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CLUSTER CATEGORIES AND THEIR RELATION TO CLUSTER ALGEBRAS, SEMI-INVARIANTS AND HOMOLOGY OF TORSION FREE NILPOTENT GROUPS

GORDANA TODOROV

Dedicated to María Inés Platzeck for her 60th birthday and Hector Merklen for his 70th birthday

Abstract. The structure of cluster categories [BMRRT] is well suited for the combinatorics of cluster algebras [FZ1] with the main correspondence being between tilting objects and clusters. Furthermore it was shown in [IOTW] that there is a close relation between domains of virtual semi-invariants and simplicial complexes associated to cluster categories. Also the same simplicial complexes associated to cluster categories are related to the Igusa-Orr pictures in the homology of nilpotent groups.

0. Introduction

The purpose of this paper is to define and relate several, quite different notions, and therefore the paper is mostly a survey paper, including many results without proofs and also some indications about work in progress. With this in mind, the paper is divided in the following way.

In Section 1 we define and state main properties of cluster categories. The results are mostly from [BMRRT].

In Section 2, cluster algebras with no coefficients are defined (2.1) and several results about their relation to cluster categories are given . We consider only acyclic case; in (2.2) the main bijection theorem between cluster variables and indecomposable objects of the cluster category is stated; in (2.3) the denominators of nonpolynomial cluster variables are described in terms of dimension vectors of exceptional indecomposable representations; in (2.4) weak positivity condition of cluster variables is stated as a consequence of the quite technical proposition, and the proof of the bijection theorem is given. Also, Cluster determines seed conjecture is proved in 2.5.

Section 3 is mostly survey section, where we first recall definitions and theorems about classical semi-invariants, and then give definition and properties of virtual representation spaces and virtual semi-invariants. We only state the main theorems and for proofs refer to [IOTW].

Finally, Section 4 contains a short summary of the results about homology of nilpotent groups and the relation to the simplicial complex of tilting object in the corresponding cluster category.

1. Cluster Categories

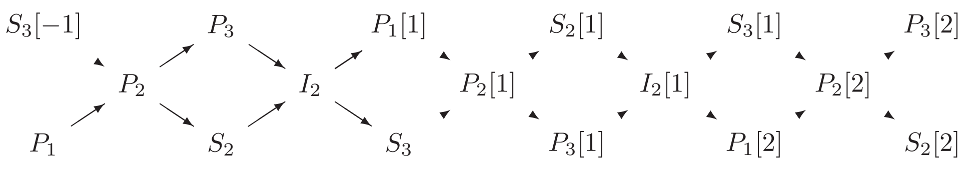
1.1. **A motivation.** The categories of finitely generated modules over hereditary artin algebras, together with their derived categories, are very well understood. Since there were clear indications (e.g.[FZ1]) that there was close relation between the indecomposable modules and cluster variables of the cluster algebras, at least in the finite representation type, via root systems, it was quite natural to study and develop the theory of certain orbit categories of the derived categories which perfectly reflect the combinatorics of cluster algebras.

1.2. **Derived category.** We recall the standard definitions and notation: Let Q be a quiver with n vertices and no oriented cycles and $\Bbbk$ a field. Then $\Lambda = \Bbbk Q$, the associated path algebra is hereditary. Let $\operatorname{mod}\Lambda$ be the category of finitely generated Λ-modules, and $D^b := D^b(\operatorname{mod}\Lambda)$ the derived category of bounded complexes in $\operatorname{mod}\Lambda$. Since Λ is hereditary the derived category is easy to describe: the objects in D^b are finite direct sums of the objects in $\cup_i\{(\operatorname{mod}\Lambda[i])\}$ and the morphisams in D^b can be described in the following way:
$\operatorname{Hom}_{D^b}(M,N) = \operatorname{Hom}_\Lambda(M,N)$ for M,N in $\operatorname{mod}\Lambda$,
$\operatorname{Hom}_{D^b}(M,N[1]) = \operatorname{Ext}^1_\Lambda(M,N)$ for M,N in $\operatorname{mod}\Lambda$,
$\operatorname{Hom}_{D^b}(M,N[i]) = 0$ for M,N in $\operatorname{mod}\Lambda$ and all $i \neq 0,1$,
$\operatorname{Hom}_{D^b}(X[i],Y[i]) = \operatorname{Hom}_{D^b}(X,Y)$ for X,Y in D^b and all $i \in \mathbb{Z}$.

1.3. **Cluster category.** In order to define the corresponding cluster category, we consider the endofunctor: $F = \tau^{-1}[1] = [1]\tau^{-1} : D^b \to D^b$, where $[1] : D^b \to D^b$ is the suspension functor, and $\tau : D^b \to D^b$ is the Auslander-Reiten functor. Then, the *cluster category* C_Λ is defined to be the orbit category, i.e. the objects in C_Λ are F-orbits, and the morphisams in C_Λ are $\operatorname{Hom}_{C_\Lambda}(\tilde{X},\tilde{Y}) := \coprod_i \operatorname{Hom}_{D^b}(X, F^iY)$, where $\tilde{X}$ and $\tilde{Y}$ are F-orbits of X and Y.

From the above definition it is easy to see that a set of representatives of the orbits may be chosen in $\operatorname{add}(\operatorname{ind}\Lambda \cup \{P_i[1]\}_{i=1}^n)$. Often we will use the same symbol for the objects in D^b and their orbits in C_Λ.

1.4. **An example.** Let Q be the Dynkin diagram of type A_3 with the following orientation and labeling of the vertices: $1 \leftarrow 2 \leftarrow 3$. Let S_1, S_2, S_3 be the non isomorphic simple Q representations, or simple $\Bbbk Q$-modules, let $P_1 = S_1, P_2, P_3$ be the nonisomorphic indecomposable projective Q representations ($\Bbbk Q$-modules), and let $I_1, I_2, I_3 = S_3$ be the nonisomorphic indecomposable injective Q representations ($\Bbbk Q$-modules). Consider the Auslander-Reiten quiver of $\operatorname{mod}\Lambda$ and D^b:



A set of representatives for the F-orbits may be chosen to be the following 9 objects: $\{P_1, P_2, P_3, S_2, I_2, S_3, P_1[1], P_2[1], P_3[1]\}$.

1.5. **Tilting objects.** Tilting objects are defined in a similar way as tilting modules. First we define $\mathrm{Ext}^1_{C_\Lambda}(\tilde{X},\tilde{Y}) := \coprod_i \mathrm{Ext}^1_{D^b}(X, F^iY) = \coprod_i \mathrm{Hom}_{D^b}(X, F^iY[1])$. Then, we define $\tilde{X}$ to be an *exceptional object* in C_Λ if $\mathrm{Ext}_{C_\Lambda}(\tilde{X},\tilde{X}) = 0$. Define a *tilting object (basic)* to be a maximal exceptional object in C_Λ without multiplicities. As stated in 1.3 we will usually use the same symbol for the objects in D^b and their orbits in C_Λ.

1.6. **Tilting seeds.** The notion of tilting seed is introduced for the sake of setting up a good correspondence between "cluster category" terminology and "cluster algebra" terminology, in order to be able to write rigorous proofs. A *tilting seed* is defined to be a pair (T, Q_T), where T is a basic tilting object and Q_T is the quiver of $\mathrm{End}_{C_\Lambda}(T)^{op}$. Notice that the tilting object T determines the quiver Q_T.

1.7. **Exchange pairs.** The following theorems give precise conditions when an indecomposable summand of a tilting object can be replaced by another indecomposable object, i.e. they "can be exchanged".

Theorem 1.7.1. [BMRRT] *Let $T = M \coprod \bar{T}$ be a basic tilting object with M indecomposable. Then, there exists exactly one object M^* not isomorphic to M, such that $T' = M^* \coprod \bar{T}$ is also a basic tilting object.*

Definition 1.7.2. Using the notation from Theorem 1.7.1 we call the indecomposable objects M and M^* an *exchange pair* with respect to T (or T').

Theorem 1.7.3. [BMRRT] *Two exceptional indecomposable objects M and M^* form an exchange pair if and only if* $\dim_{D_M} \mathrm{Ext}^1_{C_\Lambda}(M, M^*) = 1 = \dim_{D_{M^*}} \mathrm{Ext}^1_{C_\Lambda}(M, M^*)$, *where $D_M = \mathrm{End}_{C_\Lambda}(M)^{op}$ and $D_{M^*} = \mathrm{End}_{C_\Lambda}(M^*)^{op}$.*

1.8. **Tilting objects in the cluster category for A_3.** We will use the same example and the same notation as in 1.4. In the following figure the vertices are labeled by representatives of indecomposable objects in the cluster category. Vertices are connected by a line segment if there are no extensions between the two corresponding objects. Consequently, each small triangle defines a tilting object and all tilting objects are given that way. There are 14 tilting objects in this example, e.g. $(S_1 \coprod P_2 \coprod P_3)$, $(S_2 \coprod P_2 \coprod P_3)$, $(S_1[1] \coprod S_2 \coprod P_3[1])$, $(S_1[1] \coprod P_2[1] \coprod P_3[1])$ etc.

1.9. **Tilting mutations.** Throughout the paper we will use 3 different ways to express the situation from the Theorem 1.7.1:

(1) *Tilting mutation of an indecomposable object M is M^*, with respect to T*: $\nu(M) = M^*$ or more precisely $\nu_T(M) = M^*$ e.g. $\nu_T(S_1) = S_2$ with respect to the tilting object $T = S_1 \coprod P_2 \coprod P_3$.

(2) *Tilting mutation of the tilting object $T = M \coprod \bar{T}$ at M is $T' = M^* \coprod \bar{T}$*: $\bar{\nu}(T) = T'$, or $\bar{\nu}_M(T) = \bar{\nu}_M(M \coprod \bar{T}) = \nu_T(M) \coprod \bar{T} = M^* \coprod \bar{T} = T'$ e.g. $\bar{\nu}_{S_1}(S_1 \coprod P_2 \coprod P_3) = S_2 \coprod P_2 \coprod P_3$.

(3) *Tilting mutation of the tilting seed (T, Q_T) at M is the new seed $(T', Q_{T'})$*: $\tilde{\nu}_M(T, Q_T) = (\bar{\nu}_M(T), Q_{\bar{\nu}_M(T)})$

e.g. $\tilde{\nu}_{S_1}((S_1 \coprod P_2 \coprod P_3), Q_T) = (S_2 \coprod P_2 \coprod P_3), Q_{T'})$, with the extra information that $Q_T = 1 \leftarrow 2 \leftarrow 3$ and $Q_{T'} = 1' \rightarrow 2 \leftarrow 3$.

All examples refer to the cluster category in 1.8.

1.10. **A construction of exchange pairs, i.e. tilting mutation.** This construction uses the notion of minimal right approximation by a subcategory. Let $T = M \coprod \bar{T}$ be a tilting object, where M is an indecomposable summand. Let $f : B \rightarrow M$ be a minimal right $\operatorname{add} \bar{T}$-approximation of M. This approximation exists, since $\operatorname{add} \bar{T}$ is additively generated by a finitely generated module $\bar{T}$. Let

$$\rightarrow M^* \xrightarrow{g} B \xrightarrow{f} M \rightarrow$$

be the associated triangle in D^b.

1.11. **Some facts about exchange pairs.** Here we state some of the facts (without proofs) and draw the commutative diagrams from [BMRRT].

(1) M^* is indecomposable and exceptional,
(2) $M^* \xrightarrow{g} B$ is a minimal left $\operatorname{add} \bar{T}$- approximation of M^*,
(3) $\dim_{D_M} \operatorname{Ext}^1_{C_\Lambda}(M, M^*) = 1 = \dim_{D_{M^*}} \operatorname{Ext}^1_{C_\Lambda}(M, M^*)$,
(4) M and M^* form an exchange pair with respect to T (or T').
(5) Consider the AR-triangle for M^*:

$$\rightarrow M^* \rightarrow A_{M^*} \rightarrow \tau^{-1} M^* \rightarrow .$$

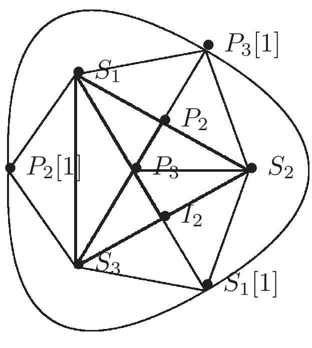


FIGURE 1. Tilting objects in the cluster category of A_3

Together with the triangle from 1.10, we get the following commutative diagram:

$$\begin{tikzcd}
 & {} \arrow[d] & & K_h \coprod L_h[-1] \arrow[d] & \\
{} \arrow[r] & M^* \arrow[r] \arrow[d, "="'] & A_{M^*} \arrow[r] \arrow[d] & \tau^{-1}M^* \arrow[r] \arrow[d, "h"] & {} \\
{} \arrow[r] & M^* \arrow[r, "g"] & B \arrow[r, "f"] & M \arrow[r] \arrow[d, "g'"] & {} \\
 & & & B' \arrow[d, "f'"] & \\
 & & & \tau^{-1}M^*[1] = FM^* \cong_{c_\Lambda} M^*. &
\end{tikzcd}$$

(6) $\dim \mathrm{Hom}_{D^b}(\tau^{-1}M^*, M) = 1$,
(7) $B' = Coker(h) \coprod Ker(h)[1] := L_h \coprod K_h[1]$,
(8) h is an isomorphism if and only if $B' = 0$,
(9) $M^* \cong \tau M$ if and only if $M^* \to B \to M$ is AR triangle if and only if $B' = 0$.
(10) The vertical maps $M \xrightarrow{g} B'$ and $B' \xrightarrow{f} M^*$ are left and right add$\bar{T}$-approximations of M and M^* (respec).
(11) By considering AR triangle for M we get a similar commutative diagram.

2. Cluster Algebras and Cluster Categories

In this section we want to indicate what correspondences exist between the categorical notions of cluster categories and combinatorial notions of cluster algebras.

Cluster algebras were defined by Fomin and Zelevinsky, in general with coefficients, however cluster categories are set to deal only with the cluster algebras without coefficients, so we will recall the definition of cluster algebras only in that case. We also point out that the "sign skew symmetrizable square matrices" correspond to valued quivers, with valuations on arrows being precisely the integer values of the entries of the corresponding matrix.

2.1. **Cluster algebras; Definitions.**

Cluster algebra $\mathcal{A}$ is a subalgebra of the field of rational functions $\mathbb{Q}(u_1, \dots, u_n)$, which is generated as algebra by cluster variables; *cluster variables* are defined to be all rational functions which appear in a certain collection of transcendence bases, called clusters; *clusters* are transcendence bases in $\mathbb{Q}(u_1, \dots, u_n)$ which are obtained from an initial cluster after applying sequences of "cluster mutations" in every possible direction, but according to the rules prescribed by the matrices B, (or equivalently, quivers Q) as we will describe in 2.1.1.

We point out that at every stage of cluster mutations, there is a new pair formed $(\underline{x}, Q)$, called a *cluster seed*, where $\underline{x}$ is the new cluster and Q is the new quiver. *Initial cluster seed* is a pair $(\underline{x}_{(0)}, Q_{(0)})$, where $\underline{x}_{(0)} = \{x_{(0),1}, \dots, x_{(0),n}\}$ is a transcendence basis, called *initial cluster* and $Q_{(0)}$ is the initial quiver (a matrix $B = (b_{ij})$).

Definition 2.1.1. Using the notation from above we now define cluster mutations (again, the same mutation expressed in 3 ways as in 1.9):

(1) *Cluster mutation of* x_i with respect to the cluster seed $(\underline{x}, Q)$ is the new cluster variable $(x_i)^*$ obtained as $x_i \cdot (x_i)^* = \prod_{b_{ij}>0} x_i^{b_{ij}} + \prod_{b'_{ij}<0} x_i^{b'_{ij}}$.
Notation $\mu(x_i) = (x_i)^*$.

(2) *Cluster mutation of the cluster* $\underline{x} = \{x_1, \dots, x_n\}$ at x_i is a new cluster $\underline{x}' = \{x_1, \dots, (x_i)^*, \dots x_n\} = \{x_1, \dots, \mu_i(x_i), \dots x_n\}$
Notation: $\bar{\mu}_i(\underline{x}) = \{x_1, \dots, \mu_i(x_i), \dots x_n\} = \underline{x}'$

(3) *Cluster mutation of the cluster seed* $(\underline{x}, Q)$ *at* x_i is the new cluster seed $(\underline{x}', Q')$.
Notation: $\tilde{\mu}_i(\underline{x}, Q) = (\bar{\mu}_i(\underline{x}), \vec{\mu}_i Q) = (\underline{x}', Q')$, where $\vec{\mu}_i(Q) = Q'$ new quiver obtained in quite a complicated way (see [FZ1]).

In general, this process of creating new cluster variables will never stop, however there is a precise classification of cluster algebras which have only finitely many cluster variables, as stated in the following theorem.

Theorem 2.1.2. [FZ2] *A cluster algebra has finitely many cluster varibles, if and only if a Dynkin diagram appears as one of the quivers in the mutation process.*

2.2. **Bijections.**

Starting with a fixed quiver Q (no oriented cycles), one obtains a cluster algebra $\mathcal{A}(Q)$, with all of its notions: cluster seeds, clusters, cluster variables, cluster mutations. From the same quiver one can construct cluster category $C_{\Bbbk Q}$, with the tilting seeds, tilting object, indecomposable objects and tilting mutations. The following table contains main "cluster algebra" and "cluster category" notions, NOT as a theorem, but just to indicate where the correspondences should be. However, the theorems are stated afterwards.

Q quiver

$\mathcal{A}(Q)$-cluster algebra	——	C_Λ-cluster category, where $\Lambda = \Bbbk Q$
$(\underline{x}_{(0)}, Q_{(0)})$-initial cluster seed	——	$(P_1[1] \coprod \cdots \coprod P_n[1], Q)$-initial tilting seed
$(\underline{x}, Q)$-any cluster seed	——	(T, Q_T)-any tilting seed
$\underline{x} = \{x_1, \dots, x_n\}$-any cluster	——	$T = T_1 \coprod \cdots \coprod T_n$-any basic tilting object
x_i-any cluster variable	——	M-any indecomposable exceptional object

Theorem 2.2.1. [FZ1], [MRZ] *Suppose* Q *is a Dynkin diagram. Then there exists a one-to-one correspondence*
$\{$*cluster variables*$\} \to \{$*isoclasses of indecomposable objects* $\}$.

Theorem 2.2.2. [BMRRT] *Suppose* Q *is a simply laced Dynkin diagram. Then there exist one to one correspondences (the second one being induced by the first):*
$\{$*cluster variables*$\} \to \{$*isoclasses of indecomposable objects*$\}$,
$\{$*clusters*$\} \to \{$*basic tilting objects*$\}$.

It was conjectured in the same paper that similar one-to-one correspondences exist between cluster variables and exceptional indecomposable objects for all simply laced diagrams with no oriented cycles. The proof of this follows from the existence of two particular surjective maps, which are inverse bijections:
$\alpha : \{\text{cluster variables}\} \to \{\text{isoclasses of indecomposable exceptional objects}\}$,
$\beta : \{\text{isoclasses of indecomposable exceptional objects}\} \to \{\text{cluster variables}\}$.

The map β is the Caldero-Chapoton map from [CK2], and the existence of map α is given in the following theorem.

Theorem 2.2.3. [BMRT] *Let Q be simply laced diagram with no oriented cycles.*
(a) There exist well defined maps:
$\alpha : \{\textit{cluster variables}\} \to \{\textit{isoclasses of indecomp. exceptional objects}\}$,
$\bar{\alpha} : \{\textit{clusters}\} \to \{\textit{tilting objects}\}$,
$\tilde{\alpha} : \{\textit{cluster seeds}\} \to \{\textit{tilting seeds}\}$.
(b) All three maps: $\alpha, \bar{\alpha}, \tilde{\alpha}$ are onto.

We will indicate main steps of the proof of the above theorem, since they include several other interesting results about the denominators of the cluster variables 2.3.4, weak positivity condition 2.4.6 and proof of the Fomin-Zelevinsky conjecture that cluster determines the entire cluster seed after cluster mutations 2.5.1.

Theorem 2.2.4. Bijection *(*[CK2],[BMRT]*) Let Q be simply laced diagram with no oriented cycles. Then there exist a bijection between cluster variables and exceptional indecomposable objects, inducing a bijection between clusters and tilting objects.*

An illustration on A_3. We illustrate the above correspondence by using the same Figure 1 as we did for the objects, tilting objects and exchange pairs of the cluster category, but label the vertices now by the corresponding cluster variables.

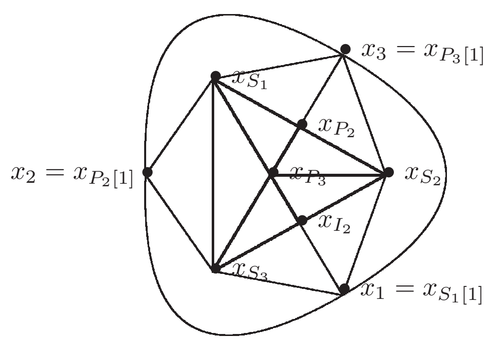


FIGURE 2. Clusters in the cluster algebra of A_3

- Vertices are labeled by cluster variables and cluster variables are labeled by the corresponding representatives of indecomposable objects in the cluster category.
- Vertices are connected by a line if the compatibility degree between the cluster variables is 0 which corresponds to no extensions between the corresponding two objects.
- Each small triangle defines a cluster and a basic tilting object.
- All clusters and all tilting objects are given that way.
- There are 14 clusters and 14 basic tilting objects in this example.
- The initial cluster $\{x_1, x_2, x_3\}$ is chosen in such a way, that the denominators of all other cluster variables are given by the dimension vectors of the corresponding $\Bbbk Q$-modules (see theorems in next section).

2.3. **Denominators of cluster variables.**

From this section, we only need notation, and the famous "Laurent phenomenon" theorem of Fomin and Zelevinsky. However we will also state known results about the monomials which appear in the denominators of the cluster variables.

Theorem 2.3.1. [FZ1] *(Laurent phenomenon) The denominators of all cluster variables, when expressed in terms of the initial cluster, and then reduced, are monomials.*

Notation: Let $m = x_1^{k_1} \dots x_n^{k_n}$ be a monomial in variables $x_1, \dots, x_n$. Denote the exponent vector by $\epsilon(m) = (k_1, \dots, k_n) \in \mathbb{Z}^n_{\geq 0}$.

Theorem 2.3.2. [FZ1] *Let Q be a Dynkin diagram with alternating orientation. For each cluster variable in reduced form f/m, there is an indecomposable module M, such that $\epsilon(m) = \underline{dim} M$.*

Theorem 2.3.3. [CCS2] *Let Q be any Dynkin diagram. For each cluster variable in reduced form f/m, there is an indecomposable module M, such that $\epsilon(m) = \underline{dim} M$.* (Also [RT] and [CK1] have the same result.)

Theorem 2.3.4. [BMRT] *Let Q be any finite simply laced diagram with no oriented cycles. For each cluster variable in reduced form f/m, there is an indecomposable exceptional module M, such that $\epsilon(m) = \underline{dim} M$.*

Proof. This follows from 2.4.6. □

2.4. **Weak Positivity Condition and Conditions ($*$, $\bar{*}$, $\tilde{*}$).**

These are essential notions for the proofs of the existence of the maps α, $\bar{\alpha}$, $\tilde{\alpha}$, i.e. that the objects $\alpha(x)$, $\bar{\alpha}(\underline{x'})$, $\tilde{\alpha}((\underline{x'}, Q'))$ satisfy the desired conditions: $\alpha(x)$ is an indecomposable exceptional object, $\bar{\alpha}(\underline{x'})$ is a tilting object and $((\underline{x'}, Q'))$ is a tilting seed.

Definition 2.4.1. A polynomial f in variables $\{x_1, \dots, x_n\}$ is said to satisfy the *weak positivity condition* if $f(e_i) > 0$ for all $e_i = (1, 1, \dots, 1, 0, 1, \dots, 1, 1)$, where 0 is at the i-th place.

Remark: If f satisfies the weak positivity condition, then f does not have any non-constant monomial factors.

Definition 2.4.2. Let $(x_1, \ldots, x_n)$ be an initial cluster for a cluster algebra given by a quiver Q. A cluster variable x is said to satisfy *condition* $(*)$ if:

(1) either $x = f/m$ or $x = fx_i$, where f is a polynomial in $\{x_1, \ldots, x_n\}$ and it satisfies positivity condition and
(2) if $x = f/m$, then there exists an indecomposable exceptional $\Bbbk Q$-module M, such that $\epsilon(m) = \underline{dim}M$.

Definition 2.4.3. If a cluster variable x satisfies condition $(*)$, we (can) define:

$$\alpha(x) := \begin{cases} M & \text{if } x = f/m \text{ and } \epsilon(m) = \underline{dim}M \\ P_i[1] & \text{if } x = fx_i. \end{cases}$$

Remark: We want to define α on all cluster variables. So far, we defined only on cluster variables satisfying $(*)$. So, we will prove that ALL cluster variables satisfy condition $(*)$ if the cluster algebra is acyclic with no coefficients.

This will be proved after the Proposition 2.4.4 and Corollary 2.4.6, however the proof involves clusters and cluster seeds as well. The corresponding conditions on clusters and cluster seeds will be denoted by: $(\bar{*})$,$(\tilde{*})$ and we define them now.

Properties: $(*)$, $(\bar{*})$, $(\tilde{*})$ Definitions and properties.
$(*)$ is a property of a cluster variable x'_i consisting of two parts:

- $x'_i = f/m$ or $x'_i = fx_i$, where f satisfies positivity condition, and
- $\epsilon(m) = \underline{dim}M$, for some indecomposable exceptional module M.
  Notice: If x'_i satisfies $(*)$, then $\alpha(x'_i)$ is defined (i.e. it is an indecomposable exceptional object)(as in 2.4.2 and 2.4.3).

$(\bar{*})$ is a property of a cluster $\underline{x'} = \{x'_1, \ldots, x'_n\}$ consisting of two parts:

- each cluster variable x'_i satisfies $(*)$, and
- $\alpha(x'_1) \coprod \cdots \coprod \alpha(x'_n)$ is a tilting object.
  Notice: If $\underline{x'}$ satisfies $(\bar{*})$, then $\bar{\alpha}(\underline{x'})$ can be defined as
  $\bar{\alpha}(\underline{x'}) = \alpha(x'_1) \coprod \cdots \coprod \alpha(x'_n)$, which is a tilting object.

$(\tilde{*})$ is a property of a cluster seed $(\underline{x'}, Q')$ consisting of two parts:

- the cluster $\underline{x'}$ satisfies $(\bar{*})$, and
- $Q_{(\bar{\alpha}(\underline{x'}))} = Q'$.

  Notice: If $(\underline{x'}, Q')$ satisfies $(\tilde{*})$, then $\tilde{\alpha}(\underline{x'}, Q')$ can be defined as
  $\tilde{\alpha}(\underline{x'}, Q') = (\bar{\alpha}(\underline{x'}), Q')$ which is a tilting seed.

Proposition 2.4.4. *Let $(\underline{x'}, Q')$ be a cluster seed satisfying $(\tilde{*})$. Consider a cluster mutation. Then:*

(1) a**:** *The new cluster variable satisfies $(*)$.*
  $\bar{a}$**:** *The new cluster satisfies $(\bar{*})$.*
  $\tilde{a}$**:** *The new cluster seed satisfies $(\tilde{*})$.*
(2) a**:** *α commutes with mutations, i.e. $\alpha \circ \mu = \nu \circ \alpha$.*

$\bar{a}$**:** $\bar{\alpha}$ *commutes with mutations, i.e.* $\bar{\alpha} \circ \bar{\mu} = \bar{\nu} \circ \bar{\alpha}$.
$\tilde{a}$**:** $\tilde{\alpha}$ *commutes with mutations, i.e.* $\tilde{\alpha} \circ \tilde{\mu} = \tilde{\nu} \circ \tilde{\alpha}$.

Proof. We need to set up a bit more detailed and precise notation.
Let $(\underline{x}', Q')$ be a cluster seed, where $\underline{x}' = \{x'_1, \dots, x'_n\}$ is a cluster.
Let μ be a cluster mutation at x'_i.
Since x'_i satisfies $(*)$, there exists an object M such that $\alpha(x'_i) = M$.
Denote x'_i by x_M.
Let $(x_M)^*$ be the new cluster variable defined via cluster mutation μ.
Then $\mu(\underline{x}') = \mu(\{x'_1, \dots, x'_i \dots, x'_n\}) = \mu(\{x'_1, \dots, x_M \dots, x'_n\}) = \{x'_1, \dots, (x_M)^* \dots, x'_n\}$.
<u>Proof of *a:* WTS $(x_M)^*$ satisfies $(*)$</u>. Let $T = T_1 \coprod \dots \coprod T_n = \alpha(x'_1) \coprod \dots \coprod \alpha(x'_n)$.
It is a tilting object since $\underline{x}'$ satisfies $(\bar{*})$.
Consider tilting mutation of T at M, i.e. exchange M with another object M^*.
Then $T' = T_1 \coprod \dots \coprod M^* \coprod \dots \coprod T_n$ is the new tilting object.
Let $M^* \to B \to M$ and $M \to B' \to M^*$ be the exchange triangles.
Notation and results from ([BMR2], 6.2) imply:
$x_M (x_M)^* = x_B + x_{B'} = \prod x_{B_i} + \prod x_{B'_i}$ if $B = \coprod B_i$ and $B' = \coprod B_i$.
At this point we brake the proof into several parts, depending on whether the representatives of M and M^* are modules or shifted projectives.
Case I: Both M and M^* are modules.

Step 1: $0 \to M^* \to B \to M \to 0$ is an exact sequence of modules.
Step 2: Describe B', i.e. representatives of the summands of B' in the fundamental domain $\mathrm{add}(\mathrm{ind}\Lambda \cup \{P_i[1]\}_{i=1}^n)$. Consider the following commutative diagram, together with the AR-triangles as mentioned at the beginning:

$$\begin{tikzcd}
 & {} \arrow[dd] & & K_h \coprod L_h[-1] \arrow[dd] & \\
 & & & & 0 \\
 & K_g \coprod L_g[-1] \arrow[dd] & & \tau^{-1}M^* \arrow[ur] \arrow[dd, "h"] & \\
 & & A_{M^*} \arrow[ur] \arrow[d] & & 0 \\
0 \arrow[r] & M^* \arrow[r] \arrow[ur] \arrow[dd, "g"] & B \arrow[r] \arrow[d] & M \arrow[r] \arrow[ur] \arrow[dd] & 0 \\
0 \arrow[ur] & & A_{\tau M} \arrow[ur] & & \\
 & \tau M \arrow[ur] \arrow[dd] & & B' = K_h[1] \coprod L_h \arrow[dd] & \\
0 \arrow[ur] & & & & \\
 & {} & & \tau^{-1}M^*[1] = FM^* &
\end{tikzcd}$$

$\dim \operatorname{Ext}(M, M^*) = 1$ implies $\dim \operatorname{Hom}(\tau^{-1}M^*, M) = 1$ and $\tau^{-1}g = h$
$\tau^{-1}(K_g \coprod L_g[-1])[1] \cong_{D^b} (K_h[1] \coprod L_h)$
Let $L_g = I \coprod Z$ with I injective and Z with no injective summands.
Let $K_h = P \coprod Y$ with P projective and Y with no projective summands.
$\tau^{-1}K_g[1] \coprod \tau^{-1}I \coprod \tau^{-1}Z \cong_{D^b} (L_h \coprod P[1] \coprod Y[1])$

By comparing shifts of nonprojectives, shifts of projectives and actual modules, get the following isomorphisms in the derived category:
$\tau^{-1}K_g[1] \cong_{D^b} (Y[1])$, $P(SocI)[1] = \tau^{-1}I \cong_{D^b} P[1]$, $\tau^{-1}Z \cong_{D^b} (L_h)$
$\tau^{-1}K_g[1] \cong_{D^b} (Y[1])$ implies $FK_g \cong_{D^b} (Y[1])$ and so $K_g \cong_{C} (Y[1])$
Finally: $B' \cong_{C} (L_h \coprod K_g \coprod P[1])$

Step 3: All x_M, x_B, $x_{B'}$ satisfy condition $(*)$ by assumption.
$x_M = (f_M/m_M)$, $x_B = (f_B/m_B)$, $x_{K_g} = (f_{K_g}/m_{K_g})$, $x_{L_h} = (f_{L_h}/m_{L_h})$,
$x_{P[1]} = (c_{(P/rP)} f_{P[1]})$ where $\epsilon(c_{(P/rP)}) = \underline{\dim}(P/rP)$
f_M, f_B, f_{K_g}, f_{L_h}, $f_{P[1]}$ satisfy positivity condition
$\epsilon(m_M) = \underline{\dim}M$, $\epsilon(m_B) = \underline{\dim}B$, $\epsilon(m_{K_g}) = \underline{\dim}(K_g)$, $\epsilon(m_{L_h}) = \underline{\dim}(L_h)$.

Step 4: $(m_{K_g} m_{L_h}) | m_B$ or $(m_{K_g} m_{L_h}) = (m_B/m')$ Reason:
$\underline{\dim}(K_g) + \underline{\dim}(L_h) + \underline{\dim}(-) + \underline{\dim}(-) = \underline{\dim}M^* + \underline{\dim}M = \underline{\dim}B$

Step 5: Compute $(x_M)^*$.
$(x_M)^* = \frac{(x_B + x_{B'})}{(x_M)} = \frac{(f_B + f_{K_g} \cdot f_{L_h} \cdot m' \cdot c_{(P/rP)} \cdot f_{P[1]})(1/f_M)}{(m_B/m_M)}$

Step 6: $(x_M)^*$ satisfies condition $(*)$. Reason:
f_M divides the rest of the numerator (Laurent phenomenon).
Numerator is a polynomial which satisfies positivity condition.
Numerator does not have any non-constant monomial factors.
Denominator is monomial m_B/m_M, with $\epsilon(m_B/m_M) = \underline{\dim}M^*$.
M^* is an indecomposable exceptional module.

Step 7: $\alpha((x_M)^*) = M^*$. This is exactly the statement that α commutes with mutations, i.e. 2)a of the proposition.

Case II: M is not a module, i.e. $M = P[1]$.
First, we point out that in this case M^* must be a module, since $\operatorname{Ext}(M, M^*) \neq 0$. We will just set up the exchange triangles and commutative diagrams, and the rest is similar kind of analysis, but somewhat more complicated.

$M = P[1]$ implies that $B = C \coprod P'[1]$, with C non-projective and P' projective.
$M^*[1] \cong P/P'[1]$ since $\dim \operatorname{Hom}(M, M^*[1]) = 1$

$$\begin{tikzcd}
 & {} \arrow[d] & & {} \arrow[d] & {} \\
 & B' \arrow[dd] & & \tau^{-1}(P/P') \arrow[ur] \arrow[dd, "h"] & \\
 & & A_{P/P'} \arrow[ur] \arrow[d] & & {} \\
P \arrow[r] & P/P' \arrow[r] \arrow[ur] \arrow[dd, "g"'] & C\coprod P'[1] \arrow[r] \arrow[d] & P[1] \arrow[r] \arrow[ur] \arrow[dd] & (P/P')[1] \\
{} \arrow[ur] & & A_I \arrow[ur] & & \\
{} \arrow[ur] & I \arrow[ur] \arrow[d] & & B' \arrow[d] & \\
 & {} & & \tau^{-1}(P/P')[1] = F(P/P') &
\end{tikzcd}$$

Case III: M a module, $M^* = P[1]$. Similar to the previous case.

This finishes the proof of a : the new cluster variable $(x_M)^*$ satisfies condition $(*)$ and also $\alpha((x_M)^*) = M^*$.

Proof of $\bar{a}$: The new cluster $\{x'_1, x'_2, \ldots, (x_M)^*, \ldots, x'_{n-1}, x'_n\}$ satisfies $(\bar{*})$.

- Each cluster variable $\{x'_1, x'_2, \ldots, (x_M)^*, \ldots, x'_{n-1}, x'_n\}$ satisfies $(*)$ condition by assumption and part a.
- $\alpha(x'_1) \coprod \alpha(x'_2) \ldots \alpha((x_M)^*) \ldots \alpha(x'_{n-1}) \coprod \alpha(x'_n) = T_1 \coprod T_2 \ldots M^* \ldots T_{n-1} \coprod T_n$ is a tilting object.

Proof of $\tilde{a}$: The new cluster seed $(\{x'_1, x'_2, \ldots, (x_M)^*, \ldots, x'_{n-1}, x'_n\}, Q")$ satisfies $(\tilde{*})$.

- The new cluster $\{x'_1, x'_2, \ldots, (x_M)^*, \ldots, x'_{n-1}, x'_n\}$ satisfies $(\bar{*})$ by $\bar{a}$.
- The quiver of $\mathrm{End}(T_1 \coprod T_2 \ldots M^* \ldots T_{n-1} \coprod T_n)^{op}$ is equal to $Q"$ by [BMR].

This finishes the proof of the proposition. $\square$

Corollary 2.4.5. *Let $(\underline{x}', Q')$ be a cluster seed satisfying condition $(\tilde{*})$. Let $\tilde{\mu}$ be a cluster mutation and $\tilde{\nu}$ the corresponding tilting mutation. Then: $\tilde{\alpha} \circ \tilde{\mu} = \tilde{\nu} \circ \tilde{\alpha}$.*

Corollary 2.4.6. *Every cluster seed satisfies* ($\tilde{*}$)*, every cluster satisfies* ($\bar{*}$) *and every cluster variable satisfies* ($*$)*. In particular, every cluster variable is either of the form* f/m *or* fx_i*, for some weakly positive polynomial* f*.*

Proof. The initial seed $(\{x_1 \dots x_n\}, Q)$ satisfies ($\tilde{*}$) condition. Since every cluster seed can be reached by a finite number of cluster mutations, and by the proposition every cluster seed at each step satisfies ($\tilde{*}$) condition it follows that all cluster seeds satisfy ($\tilde{*}$) condition. □

Proof. of the Theorem 2.2.3(a): By the above corollary it follows that everything satisfies appropriate ($*$), ($\bar{*}$), ($\tilde{*}$), and by the comments in the definitions of ($*$), ($\bar{*}$), ($\tilde{*}$), all three maps (α), ($\bar{\alpha}$), ($\tilde{\alpha}$) are defined. □

Proof. of the Theorem 2.2.3(b): We will show that $\tilde{\alpha}$ is onto. Let (T, Q_T) be a tilting seed. There exists a finite number of tilting mutations from the initial tilting seed to (T, Q_T). Consider the same sequence of cluster mutations from the initial cluster seed. Use Corollary 2.4.5. □

2.5. **Fomin-Zelevinsky Conjecture: Cluster Determines Seed.**

The following was conjectured by Fomin and Zelevinsky (for any cluster algebra) and we prove it for the acyclic case with no coefficients.

Theorem 2.5.1 (BMRT)**.** *Let* $(\underline{x}', Q')$ *and* $(\underline{x}', Q")$ *be cluster seeds for an acyclic cluster algebra with no coefficients. Then* $Q' = Q"$*.*

Proof. $\tilde{\alpha}(\underline{x}', Q') = (\bar{\alpha}(\underline{x}'), Q')$ and $\tilde{\alpha}(\underline{x}', Q") = (\bar{\alpha}(\underline{x}'), Q")$ are tilting seeds. But, $Q' = Q"$ since tilting seed is determined by its tilting object. □

3. Semi-invariants and Cluster Categories

We first recall the definition and some of the classical theorems about semi-invariants, as done by Kac, Schofield, Derksen-Weyman for non-negative integral vectors. After that we state theorems for "mixed signs" integral vectors as done in [IOTW]. In particular, in the Dynkin diagram case, we state the relation between the simplicial complex of the partial tilting objects and the domains of semi-invariants .

3.1. **Definitions; Representations and Semi-invariants.** Let $Q = (Q_0, Q_1)$ be a simply laced quiver, where Q_0 denotes the set of the vertices of Q, and Q_1 is the set of the arrows of Q. Assume Q has no oriented cycles. Let $\Bbbk$ be an algebraically closed field. Let n be the number of the vertices in Q_0.

3.2. **Representation space for** $\underline{\alpha} \in \mathbb{N}^n$**.** The representation space for a non-negative integral vector $\underline{\alpha} = (\alpha_v)$ is the affine space:

$$R(\underline{\alpha}) = \prod_{(u \to v) \in Q_1} \mathrm{Hom}_{\Bbbk}(\Bbbk^{\alpha_u}, \Bbbk^{\alpha_v}).$$

Elements of $R(\underline{\alpha})$ will be called based representations with *dimension vector* $\underline{\alpha}$. Every element of $R(\underline{\alpha})$ can be viewed as a collection of $\alpha_v \times \alpha_u$-matrices, one for each arrow $a : u \to v$. The group:

$$G(\underline{\alpha}) = \prod_{v \in Q_0} Gl_{\alpha_v}(\Bbbk)$$

acts on $R(\underline{\alpha})$, by $\underline{g}M_a := (g_{ha})M_a(g_{ta})^{-1}$. We use the convention that $Gl_0(\Bbbk)$ is the trivial group.

3.3. **Euler bilinear form.** The vertices of the quiver are partially ordered by $u < v$ if there is a directed path from u to v and we choose a fixed extension of this partial ordering to a total ordering. The *Euler matrix* E is defined as $n \times n$ matrix with rows and columns labeled by Q_0 (written in the order described above), with the diagonal entries equal to 1 and the entry $E_{u,v} = -$(the number of arrows from u to v) for $u \neq v$. The matrix E gives a nonsymmetric bilinear form

$$\langle \underline{\alpha}, \underline{\beta} \rangle := \underline{\alpha}^t E \underline{\beta}$$

on $\mathbb{Z}^n$, which we call the *Euler form.*

3.4. **Some useful facts about the Euler form and Euler matrix.**

(1) $\langle \underline{\dim} M, \underline{\dim} M' \rangle = \dim_{\Bbbk} \operatorname{Hom}_Q(M, M') - \dim_{\Bbbk} \operatorname{Ext}^1_Q(M, M')$, i.e. $\langle \underline{\alpha}, \underline{\alpha}' \rangle = \dim_{\Bbbk} \operatorname{Hom}_Q(M, M') - \dim_{\Bbbk} \operatorname{Ext}^1_Q(M, M')$ for all representations M and M' such that $\underline{\alpha} = \underline{\dim} M$ and $\underline{\alpha}' = \underline{\dim} M'$.
(2) The row of E corresponding to the vertex v, consists of coefficients of indecomposable projectives in the expression of $S(v)$.
(3) $E^t(\underline{\alpha})$ gives coefficient vector of $\underline{\alpha}$ in terms of $\underline{\dim} P(w)$s.
(4) $\underline{\alpha} - E^t(\underline{\alpha})$ always has non-negative coefficients, e.g. if $\underline{\alpha} = \underline{\dim} S(v)$ then $\underline{\alpha} - E^t(\underline{\alpha}) = \underline{\dim}(radP(v))$.

3.5. **Notation.** Let $\underline{\alpha} \in \mathbb{N}^n$. We denote by $P(\underline{\alpha})$ the following projective representation: $P(\underline{\alpha}) = \coprod_{v \in Q_0} P(v)^{\alpha_v}$. Notice that $P(\underline{\dim} S(v)) = P(v)$.

3.6. **Canonical projective presentations.** Recall, the *canonical projective presentation* of a representation M, with $\underline{\dim} M = \underline{\alpha}$ is:

$$0 \to P_1 \xrightarrow{c_M} P_0 \to M \to 0$$

where $P_0 = P(\underline{\alpha}) = \coprod_v P(v)^{\alpha_v}$ and $P_1 = \coprod_{u \to v} P(v)^{\alpha_u}$. The mapping $c_M : P_1 \to P_0$ sends $P(v)^{\alpha_u}$ to $P(u)^{\alpha_u}$ by the negative inclusion map; so, the generator of each $P(v)$ maps to -(the image in $P(u)_v$ of the generator of the corresponding $P(u)$) and c_M sends $P(v)^{\alpha_u}$ to $P(v)^{\alpha_v}$ by the mapping $M_a : M_u \to M_v$ induced by the arrow $a : u \to v$.

Remark 3.6.1. *The representation P_1 can also be described as:*
$P_1 = \coprod_v P(v)^{(\sum_{u \to v} \alpha_u)} = P(\underline{\alpha} - E^t\underline{\alpha})$, *and* $c_M \in \operatorname{Hom}_Q(P(\underline{\alpha}), P(\underline{\alpha} - E^t\underline{\alpha}))$.

3.7. **Classical results on semi-invariants of quivers.**

We recall now the notion of semi-invariants of a group acting on a variety and state the classical results about semi-invariants on the representation spaces of quivers by Kac, Schofield and Derksen-Weyman.

3.8. **Definition of semi-invariants.** For an algebraic group G acting on a variety X, an element f of the coordinate ring of X is called a *semi-invariant*, if there exists a character χ of G such that for all $g \in G$ and all $v \in X$:

$$f(g \cdot v) = \chi(g) f(v).$$

3.9. **Semi-invariants of quivers, $\underline{\alpha} \in \mathbb{N}^n$.** The group $G(\underline{\alpha})$ acts on the representation space $R(\underline{\alpha})$ (see 3.2). Since the group $G(\underline{\alpha})$ is the product of general linear groups, the character $\chi(g)$ is the product $(det(g_1))^{\sigma_1} \dots (det(g_n))^{\sigma_n}$. The integral vector $\underline{\sigma} = (\sigma_1, \dots, \sigma_n) \in \mathbb{Z}^n$ is called the *weight* of the semi-invariant. Note that if $g_i \in GL_0(\Bbbk)$ then σ_i is indeterminate.

3.10. **Fundamental theorems for semi-invariants of quivers.** The First fundamental theorem (FFT) states that all semi-invariants of quivers are generated by determinants, the Saturation theorem describes all nonnegative vectors with semi-invariants of a given weight and the third theorem describes Generic decomposition of any non-negative integral vector $\underline{\alpha} \in \mathbb{N}^n$.

Theorem 3.10.1 (FFT,[S, DW1])**.** *Let Q be a quiver and $\underline{\alpha} \in \mathbb{N}^n$. Then the ring of semi-invariants on $R(\underline{\alpha})$ is generated by $\{det(c_M, N)\}$, where*

$$(c_M, N) : \mathrm{Hom}_Q(P_0, N) \to \mathrm{Hom}_Q(P_1, N)$$

are induced by the canonical presentation of $M \in R(\underline{\alpha})$, and representations N satisfy $\langle \underline{\alpha}, \underline{dim} N \rangle = \underline{\alpha}^t E \underline{dim} N = 0$. Furthermore, the weight of the semi-invariant $det(c_M, N)$ is $E\underline{dim}N$.

Remark 3.10.2. *The condition $\langle \underline{\alpha}, \underline{dim} N \rangle = \underline{\alpha}^t E \underline{dim} N = 0$ is equivalent to saying that the matrix of (c_M, N) is square.*

Definition 3.10.3. *Let $\underline{\sigma} \in \mathbb{Z}^n$. The* support *of $\underline{\sigma}$ is defined as*

$$supp(\underline{\sigma}) := \{\underline{\alpha} \in \mathbb{N}^n \mid R(\underline{\alpha}) \text{ has a semi-invariant of weight } \underline{\sigma}\}.$$

Theorem 3.10.4 (Saturation,[DW])**.** *Let $\underline{\beta} \in \mathbb{N}^n$. Then:*

$$supp(E\underline{\beta}) = \mathbb{N}^n \cap D(\underline{\beta}), \text{ with}$$

$D(\underline{\beta}) = \{\underline{\alpha} \in \mathbb{R}^n | \langle \underline{\alpha}, \underline{\beta} \rangle = 0\} \cap (\cap_{\underline{\beta}' \hookrightarrow \underline{\beta}} \{\underline{\alpha} \in \mathbb{R}^n | \langle \underline{\alpha}, \underline{\beta}' \rangle \leq 0\})$, where $\underline{\beta}' \hookrightarrow \underline{\beta}$ means that the general representation of dimension $\underline{\beta}$ has a subrepresentation of dimension $\underline{\beta}'$.

Before stating the generic decomposition theorem, recall the following definition: for $\underline{\alpha}, \underline{\beta} \in \mathbb{N}^n$,

$$ext(\underline{\alpha}, \underline{\beta}) := \min\{\dim_{\Bbbk} \mathrm{Ext}(A, B) | \underline{\dim} A = \underline{\alpha}, \underline{\dim} B = \underline{\beta}\}$$

($\Bbbk$ is algebraically closed). Since $\dim \mathrm{Ext}$ is upper semicontinuous this minimum is attained for generic modules of these dimensions. So, this definition agrees with the usual definition, i.e. $ext(\underline{\alpha}, \underline{\beta}) = \dim_{\Bbbk} Ext_Q(A, B)$, where A, B are generic representations with $\underline{dim}A = \underline{\alpha}$ and $\underline{dim}B = \underline{\beta}$.

Theorem 3.10.5 (Generic decomposition,[K]). *Any $\underline{\alpha} \in \mathbb{N}^n$ has a unique decomposition of the form $\underline{\alpha} = \Sigma\underline{\alpha}_i$ where $ext(\underline{\alpha}_i, \underline{\alpha}_j) = 0$ for all $i \neq j$ and each $\underline{\alpha}_i$ is a Schur root. Furthermore, general representation M with $\underline{dim}M = \underline{\alpha}$ decomposes as $M \cong \coprod M_i$ with $\underline{\dim} M_i = \underline{\alpha}_i$ where M_i are indecomposable representations which do not extend each other.*

3.11. **Virtual representation spaces and Virtual semi-invariants.**

In this section we deal with integral vectors (not necessarily non-negative), define non-canonical generalized representation spaces and the virtual representation space as the direct limit of those. Similarly, we define semi-invariants on the generalized representation spaces and virtual semi-invariant as the induced map on the virtual representation space.

3.12. **Projective decomposition of integral vectors.** Let $\underline{\alpha} \in \mathbb{Z}^n$ and let $E^t\underline{\alpha} = \underline{\gamma}_0 - \underline{\gamma}_1$ with $\underline{\gamma}_i \in \mathbb{N}^n$. We refer to $(\underline{\gamma}_0, \underline{\gamma}_1)$ as a *projective decomposition* of $\underline{\alpha}$. Note that there is a unique *minimal projective decomposition* where $\underline{\gamma}_0, \underline{\gamma}_1$ have disjoint supports.

3.13. **Generalized representation spaces for integral vectors.** Let $\underline{\alpha} \in \mathbb{Z}^n$. For each projective decomposition $(\underline{\gamma}_0, \underline{\gamma}_1)$ of $\underline{\alpha}$, a non-canonical *generalized representation space* is defined as

$$R^{(\underline{\gamma}_0, \underline{\gamma}_1)}(\underline{\alpha}) := \mathrm{Hom}_Q(P(\underline{\gamma}_1), P(\underline{\gamma}_0)).$$

3.14. **Stabilization maps.** Given any $\underline{\gamma}_0, \underline{\gamma}_1, \underline{\in}\mathbb{N}^n$ we have the stabilization map: $\mathrm{Hom}_Q(P(\underline{\gamma}_1), P(\underline{\gamma}_0)) \to \mathrm{Hom}_Q(P(\underline{\gamma}_1) \coprod P(\underline{}), P(\underline{\gamma}_0) \coprod P(\underline{}))$, which sends ψ to $\psi \coprod 1_{P(\underline{\gamma})}$.

3.15. **Virtual representation spaces for integral vectors.** Let $\underline{\alpha} \in \mathbb{Z}^n$. Then all representation spaces $R^{(\underline{\gamma}_0, \underline{\gamma}_1)}(\underline{\alpha})$ form a directed system with the above stabilization maps. We define *the virtual representation space* as the direct limit:

$$R^{vir}(\underline{\alpha}) = \varinjlim R^{(\underline{\gamma}_0, \underline{\gamma}_1)}(\underline{\alpha}).$$

3.16. **Semi-invariants on Generalized representation spaces.** The space $R^{(\underline{\gamma}_0, \underline{\gamma}_1)}(\underline{\alpha}) = \mathrm{Hom}_Q(P(\underline{\gamma}_1), P(\underline{\gamma}_0))$ is an affine space with the natural action of the group $\mathrm{Aut}(P(\underline{\gamma}_0)) \times (\mathrm{Aut}(P(\underline{\gamma}_1)))^{op}$ which is given by $(g_0, g_1)\psi := g_0\psi g_1$ for each $\psi \in \mathrm{Hom}_Q(P(\underline{\gamma}_1), P(\underline{\gamma}_0))$. (We point out that this is a different action then in 3.2).

Since $R^{(\underline{\gamma}_0, \underline{\gamma}_1)}(\underline{\alpha})$ is an affine space its coordinate ring is a polynomial ring, hence polynomial semi-invariants are defined as in 3.8.

3.17. **Virtual semi-invariants.** A *virtual semi-invariant* on $R^{vir}(\underline{\alpha})$ is a function f^{vir} induced by a family of semi-invariants $f^{(\underline{\gamma}_0,\underline{\gamma}_1)}$ on the representation spaces $R^{(\underline{\gamma}_0,\underline{\gamma}_1)}(\underline{\alpha})$, which are compatible with stabilization maps (3.14).

3.18. **Characters .** Notice that for $P(\underline{\gamma}) = \coprod_v P(v)^{\gamma_v}$, each element of $\mathrm{Aut}(P(\underline{\gamma}))$ can be written as an $n \times n$ upper triangular matrix $g = (g_{uv})$ with $g_{uv} \in \mathrm{Hom}_Q(P(v)^{\gamma_v}, P(u)^{\gamma_u})$ and diagonal entries $g_{vv} \in \mathrm{Aut}(P(v)^{\gamma_v}) \cong GL_{\gamma_v}(\Bbbk)$. The reason is that the vertices are partially ordered by $u < v$ if there is a directed path from u to v and this partial ordering is extended to a total ordering to write the matrix.

Remark 3.18.1. *Given a projective representation $P(\underline{\gamma})$ then any rational character of the group $\mathrm{Aut}(P(\underline{\gamma}))$ is of the form: $\chi_{\underline{\sigma}}(\underline{g}) = \prod_{v\in Q_0} \det(g_{vv})^{\sigma_v}$, with the vector $\underline{\sigma} \in \mathbb{Z}^n$; and $\underline{\sigma} \in \mathbb{N}^n$ for polynomial characters. As before, σ_v is indeterminate if $\underline{\gamma}_v = 0$.*

Remark 3.18.2. *(a) A polynomial SI on $R^{(\underline{\gamma}_0,\underline{\gamma}_1)}(\underline{\alpha}) = \mathrm{Hom}_Q(P(\underline{\gamma}_1), P(\underline{\gamma}_0))$ is a polynomial function f so that: $f((g_0, g_1)\psi) = \chi(g_0, g_1)f(\psi)$ for some character χ, all $\psi \in \mathrm{Hom}_Q(P(\underline{\gamma}_1), P(\underline{\gamma}_0))$ and all $(g_0, g_1) \in \mathrm{Aut}(P(\underline{\gamma}_0)) \times (\mathrm{Aut}(P(\underline{\gamma}_1)))^{op}$.*

(b) Furthermore $\chi(g_0, g_1) = \chi^0(g_0)\chi^1(g_1)$, with χ^0 and χ^1 characters of $\mathrm{Aut}(P(\underline{\gamma}_0))$ and $(\mathrm{Aut}(P(\underline{\gamma}_1)))^{op}$ respectively, and therefore of the form $\chi^0_{\underline{\sigma}^0}$ and $\chi^1_{\underline{\sigma}^1}$ for $\underline{\sigma}^0, \underline{\sigma}^1 \in \mathbb{N}^n$.

3.19. **Weights of semi-invariants.** If $\underline{\gamma}_0, \underline{\gamma}_1$ are sincere (nonzero at every vertex) then $\underline{\sigma}^0 = \underline{\sigma}^1$ above, call it $\underline{\sigma}$. The well-defined $\underline{\sigma} \in \mathbb{N}^n$ is called the *weight* of the SI f. Since the collection of $R^{(\underline{\gamma}_0,\underline{\gamma}_1)}$ for sincere pairs $\underline{\gamma}_0, \underline{\gamma}_1$ is cofinal in the directed system, the weights are well defined for virtual SI and have non-negative coordinates.

3.20. **A construction of semi-invariants for $\underline{\alpha} \in \mathbb{Z}^n$.** We show that the determinants also give generalized SI and induce virtual SI.

Proposition 3.20.1. *Let $E^t\underline{\alpha} = \underline{\gamma}_0 - \underline{\gamma}_1$ be a projective decomposition of $\underline{\alpha} \in \mathbb{Z}^n$. Let N be a representation with $\langle \underline{\alpha}, \underline{\dim} N\rangle = 0$. Then:*

1. *The function $f_N^{(\underline{\gamma}_0,\underline{\gamma}_1)} := \det(\mathrm{Hom}(\ , N))$ is a polynomial SI on the generalized representation space $R^{(\underline{\gamma}_0,\underline{\gamma}_1)}(\underline{\alpha})$,*
2. *Weight($f_N^{(\underline{\gamma}_0,\underline{\gamma}_1)}$)= $\underline{\dim} N$.*
3. *Furthermore, $f_N^{(\underline{\gamma}_0,\underline{\gamma}_1)}$ is compatible with stabilizations.*
4. *The induced semi-invariant f_N^{vir} on the virtual representation space $R^{vir}(\underline{\alpha})$ has also weight equal to $\underline{\dim} N$.*

Definition 3.20.2. *The support of f_N is defined to be the set of all $\alpha \in \mathbb{Z}^n$ so that $f_N \neq 0$ on $R^{(\underline{\gamma}_0,\underline{\gamma}_1)}(\underline{\alpha})$ for some projective decomposition $E^t\underline{\alpha} = \underline{\gamma}_0 - \underline{\gamma}_1$*

Theorem 3.20.3. (Virtual Saturation Theorem) *The support of f_N is equal to*

$$\mathbb{Z}^n \cap D(\underline{\beta})$$

if N is the general representation of dimension $\underline{\beta}$.

Theorem 3.20.4. (Virtual Generic Decomposition) *Any $\underline{\alpha} \in \mathbb{Z}^n$ has a unique decomposition of the form*

$$\underline{\alpha} = \sum \underline{\beta}_i$$

where

(1) *$ext(\underline{\beta}_i, \underline{\beta}_j) = 0$ for all $i \neq j$,*
(2) *Each $\underline{\beta}_i$ is either a Schur root in degree) or negative indecomposable projective in degree 1.*

3.21. **The finite case - Dynkin diagrams.**

We show that the simplicial complex of the clusters can be identified with the domains of the generalized semi-invariants, which will also be identified with the Igusa-Orr pictures for the sets of simple roots [IO].

3.21.1. **Simplicial complex of exceptional (or partial tilting) objects.** The *simplicial complex* of the Dynkin diagram is defined to be the simplicial complex $T(Q)$, with vertex set

$$\Phi_+ \cup \{\text{negative projective roots}\},$$

and faces are the spans of subsets of vertices, corresponding to the exceptional objets in the cluster category.

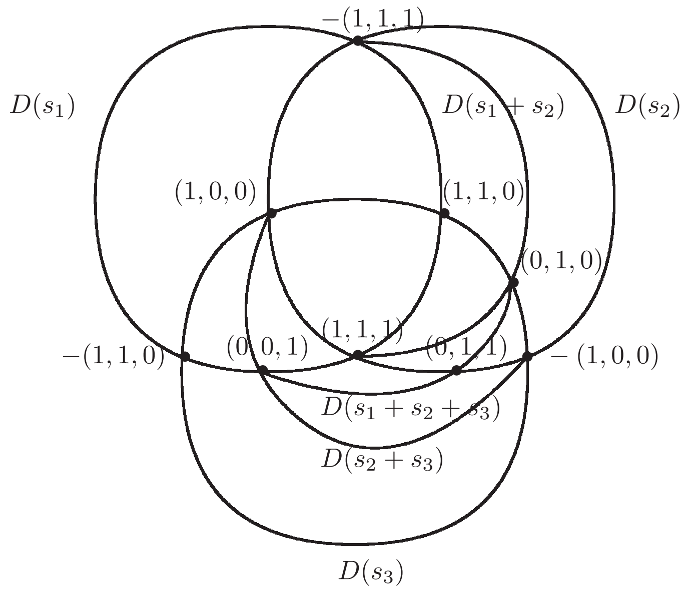
Theorem 3.21.1. (Simplicial complex and domains of semi-invariants theorem) [IOTW]. *Let Q be a Dynkin diagram of type A, D, E. Then:*

(1) *The simplicial complex of the exceptional objects is homeomorphic to the $n-1$ sphere.*
(2) *The image of the $n-2$ skeleton of $T(Q)$ in S^{n-1} is the union:*

$$\bigcup_{\beta \in \Phi_+} D(\beta) \cap S^{n-1}.$$

3.21.2. **Illustration on the example of A_3.** The image of the 1 skeleton of $T(A_3)$ in S^2 is the union:

$$\bigcup_{\beta \in \Phi_+} D(\beta) \cap S^2.$$



- The quiver is $1 \leftarrow 2 \leftarrow 3$ with Euler matrix

$$E = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

- Circles and semicircles, labeled by $D(\beta)$, denote the domains of semiinvariants (with weight β).
- Vertices are labeled by the positive roots Φ_+ and negative projective roots.
- Positive roots can also be viewed as dimension vectors of indecomposable representations by Gabriel's theorem, e.g. $(1,1,0) = \underline{\dim}(P_2)$.
- For example: $Rep((1,1,0))$ has semi-invariants of weights $s_1 = (1,0,0)$ and $s_3 = (0,0,1)$.
- For example: $Rep((1,1,0))$, $Rep((1,1,1))$, $Rep((0,0,1))$,$Rep(-(1,1,0))$, $Rep(-(1,1,1))$ all have semi-invariants of weights $s_1 = (1,0,0)$ since $Es_1 = (1,-1,0)$ making $\alpha^T E s_1 = 0$ iff $\alpha_1 = \alpha_2$.
- The semiinvariant σ_{s_1} on a representation

$$V : V_1 \xleftarrow{f_{12}} V_2 \xleftarrow{f_{23}} V_3$$

is $\sigma_{s_1}(V) = \det f_{12}$. This is only well-defined if coordinates $\phi_i : V_i \xrightarrow{\approx} \alpha_i \Bbbk$ are chosen. If these are changed by $g_i \in GL(\alpha_i, \Bbbk)$ then $\det f_{12}$ becomes

$$\det(g_1 f_{12} g_2^{-1} = (\det g_1)(\det g_2)^{-1} \det f_{12}$$

So, this semiinvariant has weight $(1,-1,0) = Es_1$.

- In the case $\beta = s_1 + s_2 = (1,1,0)$ we have $s_1 \hookrightarrow s_1 + s_2$. So, $D(s_1 + s_2)$ does not contain points α where $\alpha^T E s_1 > 0$. So, $D(s_1 + s_2)$ contains no points inside the $D(s_1)$ circle.
- Furthermore, any integral vector belongs to one of the simplices, and the generic decomposition is given in terms of the vertices of that simplex.

4. Homology of Nilpotent, Torsion-free Groups

4.1. **Homology of groups and Lie algebras.**

4.1.1. **Homology of a group.** The homology of a group G is defined by

$$H_k(G;\mathbb{Z}) := Tor_k^{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z})$$

More explicitly, we need to choose a free $\mathbb{Z}G$ resolution $C_*(G)$ of $\mathbb{Z}$. Then $H_*(G;\mathbb{Z})$ is the homology of the complex $C_*(G) \otimes_{\mathbb{Z}G} \mathbb{Z}$. One standard complex is given by the *bar resolution* where $C_k(G)$ is the free $\mathbb{Z}G$ module generated by G^k and the boundary map $d : C_k(G) \to C_{k-1}(G)$ is given by

$$d(g_1, \cdots, g_k) = g_1(g_2, \cdots, g_k) + \sum (-1)^i (g_1, \cdots, g_i g_{i-1}, \cdots, g_k) + (-1)^k (g_1, \cdots, g_{k-1})$$

This chain complex is infinitely generated. In the case when G is torsion-free, nilpotent, there is a smaller chain complex in which $C_k(G)$ is generated by the set of all k'-element subsets of the nilpotent basis $\mathcal{B}$ of G, see 4.2.7. There are various descriptions of this complex ([CP1], [IO]). One of the objectives of this work is to give a new canonical description of this smaller complex using root systems and semi-invariants in the case when G is a monomial group (including groups of Dynkin type).

4.1.2. **Homology of a group - topological definition.** We recall, but will not use that the homology of the group G is defined topologically by $H_k(G;\mathbb{Z}) = H_k(BG;\mathbb{Z})$ where BG is the classifying space of the discrete group G, also known as the Eilenberg-MacLane space $BG = K(G,1)$.

4.1.3. **Rational homology of a group.** The rational homology of a group G is $H_*(G;\mathbb{Q}) = H_*(G;\mathbb{Z}) \otimes \mathbb{Q}$.

4.1.4. **Homology of a Lie algebra.** The homology of a Lie algebra L, $H_*(L)$, is defined to be the homology of its Koszul complex $C_*(L)$ given by $C_k(L) := \Lambda^k L$ with boundary $d : C_k(L) \to C_{k-1}(L)$ given by

$$d(x_1 \wedge \cdots \wedge x_k) = \sum_{i<j} (-1)^{i+j} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_k$$

4.1.5. **Rational homology of a Lie algebra.** The rational homology of a Lie algebra L is $H_*(L;\mathbb{Q}) = H_*(L) \otimes \mathbb{Q} \cong H_*(L \otimes \mathbb{Q})$.

4.1.6. **Lie algebra L associated to a group G.** To every group G there is an associated graded Lie algebra $L(G) = \bigoplus L_i(G)$ given by $L_i(G) = G_i/G_{i+1}$ with Lie bracket

$$[\ ,\ ] : L_i(G) \otimes L_j(G) \to L_{i+j}(G)$$

given by the commutator: $[gG_{i+1}, hG_{j+1}] := [g,h]G_{i+j+1}$.

4.2. **Torsion-free Nilpotent groups.**

4.2.1. **Definition of Torsion-free, Nilpotent groups.** Let G be a group and $G = G_0 \supset G_1 \supset \cdots \supset G_i \supset G_{i+1} \supset \dots$ the chain of subgroups defined as $G_{i+1} = [G, G_i]$, where $[\ ,\ ]$ denotes the group commutator operation, i.e. $[g,h] = ghg^{-1}h^{-1}$ for all $g, h \in G$. The group G is said to be:

- *Torsion free* if G_i/G_{i+1} is finitely generated free abelian group for all i.
- *Nilpotent* if there exists an s such that $G_s = 0$.

4.2.2. **Theorem of Nomizu.**

Theorem 4.2.1 (Nomizu)**.** [N54] *The rational homology of a torsion-free nilpotent group G is isomorphic to the homology of the Lie algebra $L(G) \otimes \mathbb{Q}$.*

4.2.3. **Remark.** Nomizu [N54] gave the first example of a nilpotent torsion free group whose integral homology is not isomorphic to that of the associated Lie algebra. Dwyer [D] points out that the upper triangular matrix group (the nilpotent group associated to the Dynkin diagram A_n) has p-torsion in its homology for all $p < n$. The rational homology is given by the following theorem of Bott and Kostant.

Theorem 4.2.2. [Bo][Ko] *The rank of the k-th homology of the nilpotent subalgebra of a semisimple Lie algebra spanned by the positive roots is equal to the number of elements of the Weyl group of weight k.*

For example, the rank of $H_k(G(A_n)\mathbb{Q})$ is equal to the number of elements σ of the symmetric group on $n+1$ letters for which there are k pairs $i < j$ with $\sigma(i) > \sigma(j)$.

4.2.4. **Koszul complex.** The Koszul complex gives both the integral and rational homology for any Lie algebra L and in particular for the Lie algebra $L(G)$ and therefore, by Nomizu's theorem, it gives the rational homology of G as well.

4.2.5. **Cenkl and Porter.** In [CP1] they give an algorithm for constructing the integral cochain complex for the integral cohomology of G. (The integral cohomology determines the integral homology.) In [CP1] they construct a nilpotent torsion free group with a prescribed Lie algebra (the reverse of the usual construction).

4.2.6. **A motivation from topology.** Kent Orr proved that all Milnor $\bar{\mu}$-invariants of links are represented by elements of H_3 of the fundamental group of the complement of the link. Igusa and Orr proved that H_3 has no torsion for nilpotent torsion free groups, using a particular complex in which boundaries are given by "pictures".

4.2.7. **Remark.** Each nilpotent torsion free group has a finite basis $\mathcal{B} = \{b_1, \dots, b_m\}$ which can be obtained as the union of preimages of the bases of the free abelian groups G_i/G_{i+1} for $0 \le i \le s$. These elements form a basis in the sense that every element in the group can be expressed uniquely in the form $b_1^{k_1} b_2^{k_2} \cdots b_m^{k_m}$ where $k_1, \cdots, k_m$ are integers.

- Let $\{s_1, \ldots, s_n\}$ be preimages of a set of basis elements of G_0/G_1.
- Then $\{s_1, \ldots, s_n\}$ is a minimal set of generators for the group G, call these elements *simple generators*.
- The number n does not depend on the choice of the basis.

4.2.8. **Igusa-Orr complex and "pictures".** This is a particular $\mathbb{Z}[G]$ resolution of $\mathbb{Z}$, which we will not describe precisely, but the following are some of the important facts about it:

- $C_k(G)$ is freely generated by the sets of k-element subsets of the basis $\mathcal{B}$.
- The boundary $d : C_k(G) \to C_{k-1}(G)$ is recursively defined for each subset of $\mathcal{B}$.
- Boundary is Koszul boundary plus additional terms.
- Boundary for each subset of $\mathcal{B}$ corresponds to a "picture", i.e. a cell decomposition of sphere, satisfying certain necessary and sufficient conditions.
- Left and right commutators in the group give the most efficient "collecting process".
- Different forms of commutators. The standard form of commutator is the left commutator $aba^{-1}b^{-1}$, the original Igusa-Orr pictures use the right commutator $a^{-1}b^{-1}ab$, and for the Cluster-Semiinvarian pictures it is more convenient to use the middle commutator $b^{-1}aba^{-1}$.

4.3. **Monomial groups.**

4.3.1. **Definition of Monomial groups.** These are special torsion free monomial groups, for which Igusa-Orr pictures are related to the Cluster-Semi-invariant pictures. A group G is called *monomial* if it is:

(1) Torsion free,
(2) Nilpotent and
(3) There exists a basis $\mathcal{B} = \{b_1, \ldots, b_m\}$ satisfying:
    a: $[b_i, b_j] \subset \mathcal{B} \cup \mathcal{B}^{-1} \cup [1]$
    b: For any three elements at least two commute
    c: For any $b_i, b_j \in \mathcal{B}$, $[b_i, b_j]$ commutes with both b_i and b_j

4.3.2. **Maximal monomial groups.** These will be monomial groups which are maximal in the following sense (but we need to point out a few facts):

- The number of elements in $\{s_1 \ldots, s_n\}$ is an invariant of the group G.
- Monomial groups with n simple generators, for a fixed n, are partially ordered by epimorphisms of groups, which send simple generators to simple generators (or their inverse).
- Maximal elements with respect to this partial order are called *Maximal monomial groups*.

4.3.3. **Dynkin diagrams and Maximal monomial groups.** To each simply laced Dynkin diagram Q with the set of positive roots Φ_+, we associate a group G defined by:

- Generators: $\{\Theta(\alpha) | \alpha \in \Phi_+\}$ and

- Relations: $\{[\Theta(\alpha), \Theta(\beta)] = \Theta(\alpha+\beta)^{\epsilon(\alpha,\beta)}$ where $\epsilon(\alpha,\beta) = (-1)^{E(\alpha,\beta)}$ if $\alpha+\beta \in \Phi_+$ and 0 otherwise$\}$. Here $E(\alpha,\beta)$ denotes the Euler form.

We have the following facts, questions (and work in progress):

- Simply laced Dynkin diagrams define maximal monomial groups.
- Are all maximal monomial groups given by simply laced Dynkin diagrams (A,D,E) as above?
- Description of the groups (with appropriate modifications of the condition (3)), which will correspond to the other Dynkin diagrams (B,C,F,G).

4.4. **Pictures.**

4.4.1. **An A_3 example of Igusa-Orr picture.**

- Circles and semicircles are labeled by positive roots $\alpha \in \Phi_+$ for the Dynkin diagram A_3. Any positive root α is expressed as a sum of simple roots $\{s_1, s_2, s_3\}$
- For the middle-commutator I-O picture the same labels are used for the elements of the associated nilpotent group G, i.e. any positive root α actually stands for $\Theta(\alpha)$, see 4.3.3.
- A free $\mathbb{Z}G$ resolution of $\mathbb{Z}$ is given by C_k = free module generated by subsets of Φ_+ with k elements, and with boundary given by the pictures.
- The following picture describes the boundary $d(s_1, s_2, s_3)$ as:
- $d(s_1, s_2, s_3) = -(s_1, s_2) + (s_1, s_3) - (s_2, s_3) - \Theta(s_2)\Theta(s_1+s_2)(s_1, s_2) + \Theta(s_1)(s_2, s_3) + \Theta(s_3)(s_1, s_2+s_3) - \Theta(s_3)\Theta(s_2+s_3)(s_2, s_1+s_2+s_3) + \Theta(s_2)(s_1+s_2, s_1+s_2+s_3) + \Theta(s_3)\Theta(s_2+s_3)\Theta(s_1+s_2+s_3)(s_1, s_2)$
- Each term in the sum corresponds to a vertex, and is obtained by "reading" the picture.

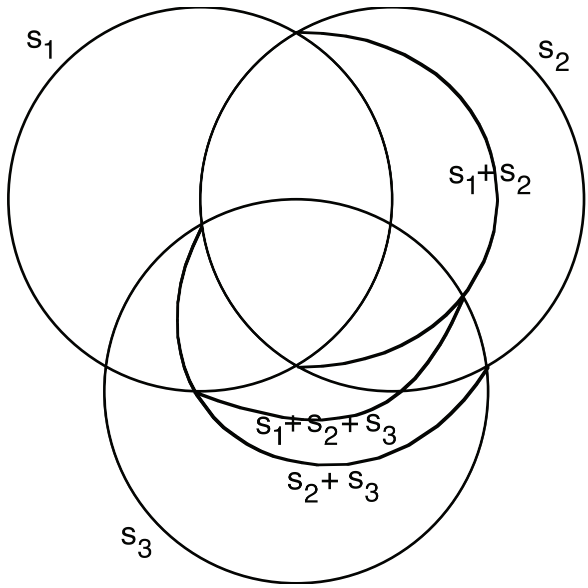
- Circles and semicircles are labeled by positive roots Φ_+ for the Dynkin diagram A_3. Labeling agrees with the same labeling of the same picture by the domains of semiinvariants.

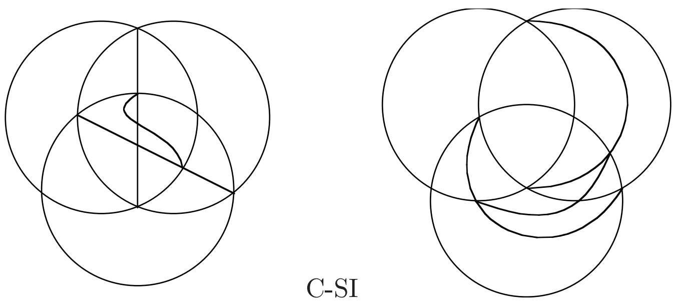
4.4.2. **I-O vs C-SI pictures.** The original, right commutator Igusa-Orr and Cluster-Semiinvariant pictures are given below for the example of Dynkin diagram A_3.

- I-O picture has 12 tri-angles, 1 quadr-angle, 1 bi-angle
- C-SI picture is a simplicial complex, triangulation of a shpere with 14 tri-angles.

4.4.3. **Some facts.**

- Igusa-Orr pictures are Koszul boundary plus higher terms.
- Cluster-Semiinvariant pictures are modified Koszul boundary plus higher terms.
- Cluster-Semiinvariant pictures are more natural - simplicial complex.
- Plan to re-do Igusa-Orr algorithm to construct all Cluster-Semiinvariant pictures.





I-O C-SI

- Cluster-Semiinvariant pictures give the middle commutator Igusa-Orr pictures for the particular subset consisting of the simple generators $\{s_1, \ldots, s_n\}$ of the group G, which actually correspond to the set of simple roots in Φ_+.
- We believe that Cluster-Semiinvariant pictures can be used to construct, in a functorial way, all middle-commutator I-O pictures (for all subsets of Φ_+.

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Gordana Todorov
Northeastern University
Department of Mathematics
360 Huntington Avenue
Boston, MA 02115, USA
todorov@neu.edu

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