative ring k, we shall denote $\mathcal{D}(A)$ for the derived category $\mathcal{D}(\mathrm{Mod}-A)$. Given two k-algebras A and B the natural question is: when are $\mathcal{D}(A)$ and $\mathcal{D}(B)$ equivalent categories? (triangle equivalent?). Of course, if A and B are Morita equivalent (that is, Mod-A and Mod-B are k-linearly equivalent) then $\mathcal{D}(A)$ and $\mathcal{D}(B)$ are equivalent. Are there other equivalences ? The answer is yes. In fact, we shall present some examples later.

Rickard developed [Rick89], [Rick91] a Morita theory for derived categories based on the notion of tilting complex. As we shall see this is a generalization of the notion of tilting module. A summary of the history of the subject is developed for example in [KZ98]. Keller's approach [Ke94] is a little different and we will follow it. This is also the point of view of [DG02].

DEFINITION 6.1.

(1) *A functor F between triangulated categories $\mathcal{S}$ and $\mathcal{T}$ is said exact if it commutes whith shifts and preserves distinguished triangles, that is, F is equipped with a natural isomorphism $i_X : F(X[1]) \to F(X)[1]$, $\forall X \in Obj(\mathcal{S})$ such that for every distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in $\mathcal{S}$,

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{i_X \circ F(h)} F(X)[1]$$

is a distinguished triangle in $\mathcal{T}$.

(2) *An equivalence between two triangulated categories is an equivalence of categories which is exact and whose inverse functor is also exact.*

REMARK 6.2. *If $F : \mathcal{S} \to \mathcal{T}$ is exact then it is automatically additive.*

We know from classical Morita theory that two k-algebras A and B are Morita equivalent if and only if there exists a bimodule ${}_AP_B$ such that it is finitely generated projective, balanced and generator A-module and $B \simeq \mathrm{End}_A(P)$ as a B-bimodule. We have that P is a generator for Mod-A, that is for every $M \in$ Mod-A, there exists a set I and an epimorphism $P^{(I)} \to M$. For example, A is a generator A-module but this progenerator gives the trivial equivalence. The fact that P is f.g. projective implies that P is a direct summand of $A^{(n)}$, for some $n \in \mathbb{N}$, in particular $\mathrm{Hom}_A(P, -)$ commutes with direct sums. The notion of tilting complex appears naturally if we look at the properties verified by $A[0]$ in $\mathcal{D}(A)$.

The free rank one A-module A considered as a cochain complex concentrated in degree 0 verifies that, $\forall n \neq 0$,

$$\mathrm{Hom}_{\mathcal{D}(A)}(A[n], A) = \mathrm{Hom}_{\mathcal{D}(A)}(A, A[-n]) = \mathrm{Ext}_A^{-n}(A, A) = 0.$$

DEFINITION 6.3. *A complex X in $\mathcal{D}(A)$ is a generator of this category if and only if the smallest full triangulated subcategory of $\mathcal{D}(A)$ containing X and closed by infinite direct sums is $\mathcal{D}(A)$.*

EXAMPLE 6.4. *$A[0]$ is a generator of $\mathcal{D}(A)$.*

DEFINITION 6.5. *A tilting complex $T^\bullet$ for a k-algebra A is a bounded cochain complex of f.g. projective A-modules which generates the derived category $\mathcal{D}(A)$ and such that the graded ring of endomorphisms $\mathrm{Hom}_{\mathcal{D}(A)}(T,T)$ is concentrated in degree 0.*

As a special case of tilting complexes we have the tilting modules over a finite dimensional k-algebra (k is a field) (when thinking them as their projective resolutions).

DEFINITION 6.6. *Let A be a finite dimensional k-algebra and T a finitely generated A-module. We say that T is a tilting module if*

(1) $\mathrm{pdim}_A(T) \leq 1$.
(2) $\mathrm{Ext}^1_A(T,T) = 0$ *(that is, there are no self-extensions of T).*
(3) *There exists a short exact sequence of A-modules*
$$0 \to A \to T_1 \to T_2 \to 0$$
such that T_i are direct summands of finite direct sums of T (that is, $T_i \in \mathrm{add}(T)$) for $i = 1, 2$.

REMARK 6.7. *If T is a projective A-module the first and second conditions are automatic. In that case the exact sequence in 3 splits, so A is a direct summand of T^n, for some n. The third condition is verified if and only if T is a generator in* Mod-A. *As a consequence $A \sim_M \mathrm{End}_A(T)$.*

When A is self-injective, every A-module of finite projective dimension is projective. Then for these algebras, tilting complexes are the same as Morita equivalences. The following theorem is due to Rickard [Rick89]

THEOREM 6.8. *Given two k-algebras A and B such that A or B is k-flat, then the following are equivalent:*

(1) *The unbounded derived categories of A and B are equivalent as triangulated categories.*
(2) *There is a tilting complex T in $\mathcal{D}(A)$ whose endomorphism ring $\mathrm{Hom}_{\mathcal{D}(A)}(T,T)$ is isomorphic to B.*
(3) *There exists a cochain complex of B-A-bimodules M such that the derived tensor product*
$$- \otimes^L_B M : \mathcal{D}(B) \to \mathcal{D}(A)$$
is an equivalence of categories.

We shall give a proof of this theorem following Keller. For this we must recall some preliminaries first.

7. Derived category of a differential graded algebra

7.1. **DG algebras.** Let $A = \oplus_{p\in\mathbb{Z}} A^p$ be a differential graded algebra (DG algebra for short), that is, A is $\mathbb{Z}$-graded algebra with a morphism $d : A \to A$ of degree one such that $d(ab) = d(a)b + (-1)^p a d(b)$, if $a \in A^p$, $b \in A$. The map d is called differential.

EXAMPLE 7.1.

(1) *Let* $A = A^0$ *be a* k*-algebra, and* $d = 0$.

(2) *Given a* k*-algebra* B *and a complex* $(X^\bullet, d)$ *of* B*-modules, let us take* $A = \mathrm{Hom}_B(X, X)$*, that is,* $A = \oplus_{p\in\mathbb{Z}} A^p$ *with* $A^p = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_B(X^i, X^{i+p})$ *and differential defined by* $(\delta(f))^i = d \circ f^i - (-1)^p f^{i+1} \circ d$*, for* $f = (f^i)_{i\in\mathbb{Z}} \in A^p$.

7.2. **DG modules.** Let A be a DG algebra.

DEFINITION 7.2. *A* <u>*differential graded* A*-module*</u> *(DG* A*-module for short) is a* $\mathbb{Z}$*-graded right* A*-module* $M = \oplus_{p\in\mathbb{Z}} M^p$ *together with a* k*-linear differential* $d : M \to M$ *(of degree one) such that*

$$d(ma) = d(m)a + (-1)^p m d(a), \qquad \forall m \in M^p, a \in A.$$

A <u>*morphism*</u> *of DG modules* M *and* N *is an* A*-linear map* $f : M \to N$ *of degree zero wich commutes with the differentials.*

EXAMPLE 7.3.

(1) *The DG modules for the DG algebra* A *of the first example of the previous subsection are the same as the cochain complexes of* A*-modules.*

(2) *Let* A *be the DG algebra of the second example of the previous subsection. Each complex* $(N^\bullet, d)$ *of* B*-modules gives rise to a DG* A*-module* $\mathrm{Hom}_B(X, N)$ *with* A*-action*

$$(g^j)_{j\in\mathbb{Z}}(f^i)_{i\in\mathbb{Z}} = (g^{i+p} \circ f^i)_{i\in\mathbb{Z}}, \qquad \forall (g^j) \in \mathrm{Hom}_A(X, N), (f^i) \in \mathrm{Hom}_A(X, X)^p.$$

Also, $(X^\bullet, d)$ *is a DG* A*-module by means of the action:*

$$\mathrm{Hom}_A(X, X) \otimes X \to X, \qquad (f^i)(x^j) = (f^i(x^i)).$$

7.3. **The homotopy category** $\mathcal{H}(A)$**.** We recall that the homotopy category $\mathcal{H}(A)$ is defined as follows.

Its class of objects is given by $\mathrm{Obj}(\mathcal{H}(A)) = \mathrm{Obj}(DG\,\mathrm{Mod}\text{-}A)$ and the set of morphisms $\mathrm{Hom}_{\mathcal{H}(A)}(M, N)$ is defined as $\mathrm{Hom}_A(M, N)/\sim$ where $\sim$ is the equivalence relation given by identifying homotopic maps.

There is a shift operator

$$S : \mathcal{H}(A) \to \mathcal{H}(A)$$

defined by

$$(S(M))^p = M^{p+1}$$

and $d_{S(M)} = -d_M$, $\forall M \in Obj(\mathcal{H}(A))$.

We recall that endowed with S and all the triangles isomorphic to the standard triangles, $\mathcal{H}(A)$ becomes a triangulated category.

EXAMPLE 7.4. *For the first example of the previous subsection we have that* $\mathcal{H}(A)$ *is the standard homotopy category of cochain complexes of* A*-modules.*

7.4. **The derived category.** We also recall that $\mathcal{D}(A) = \mathcal{H}(A)[\Sigma^{-1}]$, where Σ denotes the class of all the homotopy classes of quasi-isomorphisms.

EXAMPLE 7.5. *For the first example of the previous subsection we have that $\mathcal{D}(A)$ may be identified with the standard derived category of cochain complexes of A-modules.*

REMARK 7.6. *We notice that $\mathcal{D}(A)$ has infinite direct sums (ordinary sums of DG A-modules).*

Consider now the free DG A-module A and let M be a DG A-module. Then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{H}(A)}(A, M) &\to H^0(M) \\ \bar{f} &\mapsto \bar{f}(1) \end{aligned}$$

is bijective. In particular, each quasi-isomorphism $s : M \to M'$ induces a bijection

$$\mathrm{Hom}_{\mathcal{H}(A)}(A_A, M) \to \mathrm{Hom}_{\mathcal{H}(A)}(A_A, M'). \tag{1}$$

Since $H^0(M) \simeq \mathrm{Hom}_{\mathcal{D}(A)}(A, M)$ then $\mathrm{Hom}_{\mathcal{H}(A)}(A, M) \sim \mathrm{Hom}_{\mathcal{D}(A)}(A, M)$.

DEFINITION 7.7. *A DG A-module M is said to be closed if*

$$\mathrm{Hom}_{\mathcal{H}(A)}(M, N) \sim \mathrm{Hom}_{\mathcal{D}(A)}(M, N)$$

for all N. We shall denote by $H_p(A)$ the full subcategory of $\mathcal{H}(A)$ formed by closed objects.

EXAMPLE 7.8.

1. *Free DG A-modules of finte type are closed.*
2. *Complexes of f.g. projective A-modules are closed.*
3. *Suppose that M and N are closed and let $f : M \to N$ be a morphism of DG A-modules. Consider the mapping cone $\mathrm{cone}(f) = S(M) \oplus N$. Then $\mathrm{cone}(f)$ is also closed.*

REMARK 7.9. *As a consequence $\mathcal{H}_p(A)$ is a triangulated subcategory of $\mathcal{H}(A)$.*

Why are we interested in considering the subcategory $\mathcal{H}_p(A)$?

PROPOSITION 7.10.

1. *For all $M \in \mathcal{H}(A)$ there is a quasi-isomorphism ${}_pM \to M$ with ${}_pM$ closed.*
2. *The map $M \to {}_pM$ may be completed to a triangulated functor commuting with infinite direct sums and such that it gives a triangulated equivalence $\mathcal{D}(A) \simeq \mathcal{H}_p(A)$.*
3. *$\mathcal{H}_p(A)$ is the smallest full subcategory of $\mathcal{H}(A)$ containing A and closed under infinite direct sums.*

We will say that ${}_pM$ is the projective resolution of M.

EXAMPLE 7.11. *For M and A concentrated in degree zero, choose ${}_pM$ to be the homotopy class of any projective resolution of M. Then ${}_pM$ is closed since all epimorphisms split.*

EXAMPLE 7.12.

(1) *If M is a right bounded complex, ${}_pM$ is a "projective resolution" of the complex M (see* [Ha66]*).*
(2) *For $A = A^0$, $d = 0$ and arbitrary M the description of ${}_pM$ has been obtained by Spaltenstein (*[Sp88]*).*
(3) *For A and M arbitrary, see* [Ke94].

REMARK 7.13. *The two last items of the previous proposition imply that $\mathcal{D}(A)$ coincides with the smallest full subcategory containing A and closed by infinite direct sums.*

7.5. **Left derived tensor functors.** Let A and B be two DG algebras and let ${}_BX_A$ be a DG B-A-bimodule, that is, $X = \oplus_{p\in\mathbb{Z}} X^p$ with a k-linear map $d : X \to X$ of degree one such that

$$d(bxa) = d(b)xa + (-1)^p bd(xa) = d(b)xa + (-1)^p bd(x)a + (-1)^{p+q} bxd(a),$$

for $b \in B^p$, $x \in X^q$ and $a \in A$.

Define the DG algebra $B^{\mathrm{op}} \otimes_k A$ by $(B^{\mathrm{op}} \otimes_k A)^n = \oplus_{p+q=n} B^p \otimes_k A^q$ and the map d is given by

$$d^n : (B^{\mathrm{op}} \otimes_k A)^n \to (B^{\mathrm{op}} \otimes_k A)^{n+1}$$
$$d(b \otimes a) = d_B(b) \otimes a + (-1)^p b \otimes d_A(a),$$

for $b \in B^p$ and $a \in A^q$. The product is given by

$$(b \otimes a)(b' \otimes a') = (-1)^{qp'} b'b \otimes aa',$$

for $b \in B^p$, $b' \in B^{p'}$, $a \in A^q$ and $a' \in A^{q'}$.

We notice then that X is a right DG $B^{\mathrm{op}} \otimes_k A$-module since

$$x(b \otimes a) = (-1)^{rp} bxa, \qquad \forall\, b \in B^p, x \in X^r, a \in A^q.$$

Let N be a right DG B-module. We define $N \otimes_k X$ as the DG A-module with action of A in X as before and with the DG structure given by

$$(N \otimes_k X)^m = \oplus_{p+q=m} N^p \otimes_k X^q,$$
$$d(n \otimes x) = d_N(n) \otimes x + (-1)^p n \otimes d_X(x),$$

for $n \in N^p$ and $x \in X^q$.

The k-submodule generated by all differences $nb \otimes x - n \otimes bx$ is stable under d and under multiplication by elements of A. So $N \otimes_B X$, the quotient module of $N \otimes_k X$ by this submodule, is a well defined DG A-module. Moreover, this construction is functorial in N and X.

The functor $(-)\otimes_B X : DG\text{-Mod-}B \to DG\text{-Mod-}A$ yields a triangulated functor from $\mathcal{H}(B)$ to $\mathcal{H}(A)$, denoted by the same symbol.

We define the left derived tensor product $(-) \otimes_B^L X : \mathcal{D}(B) \to \mathcal{D}(A)$ by

$$N \otimes_B^L X := {}_pN \otimes_B X.$$

Notice that $(-) \otimes_B^L X$ commutes with direct sums since ${}_p(-)$ and $(-) \otimes_B X$ do.

LEMMA 7.14. *The functor $F = (-) \otimes_B^L X$ is an equivalence if and only if the following conditions hold:*

(a) *The functor F induces bijections*

$$\mathrm{Hom}_{\mathcal{D}(B)}(B, B[n]) \simeq \mathrm{Hom}_{\mathcal{D}(A)}(X_A, X_A[n]), \qquad \forall\, n \in \mathbb{Z}.$$

(b) *The functor* $\mathrm{Hom}_{\mathcal{D}(A)}(X_A, -)$ *commutes with (infinite) direct sums.*

(c) *The smallest full triangulated subcategory of $\mathcal{D}(A)$ containing X_A and closed under (infinite) direct sums coincides with $\mathcal{D}(A)$.*

Proof. The conditions are necessary since they hold in $\mathcal{D}(B)$ for B and they must be preserved by equivalences.

In order to prove that they are sufficient let us consider the full subcategory $\mathcal{C}$ of objects V in $\mathcal{D}(B)$ for which the maps

$$\mathrm{Hom}_{\mathcal{D}(B)}(B[n], V) \to \mathrm{Hom}_{\mathcal{D}(A)}(X_A[n], F(V))$$

are bijective for all $n \in \mathbb{Z}$. This category is clearly closed under the shift and its inverse. Using the 5-lemma we check that it is a triangulated subcategory. Using (b) we get that it is closed under (infinite) direct sums. Also, using (a) we see that $\mathcal{C}$ contains B_B. Thus we must have that $\mathcal{C} = \mathcal{D}(B)$. So, the full subcategory $\mathcal{C}'$ of objects U in $\mathcal{D}(B)$ such that

$$\mathrm{Hom}_{\mathcal{D}(B)}(U, V) \to \mathrm{Hom}_{\mathcal{D}(A)}(F(U), F(V))$$

is bijective for all V in $\mathcal{D}(B)$ contains B. Again it is closed under the shift and its inverse, and also closed under (infinite) direct sums. It is a triangulated category by the 5-lemma. Thus $\mathcal{C}' = \mathcal{D}(B)$ and as a consequence F is fully faithful.

Condition (c) shows that F is surjective. □

EXAMPLE 7.15. *Suppose that $\phi : A \to B$ is a quasi-isomorphism, that is, a morphism of DG algebras inducing an isomorphism $H^\bullet(A) \to H^\bullet(B)$. Then $(-) \otimes_B^L B_A : \mathcal{D}(B) \to \mathcal{D}(A)$ is an equivalence. In fact, B_A is isomorphic to A_A in $\mathcal{D}(A)$, so the conditions (b) and (c) of the lemma before hold. In order to prove (a) consider the commutative diagram*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}(B)}(B, B[n]) \arrow[r] & \mathrm{Hom}_{\mathcal{D}(A)}(B_A, B[n]_A) \\
H^n(B) \arrow[u, "\sim"] \arrow[r, "\sim"] & \mathrm{Hom}_{H(A)}(A, B[n]_A). \arrow[u, "\sim"']
\end{tikzcd}$$

Similarly, $(-) \otimes_A^L B_B : \mathcal{D}(A) \to \mathcal{D}(B)$ is an equivalence.

We may perform compositions of left derived functors of this kind.

LEMMA 7.16. *If A, B and C are DG k-algebras, A is a flat k-module, ${}_CY_B$ is a DG C-B-bimodule, we have*

$$((-) \otimes_C^L Y) \otimes_B^L X \simeq (-) \otimes_C^L Z,$$

for some DG C-A-bimodule Z.

Proof. Take $P = {}_p({}_BX_A)$, considering ${}_BX_A$ as a DG right $B^{\mathrm{op}} \otimes_k A$-module. The morphism of functors

$$(-) \otimes^L_B P \to (-) \otimes^L_B X$$

is clearly invertible since the composition

$$P \xrightarrow{\sim} F(B) = B \otimes^L_B P \to B \otimes_B X \xrightarrow{\sim} X$$

is an isomorphism and, using a result of [Ke94], a morphism of functor between triangulated categories is invertible if and only if it is an isomorphism when one applies the functors to a generator of the category.

Also the morphism

$$N \otimes^L_B P = ({}_pN) \otimes_B P \to N \otimes_B P$$

is invertible in $\mathcal{D}(A)$ for each $N \in \mathcal{D}(B)$, using the same result and the facts that P is closed over $B^{\mathrm{op}} \otimes_k A$ and that the functor $(-) \otimes_B (B^{\mathrm{op}} \otimes_k A) \xrightarrow{\sim} (-) \otimes_k A$ preserves quasi-isomorphisms by the k-flatness of A. Thus

$$(L \otimes^L_C Y) \otimes^L_B X \simeq (L \otimes^L_C Y) \otimes_B P \xrightarrow{\sim} {}_pL \otimes_C Y \otimes_B P = L \otimes^L_C (Y \otimes_B P),$$

and we take $Z = Y \otimes_B P$. □

8. Applications to tilting theory

Let k be a commutative ring, B a k-algebra, A a flat k-algebra. Recall Rickard's theorem:

THEOREM 8.1. *Given two k-algebras A and B such that A or B is k-flat, then the following are equivalent*

(1) *The unbounded derived categories of A and B are equivalent as triangulated categories.*
(2) *There exists a cochain complex of B-A-bimodules X such that the derived tensor product*

$$- \otimes^L_B X : \mathcal{D}(B) \to \mathcal{D}(A)$$

is an equivalence of categories.

Proof. (1) $\Rightarrow$ (2)

Let $F : \mathcal{D}(B) \to \mathcal{D}(A)$ be the given triangulated equivalence. Take $T = {}_p(F(B_B))$ and $\tilde{B} = \mathrm{Hom}_A(T, T)$. There are canonical isomorphisms

$$H^n(\tilde{B}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{H}(A)}(T, T[n]), \quad \forall\, n \in \mathbb{Z}.$$

Since T is closed in $\mathcal{H}(A)$, we also have

$$\mathrm{Hom}_{\mathcal{H}(A)}(T, T[n]) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(A)}(T, T[n]) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(B)}(B, B[n]), \quad \forall\, n \in \mathbb{Z}.$$

Thus $H^n(\tilde{B})$ if $n \neq 0$ and $H^0(\tilde{B})$ may be identified with $H^0(B) = B$. If we view A as a DG algebra concentrated in degree zero we may view T as a DG $\tilde{B}$-A-bimodule. We claim that $(-) \otimes^L_{\tilde{B}} T : \mathcal{D}(\tilde{B}) \to \mathcal{D}(A)$ is an equivalence. Since F is an equivalence, conditions (b) and (c) of lemma 7.14 clearly hold. For (a) use the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}(\tilde{B})}(\tilde{B}, \tilde{B}[n]) \arrow[r] & \mathrm{Hom}_{\mathcal{D}(A)}(T, T[n]) \\ H^n(\tilde{B}) \arrow[u, "\sim"] \arrow[r, "\sim"] & \mathrm{Hom}_{H(A)}(T, T[n]). \arrow[u, "\sim"'] \end{tikzcd}$$

To establish a connection between B and $\tilde{B}$, let us introduce the DG subalgebra $C \subset \tilde{B}$ with $C^n = \tilde{B}^n$ for $n \leq -1$, $C^0 = Z^0\tilde{B}$ and $C^n = 0$ for $n \geq 1$. Note that there are canonical morphisms

$$Z^0/dZ^1 = H^0(\tilde{B}) = B \leftarrow C \to \tilde{B}$$

which are both quasi-isomorphisms. So, by example 7.15, we have a chain of equivalences

$$\mathcal{D}(B) \overset{(-)\otimes^L_B B}{\rightarrow} \mathcal{D}(C) \overset{(-)\otimes^L_C \tilde{B}}{\rightarrow} \mathcal{D}(\tilde{B}) \overset{(-)\otimes^L_{\tilde{B}} T}{\rightarrow} \mathcal{D}(A).$$

Applying the previous lemma twice we get the required complex X of B-A-bimodules. □

9. Appendix

In this section we show an example of tilting equivalence which is not a Morita equivalence. The example is due to Schwede ([Sch04]).

Consider a field k and take the k-algebra A defined by:

$$A = \{\text{upper triangular matrices in } M_3(k)\}$$

Up to isomorphism, there are three indecomposable A-projective modules:

$$P^1 = \{(y_1, y_2, y_3)|y_i \in k\},\ P^2 = \{(0, y_2, y_3)|y_i \in k\} \text{ and } P^3 = \{(0, 0, y_3)|y_i \in k\}.$$

The P^i are the projective covers of the simple modules $S^1 = P^1/P^2$, $S^2 = P^2/P^3$ and $S^3 = P^3$. In particular, S^3 is projective and $\mathrm{pdim}_A(S^i) = 1$ for $i = 1, 2$.

Let us take $T = P^1 \oplus P^2 \oplus S^2$, which is clearly not projective. The following projective resolution of S^2:

$$0 \to P^1 \oplus P^2 \oplus P^3 \to P^1 \oplus P^2 \oplus P^2 \to S^2 \to 0$$

may be used to compute $\mathrm{Ext}^1_A(T, T) = \mathrm{Ext}^1_A(S^2, T) = 0$. The module $P^1 \oplus P^2 \oplus P^3$ is A-free of rank one, then T is a tilting A-module.

The k-algebra $\mathrm{End}_A(T)$ may be identified, by a direct computation, to the subalgebra of $M_3(k)$ consisting of upper triangular matrices $(x_{ij})_{1\leq i,j\leq 3}$ such that $x_{23} = 0$.

Now A and $\mathrm{End}_A(T)$ are NOT Morita equivalent since their lattices of projective modules differ.

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