FINITE ELEMENT APPROXIMATION OF THE VIBRATION PROBLEM FOR A TIMOSHENKO CURVED ROD

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ABSTRACT. The aim of this paper is to analyze a mixed finite element method for computing the vibration modes of a Timoshenko curved rod with arbitrary geometry. Optimal order error estimates are proved for displacements and rotations of the vibration modes, as well as a double order of convergence for the vibration frequencies. These estimates are essentially independent of the thickness of the rod, which leads to the conclusion that the method is locking free. A numerical test is reported in order to assess the performance of the method.

1. INTRODUCTION

It is very well known that standard finite elements applied to models of thin structures, like beams, rods, plates and shells, are subject to the so-called *locking* phenomenon. This means that they produce very unsatisfactory results when the thickness of the structure is small with respect to the other dimensions of the structure (see for instance [4]). From the point of view of the numerical analysis, this phenomenon usually reveals itself in that the a priori error estimates for these methods depend on the thickness of the structure in such a way that they degenerate when this parameter becomes small. To avoid locking, special methods based on reduced integration or mixed formulations have been devised and are typically used to date (see, for instance, [5]).

Very likely, the first mathematical piece of work dealing with numerical locking and how to avoid it is the paper by Arnold [1], where a thorough analysis for the Timoshenko beam bending model is developed. In that paper, it is proved that locking arises because of the shear terms and a locking-free method based on a mixed formulation is introduced and analyzed. It is also shown that this mixed method is equivalent to use a reduced-order scheme for the integration of the shear terms in the primal formulation.

Subsequently, several methods to avoid locking on different models of circular arches were developed by Kikuchi [12], Loula et al. [14] and Reddy and Volpi

[15]. The analysis of the latter was extended by Arunakirinathar and Reddy in [2] to Timoshenko rods of rather arbitrary geometry. An alternative approach to deal with this same kind of rods was developed and analyzed by Chapelle in [6], where a numerical method based on standard beam finite elements was used to approximate the rod.

All the above references deal only with load problems. The literature devoted to the dynamic analysis of rods is less rich. There exist a few papers introducing finite element methods and assessing their performance by means of numerical experiments (see [10, 13] and references therein). Papers dealing with the numerical analysis of the eigenvalue problems arising from the computation of the vibration modes for thin structures are much less frequent (among them we mention [7, 8], where MITC methods for computing bending vibration modes of plates were analyzed). One reason for this is that the extension of mathematical results from load to vibration problems is not quite straightforward for mixed methods.

In this paper we adapt to the vibration problem the mixed finite element method proposed and analyzed by Arunakirinathar and Reddy in [2] for the load problem for elastic curved rods. With this purpose, we settle the corresponding spectral problem by including the mass terms arising from displacement and rotational inertia in the model, as proposed in [10]. Our assumptions on the rods are slightly weaker than those in [2]. On the one hand we allow for non-constant geometric and physical coefficients varying smoothly along the rod. On the other hand, we do not assume that the Frenet basis associated with the line of cross-section centroids is a set of principal axes. We prove that the resulting method yield optimal order approximation of displacements and rotations of the vibration modes, as well as a double order of convergence for the vibration frequencies. Under mild assumptions, we also prove that the error estimates do not degenerate as the thickness becomes small, which allow us to conclude that the method is locking free.

The outline of the paper is as follows. In Sect. 2, we introduce the basic geometric and physical assumptions to settle the vibration problem for a Timoshenko rod of arbitrary geometry. The resulting spectral problem is shown to be well posed. Its eigenvalues and eigenfunctions are proved to converge to the corresponding ones of the limit problem as the thickness of the rod goes to zero, which corresponds to a Bernoulli-like rod model. The finite element discretization with piecewise polynomials of arbitrary degree is introduced and analyzed in Sect. 3. Optimal orders of convergence are settled for the eigenfunctions, as well as a double order for the eigenvalues and, whence, for the vibration frequencies. All these error estimates are proved to be independent of the thickness of the rod, which allow us to conclude that the method is locking-free. In Sect. 4, we report a numerical test, which allows assessing the performance of the lowest-degree method.

2. The vibration problem for an elastic rod of arbitrary geometry

A curved rod in undeformed reference state is described by means of a smooth three-dimensional curve, the *line of centroids*, which passes through the centroids of cross-sections of the rod. These cross-sections are initially plane and normal to

the line of centroids. The curve is parametrized by its arc length $s \in \boldsymbol{I} := [0, L]$, L being the total length of the curve.

We recall some basic concepts and definitions; for further details see [2], for instance. We use standard notation for Sobolev spaces and norms.

The basis in which the equations are formulated is the *Frenet basis* consisting of $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$, which are the tangential, normal and binormal vectors of the curve, respectively. These vectors change smoothly from point to point and form an orthogonal basis of $\mathbb{R}^3$ at each point.

Let S denote a cross-section of the rod. We denote by (η, ζ) the coordinates in the coordinate system $\{\boldsymbol{n}, \boldsymbol{b}\}$ of the plane containing S (see Fig. 2.1).
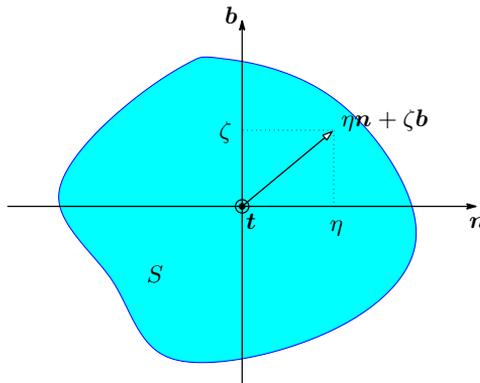
FIG. 2.1. Cross-section. Coordinate system.

The geometric properties of the cross-section are determined by the following parameters (recall that the first moments of area, $\int_S \eta \, d\eta \, d\zeta$ and $\int_S \zeta \, d\eta \, d\zeta$, vanish, because the center of coordinates is the centroid of S):

- the area of S: $A := \int_S d\eta \, d\zeta$;
- the second moments of area with respect to the axis $\boldsymbol{n}$ and $\boldsymbol{b}$: $I_{\boldsymbol{n}} := \int_S \zeta^2 \, d\eta \, d\zeta$ and $I_{\boldsymbol{b}} := \int_S \eta^2 \, d\eta \, d\zeta$, respectively;
- the polar moment of area: $J := \int_S \left(\eta^2 + \zeta^2 \right) \, d\eta \, d\zeta = I_{\boldsymbol{n}} + I_{\boldsymbol{b}}$;
- $I_{\boldsymbol{nb}} := \int_S \eta \zeta \, d\eta \, d\zeta$.

These parameters are not necessarily constant, but they are assumed to vary smoothly along the rod. For a non-degenerate rod, A is bounded above and below far from zero. Consequently, the same happens for the area moments, $I_{\boldsymbol{n}}$, $I_{\boldsymbol{b}}$ and J.

Remark **2.1.** For any planar set S, there exists an orthogonal coordinate system, named the *set of principal axes*, such that $I_{\boldsymbol{nb}}$ vanishes when computed in these coordinates. For particularly symmetric geometries of S, for instance when the cross-section of the rod is a circle or a square, $I_{\boldsymbol{nb}}$ vanishes in any orthogonal coordinate system. However, in general, there is no reason for $\boldsymbol{n}$ and $\boldsymbol{b}$ to be principal

axes, so that I_{nb} does not necessarily vanish. In any case, it is straightforward to prove that the matrix

$$\begin{pmatrix} I_n & -I_{nb} \\ -I_{nb} & I_b \end{pmatrix}$$

is always positive definite.

Vector fields defined on the line of centroids will be always written in the Frenet basis:

$$\boldsymbol{v} = v_1\boldsymbol{t} + v_2\boldsymbol{n} + v_3\boldsymbol{b}, \qquad \text{with} \quad v_1, v_2, v_3 : \boldsymbol{I} \longrightarrow \mathbb{R}.$$

We emphasize that v_1, v_2 and v_3 are not the components of $\boldsymbol{v}$ in a fixed basis of $\mathbb{R}^3$, but in the Frenet basis $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$, which changes from point to point of the curve.

Since $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ are smooth functions of the arc-length parameter s, we have that

$$\boldsymbol{v}' = v_1'\boldsymbol{t} + v_2'\boldsymbol{n} + v_3'\boldsymbol{b} + v_1\boldsymbol{t}' + v_2\boldsymbol{n}' + v_3\boldsymbol{b}'.$$

If we denote

$$\dot{\boldsymbol{v}} := v_1'\boldsymbol{t} + v_2'\boldsymbol{n} + v_3'\boldsymbol{b}, \tag{2.1}$$

then, by using the *Frenet-Serret formulas* (see, for instance, [2]), there holds

$$\boldsymbol{v}' = \dot{\boldsymbol{v}} + \Gamma^{\mathrm{t}}\boldsymbol{v}, \qquad \text{with} \quad \Gamma(s) := \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix},$$

where κ and τ are the curvature and the torsion of the rod, which are smooth functions of s, too. Therefore, $\boldsymbol{v} = v_1\boldsymbol{t} + v_2\boldsymbol{n} + v_3\boldsymbol{b} \in \mathrm{H}^1(\boldsymbol{I})^3$ if and only if $v_i \in \mathrm{L}^2(\boldsymbol{I})$ and $\dot{v}_i \in \mathrm{L}^2(\boldsymbol{I})$, $i = 1, 2, 3$.

Since we will confine our attention to elastic rods clamped at both ends, we proceed as in [2] and consider

$$\boldsymbol{\mathcal{V}} := \left\{ \boldsymbol{v} \in \mathrm{L}^2(\boldsymbol{I})^3 : \ \dot{\boldsymbol{v}} \in \mathrm{L}^2(\boldsymbol{I})^3 \text{ and } \boldsymbol{v}(0) = \boldsymbol{v}(L) = \boldsymbol{0} \right\},$$

endowed with its natural norm

$$\|\boldsymbol{v}\|_1 := \left[\int_0^L \left(|\boldsymbol{v}|^2 + |\dot{\boldsymbol{v}}|^2 \right) ds \right]^{1/2};$$

namely, $\boldsymbol{\mathcal{V}}$ is the space of vector fields defined on the line of centroids such that their components in the Frenet basis are in $\mathrm{H}_0^1(\boldsymbol{I})$.

We will systematically use in what follows the total derivative $\boldsymbol{v}' = \dot{\boldsymbol{v}} + \Gamma^{\mathrm{t}}\boldsymbol{v}$. Since $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ are assumed to be smooth functions, $\|\boldsymbol{v}'\|_0$ is a norm on $\boldsymbol{\mathcal{V}}$ equivalent to $\|\cdot\|_1$ (see [2, Theorem 3.1]). This is the reason why we denote $\|\cdot\|_1$ the norm of $\boldsymbol{\mathcal{V}}$. However, the total derivative $\boldsymbol{v}'$ should be distinguished from the vector $\dot{\boldsymbol{v}}$ of derivatives of the components of $\boldsymbol{v}$ in the Frenet basis, as defined by (2.1).

The kinematic hypotheses of Timoshenko are used for the problem formulation. The deformation of the rod is described by the displacement of the line of centroids, $\boldsymbol{u} \in \mathbb{R}^3$, and the rotation of the cross-sections, $\boldsymbol{\theta} \in \mathbb{R}^3$. The physical properties of the rod are determined by the elastic and the shear moduli E and G, respectively, the shear correcting factors k_1 and k_2, and the volumetric density ρ, all strictly

positive coefficients. These coefficients are not necessarily constant; they are allowed to vary along the rod, but they are also assumed to be smooth functions of the arc-length s.

We consider the problem of computing the free vibration modes of an elastic rod clamped at both ends. The variational formulation of this problem consists in finding non-trivial $(\boldsymbol{u}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{W}} := \boldsymbol{\mathcal{V}} \times \boldsymbol{\mathcal{V}}$ and $\omega > 0$ such that

$$\int_0^L \mathbb{E}\boldsymbol{\theta}' \cdot \boldsymbol{\psi}' \, ds + \int_0^L \mathbb{D}\left(\boldsymbol{u}' - \boldsymbol{\theta} \times \boldsymbol{t}\right) \cdot \left(\boldsymbol{v}' - \boldsymbol{\psi} \times \boldsymbol{t}\right) \, ds$$

$$= \omega^2 \left(\int_0^L \rho A \boldsymbol{u} \cdot \boldsymbol{v} \, ds + \int_0^L \rho \mathbb{J} \boldsymbol{\theta} \cdot \boldsymbol{\psi} \, ds \right) \qquad \forall (\boldsymbol{v}, \boldsymbol{\psi}) \in \boldsymbol{\mathcal{W}}, \quad (2.2)$$

where ω is the vibration frequency and $\boldsymbol{u}$ and $\boldsymbol{\theta}$ are the amplitudes of the displacements and the rotations, respectively (see [10]). The coefficients $\mathbb{D}$, $\mathbb{E}$ and $\mathbb{J}$ are 3×3 matrices, which in the Frenet basis are written as follows:

$$\mathbb{D} := \begin{pmatrix} EA & 0 & 0 \\ 0 & k_1 GA & 0 \\ 0 & 0 & k_2 GA \end{pmatrix},$$

$$\mathbb{E} := \begin{pmatrix} GJ & 0 & 0 \\ 0 & EI_{\boldsymbol{n}} & -EI_{\boldsymbol{nb}} \\ 0 & -EI_{\boldsymbol{nb}} & EI_{\boldsymbol{b}} \end{pmatrix} \qquad \text{and} \qquad \mathbb{J} := \begin{pmatrix} J & 0 & 0 \\ 0 & I_{\boldsymbol{n}} & -I_{\boldsymbol{nb}} \\ 0 & -I_{\boldsymbol{nb}} & I_{\boldsymbol{b}} \end{pmatrix}.$$

In [10], as in most references ([2, 6]), the Frenet basis is assumed to be a set of principal axes, so that $I_{\boldsymbol{nb}} = 0$ and the three matrices above are diagonal. We do not make this assumption in this paper.

Remark **2.2.** The vibration problem above can be formally obtained from the three-dimensional linear elasticity equations as follows: According to the Timoshenko hypotheses, the admissible displacements at each point $\eta \boldsymbol{n} + \zeta \boldsymbol{b} \in S$ (see Fig. 2.1) are of the form $\boldsymbol{u} + \boldsymbol{\theta} \times (\eta \boldsymbol{n} + \zeta \boldsymbol{b})$, with $\boldsymbol{u}$, $\boldsymbol{\theta}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ being functions of the arc-length coordinate s. Test and trial displacements of this form are taken in the variational formulation of the linear elasticity equations for the vibration problem of the three-dimensional rod. By integrating over the cross-sections and multiplying the shear terms by correcting factors k_1 and k_2, one arrives at problem (2.2).

It is well known that standard finite element methods applied to equations like (2.2) are subject to *numerical locking*: they lead to unacceptably poor results for very thin structures, unless the mesh-size is excessively small (see, for instance, [1]). This phenomenon is due to the different scales, with respect to the thickness of the rod, of the two terms on the left-hand side of this equation. An adequate framework for the mathematical analysis of locking is obtained by rescaling the equations in order to obtain a family of problems with a well-posed limit as the thickness becomes infinitely small.

With this purpose, we introduce the following non-dimensional parameter, characteristic of the thickness of the rod:

$$d^2 := \frac{1}{L} \int_0^L \frac{J}{AL^2}\, ds.$$

By defining

$$\lambda := \frac{\omega^2 \rho}{d^2}, \qquad \widehat{\mathbb{D}} := \frac{1}{d^2}\mathbb{D}, \qquad \widehat{\mathbb{E}} := \frac{1}{d^4}\mathbb{E}, \qquad \widehat{\mathbb{J}} := \frac{1}{d^4}\mathbb{J} \qquad \text{and} \qquad \widehat{A} := \frac{A}{d^2},$$

Problem (2.2) can be equivalently written as follows:

PROBLEM **P**: *Find non-trivial* $(\boldsymbol{u}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{W}}$ *and* $\lambda \in \mathbb{R}$ *such that*

$$\int_0^L \widehat{\mathbb{E}}\boldsymbol{\theta}' \cdot \boldsymbol{\psi}'\, ds + \frac{1}{d^2} \int_0^L \widehat{\mathbb{D}}\,(\boldsymbol{u}' - \boldsymbol{\theta} \times \boldsymbol{t}) \cdot (\boldsymbol{v}' - \boldsymbol{\psi} \times \boldsymbol{t})\, ds$$

$$= \lambda \left(\int_0^L \widehat{A}\boldsymbol{u} \cdot \boldsymbol{v}\, ds + d^2 \int_0^L \widehat{\mathbb{J}}\boldsymbol{\theta} \cdot \boldsymbol{\psi}\, ds \right) \qquad \forall\, (\boldsymbol{v}, \boldsymbol{\psi}) \in \boldsymbol{\mathcal{W}}.$$

The values of interest of d are obviously bounded above, so we restrict our attention to $d \in (0, d_{\max}]$. The coefficients of the matrices $\widehat{\mathbb{D}}$, $\widehat{\mathbb{E}}$ and $\widehat{\mathbb{J}}$, as well as $\widehat{A}$, are assumed to be functions of s which do not vary with d. This corresponds to considering a family of problems where the size of the cross-sections at all point of the line of centroids are uniformly scaled by d, while their shapes as well as the geometry of the curve and the material properties remain fixed.

Remark **2.3.** Matrices $\widehat{\mathbb{D}}$, $\widehat{\mathbb{E}}$ and $\widehat{\mathbb{J}}$ are positive definite for all $s \in \boldsymbol{I}$, the last two because of Remark 2.1. Moreover, since all the coefficients are continuous functions of s, the eigenvalues of each of these matrices are uniformly bounded below away from zero for all $s \in \boldsymbol{I}$.

Remark **2.4.** The eigenvalues λ of Problem **P** are strictly positive, because of the symmetry and the positiveness of the bilinear forms on its left and right-hand sides. The positiveness of the latter is a straightforward consequence of Remark 2.3, whereas that of the former follows from the ellipticity of this bilinear form in $\boldsymbol{\mathcal{W}}$. This can be proved by using Remark 2.3 again and proceeding as in the proof of [2, Lemma 3.4 (a)], where the same result appears for particular constant coefficients (see also [6, Proposition 1]).

We introduce the scaled shear stress $\boldsymbol{\gamma} := \frac{1}{d^2}\widehat{\mathbb{D}}\,(\boldsymbol{u}' - \boldsymbol{\theta} \times \boldsymbol{t})$ to rewrite Problem **P** as follows:

$$\left(\widehat{\mathbb{E}}\boldsymbol{\theta}', \boldsymbol{\psi}' \right) + (\boldsymbol{\gamma}, \boldsymbol{v}' - \boldsymbol{\psi} \times \boldsymbol{t}) = \lambda \left[\left(\widehat{A}\boldsymbol{u}, \boldsymbol{v} \right) + d^2 \left(\widehat{\mathbb{J}}\boldsymbol{\theta}, \boldsymbol{\psi} \right) \right] \qquad \forall\, (\boldsymbol{v}, \boldsymbol{\psi}) \in \boldsymbol{\mathcal{W}},$$

$$\boldsymbol{\gamma} = \frac{1}{d^2}\widehat{\mathbb{D}}\,(\boldsymbol{u}' - \boldsymbol{\theta} \times \boldsymbol{t}),$$

where $(\cdot, \cdot)$ denotes the $\mathrm{L}^2(\boldsymbol{I})^3$ inner product.

To analyze the approximation of this problem, we introduce the operator

$$T: \ \mathrm{L}^2(\boldsymbol{I})^3 \times \mathrm{L}^2(\boldsymbol{I})^3 \longrightarrow \mathrm{L}^2(\boldsymbol{I})^3 \times \mathrm{L}^2(\boldsymbol{I})^3,$$

defined by $T(\boldsymbol{f}, \boldsymbol{\phi}) := (\boldsymbol{u}, \boldsymbol{\theta})$, where $(\boldsymbol{u}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{W}}$ is the solution of the associated load problem:

$$\left(\widehat{\mathbb{E}}\boldsymbol{\theta}', \boldsymbol{\psi}'\right) + (\boldsymbol{\gamma}, \boldsymbol{v}' - \boldsymbol{\psi} \times \boldsymbol{t}) = \left(\widehat{A}\boldsymbol{f}, \boldsymbol{v}\right) + d^2 \left(\widehat{\mathbb{J}}\boldsymbol{\phi}, \boldsymbol{\psi}\right) \quad \forall (\boldsymbol{v}, \boldsymbol{\psi}) \in \boldsymbol{\mathcal{W}}, \quad (2.3)$$

$$\boldsymbol{\gamma} = \frac{1}{d^2}\widehat{\mathbb{D}}\left(\boldsymbol{u}' - \boldsymbol{\theta} \times \boldsymbol{t}\right). \tag{2.4}$$

The existence and uniqueness of the solution of this problem was analyzed in [2, Theorem 3.3] in case of particular constant coefficients and in [6, Proposition 2] for another equivalent formulation. Taking into account that (2.4) can be equivalently written as follows:

$$(\boldsymbol{u}' - \boldsymbol{\theta} \times \boldsymbol{t}, \boldsymbol{q}) - d^2 \left(\widehat{\mathbb{D}}^{-1}\boldsymbol{\gamma}, \boldsymbol{q}\right) = 0 \qquad \forall \boldsymbol{q} \in \boldsymbol{\mathcal{Q}} := \mathrm{L}^2(\boldsymbol{I})^3,$$

we note that the load problem falls in the framework of the mixed formulations considered in [5]. In this reference, the results from [1] are extended to cover this kind of problems. In particular, according to [5, Theorem II.1.2], to prove the well posedness it is enough to verify the classical properties of mixed problems:

i) *ellipticity in the kernel*: $\exists \alpha > 0$ such that

$$\left(\widehat{\mathbb{E}}\boldsymbol{\psi}', \boldsymbol{\psi}'\right) \geq \alpha \left(\|\boldsymbol{v}\|_1^2 + \|\boldsymbol{\psi}\|_1^2\right) \qquad \forall (\boldsymbol{v}, \boldsymbol{\psi}) \in \boldsymbol{\mathcal{W}}_0,$$

where $\boldsymbol{\mathcal{W}}_0 := \{(\boldsymbol{v}, \boldsymbol{\psi}) \in \boldsymbol{\mathcal{W}} : \boldsymbol{v}' - \boldsymbol{\psi} \times \boldsymbol{t} = 0 \text{ in } \boldsymbol{I}\}$;

ii) *inf-sup condition*: $\exists \beta > 0$ such that

$$\sup_{(\boldsymbol{0},\boldsymbol{0}) \neq (\boldsymbol{v},\boldsymbol{\psi}) \in \boldsymbol{\mathcal{W}}} \frac{(\boldsymbol{q}, \boldsymbol{v}' - \boldsymbol{\psi} \times \boldsymbol{t})}{\|\boldsymbol{v}\|_1 + \|\boldsymbol{\psi}\|_1} \geq \beta \|\boldsymbol{q}\|_0 \qquad \forall \boldsymbol{q} \in \boldsymbol{\mathcal{Q}}.$$

Property (i) has been proved in [2, Lemma 3.6] for $\widehat{\mathbb{E}}$ being the identity matrix. The extension to $\widehat{\mathbb{E}}$ positive definite uniformly in s is quite straightforward. Property (ii) has been proved in [2, Lemma 3.7]. An alternative simpler proof of an equivalent inf-sup condition appears in [6, Proposition 2].

Therefore, according to [5, Theorem II.1.2], problem (2.3)–(2.4) has a unique solution $(\boldsymbol{u}, \boldsymbol{\theta}, \boldsymbol{\gamma}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$ and this solution satisfies

$$\|\boldsymbol{u}\|_1 + \|\boldsymbol{\theta}\|_1 + \|\boldsymbol{\gamma}\|_0 \leq C \left(\|\boldsymbol{f}\|_0 + d^2 \|\boldsymbol{\phi}\|_0\right).$$

Here and thereafter, C denotes a strictly positive constant, not necessarily the same at each occurrence, but always independent of d and of the mesh-size h, which will be introduced in the next section.

Because of the estimate above and the compact embedding $\mathrm{H}^1(\boldsymbol{I}) \hookrightarrow \mathrm{L}^2(\boldsymbol{I})$, the operator T is compact. Moreover, by substituting (2.4) into (2.3), from the symmetry of the resulting bilinear forms, it is immediate to show that T is self-adjoint with respect to the 'weighted' $\mathrm{L}^2(\boldsymbol{I})^3 \times \mathrm{L}^2(\boldsymbol{I})^3$ inner product in the right-hand side of (2.3). Therefore, apart of $\mu = 0$, the spectrum of T consists of a sequence of finite-multiplicity real eigenvalues converging to zero, all with ascent 1.

Note that λ is a non-zero eigenvalue of Problem **P** if and only if $\mu := 1/\lambda$ is a non-zero eigenvalue of T, with the same multiplicity and corresponding eigenfunctions. Recall that these eigenvalues are strictly positive (cf. Remark 2.4).

Next, we define T_0 by means of the limit problem of (2.3)–(2.4) as $d \to 0$:

$$T_0 : \ \mathrm{L}^2(\boldsymbol{I})^3 \times \mathrm{L}^2(\boldsymbol{I})^3 \longrightarrow \mathrm{L}^2(\boldsymbol{I})^3 \times \mathrm{L}^2(\boldsymbol{I})^3,$$

where $T_0\left(\boldsymbol{f}, \boldsymbol{\phi}\right) := (\boldsymbol{u}_0, \boldsymbol{\theta}_0) \in \boldsymbol{\mathcal{W}}$ is such that there exists $\boldsymbol{\gamma}_0 \in \boldsymbol{\mathcal{Q}}$ satisfying:

$$\left(\widehat{\mathbb{E}}\boldsymbol{\theta}_0', \boldsymbol{\psi}'\right) + (\boldsymbol{\gamma}_0, \boldsymbol{v}' - \boldsymbol{\psi} \times \boldsymbol{t}) = \left(\widehat{A}\boldsymbol{f}, \boldsymbol{v}\right) \qquad \forall\, (\boldsymbol{v}, \boldsymbol{\psi}) \in \boldsymbol{\mathcal{W}},$$
$$\boldsymbol{u}_0' - \boldsymbol{\theta}_0 \times \boldsymbol{t} = \boldsymbol{0}.$$

The above mentioned existence and uniqueness results from [2, Theorem 3.3] and [6, Proposition 2] covers this problem as well, in case of constant coefficients. As stated above, the proofs can be readily extended to our case.

It is proved in [9] that T_t converge in norm to T_0. The next theorem follows from this fact and classical results from spectral perturbation theory (see [11]):

Lemma 2.1. *Let $\mu_0 > 0$ be an eigenvalue of T_0 of multiplicity m. Let D be any disc in the complex plane centered at μ_0 and containing no other element of the spectrum of T_0. Then, for d small enough, D contains exactly m eigenvalues of T (repeated according to their respective multiplicities). Consequently, each eigenvalue $\mu_0 > 0$ of T_0 is a limit of eigenvalues μ of T, as d goes to zero.*

Moreover, for any compact subset K of the complex plane not intersecting the spectrum of T_0, there exists $d_K > 0$ such that for all $d < d_K$, K does not intersect the spectrum of T, either.

3. Finite elements discretization

Two different finite element discretizations of the load problem for Timoshenko curved rods have been analyzed in [2] and [6]. The two methods differ in the variables being discretized: the components of vector fields $\boldsymbol{v}$ in the Frenet basis, v_1, v_2 and v_3, are discretized by piecewise polynomial continuous functions in [2]; instead, in [6], the discretized variable is the vector field $\boldsymbol{v} = v_1\boldsymbol{t} + v_2\boldsymbol{n} + v_3\boldsymbol{b}$. We follow the approach in [2].

Consider a family $\{\mathcal{T}_h\}$ of partitions of the interval $\boldsymbol{I}$, $\mathcal{T}_h : \ 0 = s_0 < s_1 < \cdots < s_n = L$, with mesh-size $h := \max_{j=1,\ldots,n}(s_j - s_{j-1})$. We define the following finite element subspaces of $\boldsymbol{\mathcal{V}}$ and $\boldsymbol{\mathcal{Q}}$, respectively:

$$\boldsymbol{\mathcal{V}}_h := \left\{\boldsymbol{v} \in \boldsymbol{\mathcal{V}} : \ v_i|_{[s_{j-1}, s_j]} \in \mathcal{P}_r, \ j = 1, \ldots, n, \ i = 1, 2, 3\right\},$$
$$\boldsymbol{\mathcal{Q}}_h := \left\{\boldsymbol{q} \in \boldsymbol{\mathcal{Q}} : \ q_i|_{[s_{j-1}, s_j]} \in \mathcal{P}_{r-1}, \ j = 1, \ldots, n, \ i = 1, 2, 3\right\},$$

where v_i, $i = 1, 2, 3$, are the components of $\boldsymbol{v}$ in the Frenet basis, $\mathcal{P}_k$ are the spaces of polynomials of degree lower or equal to k, and $r \geq 1$.

Let $\boldsymbol{\mathcal{W}}_h := \boldsymbol{\mathcal{V}}_h \times \boldsymbol{\mathcal{V}}_h$. The following is the discrete vibration problem in mixed form:

PROBLEM $\mathbf{P_h}$: *Find non-trivial* $(\boldsymbol{u}_h, \boldsymbol{\theta}_h, \boldsymbol{\gamma}_h) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h$ *and* $\lambda_h \in \mathbb{R}$ *such that:*

$$\left(\widehat{\mathbb{E}} \boldsymbol{\theta}'_h, \boldsymbol{\psi}'_h \right) + (\boldsymbol{\gamma}_h, \boldsymbol{v}'_h - \boldsymbol{\psi}_h \times \boldsymbol{t}) = \lambda_h \left[\left(\widehat{A} \boldsymbol{u}_h, \boldsymbol{v}_h \right) + d^2 \left(\widehat{\mathbb{J}} \boldsymbol{\theta}_h, \boldsymbol{\psi}_h \right) \right]$$
$$\forall (\boldsymbol{v}_h, \boldsymbol{\psi}_h) \in \boldsymbol{\mathcal{W}}_h,$$

$$(\boldsymbol{u}'_h - \boldsymbol{\theta}_h \times \boldsymbol{t}, \boldsymbol{q}_h) - d^2 \left(\widehat{\mathbb{D}}^{-1} \boldsymbol{\gamma}_h, \boldsymbol{q}_h \right) = 0 \qquad \forall \boldsymbol{q}_h \in \boldsymbol{\mathcal{Q}}_h.$$

In the same manner as in the continuous case, we introduce the operator

$$T_h : \ \mathrm{L}^2(\boldsymbol{I})^3 \times \mathrm{L}^2(\boldsymbol{I})^3 \longrightarrow \mathrm{L}^2(\boldsymbol{I})^3 \times \mathrm{L}^2(\boldsymbol{I})^3,$$

defined by $T_h(\boldsymbol{f}, \boldsymbol{\phi}) := (\boldsymbol{u}_h, \boldsymbol{\theta}_h)$, where $(\boldsymbol{u}_h, \boldsymbol{\theta}_h, \boldsymbol{\gamma}_h) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h$ is the solution of the associated discrete load problem:

$$\left(\widehat{\mathbb{E}} \boldsymbol{\theta}'_h, \boldsymbol{\psi}'_h \right) + (\boldsymbol{\gamma}_h, \boldsymbol{v}'_h - \boldsymbol{\theta}_h \times \boldsymbol{t}) = \left(\widehat{A} \boldsymbol{f}, \boldsymbol{v}_h \right) + d^2 \left(\widehat{\mathbb{J}} \boldsymbol{\phi}, \boldsymbol{\psi}_h \right)$$
$$\forall (\boldsymbol{v}_h, \boldsymbol{\psi}_h) \in \boldsymbol{\mathcal{W}}_h, \quad (3.1)$$

$$(\boldsymbol{u}'_h - \boldsymbol{\theta}_h \times \boldsymbol{t}, \boldsymbol{q}_h) - d^2 \left(\widehat{\mathbb{D}}^{-1} \boldsymbol{\gamma}_h, \boldsymbol{q}_h \right) = 0 \qquad \forall \boldsymbol{q}_h \in \boldsymbol{\mathcal{Q}}_h. \quad (3.2)$$

Problem (3.1)–(3.2) falls in the framework of the discrete mixed formulations considered in [5, Section II.2.4]. In order to apply Proposition II.2.11 from this reference to conclude well-posedness of this discrete problem and error estimates, it is enough to verify the following classical properties, for h small enough:

i) *ellipticity in the discrete kernel*: $\exists \alpha_* > 0$, independent of h, such that

$$\left(\widehat{\mathbb{E}} \boldsymbol{\psi}'_h, \boldsymbol{\psi}'_h \right) \geq \alpha_* \left(\|\boldsymbol{v}_h\|_1^2 + \|\boldsymbol{\psi}_h\|_1^2 \right) \qquad \forall (\boldsymbol{v}_h, \boldsymbol{\psi}_h) \in \boldsymbol{\mathcal{W}}_{0h},$$

where $\boldsymbol{\mathcal{W}}_{0h} := \{(\boldsymbol{v}_h, \boldsymbol{\psi}_h) \in \boldsymbol{\mathcal{W}}_h : \ (\boldsymbol{q}_h, \boldsymbol{v}'_h - \boldsymbol{\psi}_h \times \boldsymbol{t}) = 0 \quad \forall \boldsymbol{q}_h \in \boldsymbol{\mathcal{Q}}_h\}$;

ii) *discrete inf-sup condition*: $\exists \beta_* > 0$, independent of h, such that

$$\sup_{(\mathbf{0},\mathbf{0}) \neq (\boldsymbol{v}_h, \boldsymbol{\psi}_h) \in \boldsymbol{\mathcal{W}}_h} \frac{(\boldsymbol{q}_h, \boldsymbol{v}'_h - \boldsymbol{\psi}_h \times \boldsymbol{t})}{\|\boldsymbol{v}_h\|_1 + \|\boldsymbol{\psi}_h\|_1} \geq \beta_* \|\boldsymbol{q}_h\|_0 \qquad \forall \boldsymbol{q}_h \in \boldsymbol{\mathcal{Q}}_h.$$

Property (i) has been proved in [2, Lemma 4.2] for $\widehat{\mathbb{E}}$ being the identity matrix and $h > 0$ sufficiently small. The extension to $\widehat{\mathbb{E}}$ positive definite uniformly in s is quite straightforward. Property (ii) has been proved in [2, Lemma 4.3]. An alternative simpler proof of this condition can be found in [9, Lemma 6.2].

On the other hand, (3.4) is obtained by adapting to our case the duality argument used to prove [6, Theorem 2]. Therefore, the following theorem follows:

Theorem 3.1. *For sufficiently small* $h > 0$, *problem* (3.1)–(3.2) *has a unique solution* $(\boldsymbol{u}_h, \boldsymbol{\theta}_h, \boldsymbol{\gamma}_h) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h$. *This solution satisfies*

$$\|\boldsymbol{u}_h\|_1 + \|\boldsymbol{\theta}_h\|_1 + \|\boldsymbol{\gamma}_h\|_0 \leq C \left(\|\boldsymbol{f}\|_0 + d^2 \|\boldsymbol{\phi}\|_0 \right),$$

where $C > 0$ *is independent of* h *and* d.

Let $(\boldsymbol{u}, \boldsymbol{\theta}, \boldsymbol{\gamma}) \in \mathcal{W} \times \mathcal{Q}$ *be the solution of* (2.3)–(2.4). *If* $\boldsymbol{f}, \boldsymbol{\phi} \in H^{k-1}(\boldsymbol{I})^3$, $1 \leq k \leq r$, *then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 \leq Ch^k \left(\|\boldsymbol{f}\|_{k-1} + d^2 \|\boldsymbol{\phi}\|_{k-1} \right), \quad (3.3)$$

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_0 + \|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_0 \leq Ch^{k+1} \left(\|\boldsymbol{f}\|_{k-1} + d^2 \|\boldsymbol{\phi}\|_{k-1} \right), \quad (3.4)$$

with $C > 0$ *independent of* h *and* d.

By adding (3.1) and (3.2), from the symmetry of the resulting bilinear forms, it is immediate to show that T_h is self-adjoint with respect to the 'weighted' $\mathrm{L}^2(\boldsymbol{I})^3 \times \mathrm{L}^2(\boldsymbol{I})^3$ inner product in the right-hand side of (3.1). Therefore, apart of $\mu_h = 0$, the spectrum of T_h consists of a finite number of finite-multiplicity real eigenvalues with ascent 1.

Once more the spectrum of the operator T_h is related with the eigenvalues of the spectral problem $\mathbf{P_h}$: λ_h is a non-zero eigenvalue of this problem if and only if $\mu_h := 1/\lambda_h$ is a non-zero eigenvalue of T_h, with the same multiplicity and corresponding eigenfunctions. It is simple to prove that these eigenvalues are strictly positive. Moreover, the eigenvalues cannot vanish. In fact, according to the expression above, since $\widehat{\mathbb{E}}$ and $\widehat{\mathbb{D}}$ are positive definite (see Remark 2.3), $\lambda_h = 0$ implies $\boldsymbol{\gamma}_h = 0$. Then, the second equation of Problem $\mathbf{P_h}$ implies that $(\boldsymbol{u}_h, \boldsymbol{\theta}_h) \in \mathcal{W}_{0h}$ and, hence, $\boldsymbol{u}_h$ and $\boldsymbol{\theta}_h$ vanish because of property (i).

Our aim is to use the spectral theory for compact operators (see [3], for instance) to prove convergence of the eigenvalues and eigenfunctions of T_h towards those of T. However, some further considerations will be needed to show that the error estimates do not deteriorate as d becomes small. With this purpose, we will use the following result:

$$\|(T - T_h)(\boldsymbol{f}, \boldsymbol{\phi})\|_1 \leq Ch \left(\|\boldsymbol{f}\|_0 + d^2 \|\boldsymbol{\phi}\|_0 \right),$$

which follows from (3.3) with $k = 1$. As a consequence of this estimate, T_h converges in norm to T as h goes to zero. Hence, standard results of spectral approximation (see for instance [11]) show that if μ is an eigenvalue of T with multiplicity m, then exactly m eigenvalues $\mu_h^{(1)}, \ldots, \mu_h^{(m)}$ of T_h (repeated according to their respective multiplicities) converge to μ.

The estimate above can be improved when the source term is an eigenfunction $(\boldsymbol{u}, \boldsymbol{\theta})$ of T. Indeed, in such a case, for all $k \geq 2$ and d sufficiently small,

$$\|\boldsymbol{u}\|_k + \|\boldsymbol{\theta}\|_k + \|\boldsymbol{\gamma}\|_{k-1} \leq C \left(\|\boldsymbol{u}\|_0 + d^2 \|\boldsymbol{\theta}\|_0 \right),$$

with C depending on k and on the eigenvalue of T associated with $(\boldsymbol{u}, \boldsymbol{\theta})$. Note that in principle the constant C should depend also on d, because the eigenvalue does it. However, according to Lemma 2.1, for d sufficiently small we can choose C independent of d. Hence, from (3.3)–(3.4) with $k = r$, we obtain:

$$\|(T - T_h)(\boldsymbol{u}, \boldsymbol{\theta})\|_1 \leq Ch^r \|(\boldsymbol{u}, \boldsymbol{\theta})\|_1, \quad (3.5)$$

$$\|(T - T_h)(\boldsymbol{u}, \boldsymbol{\theta})\|_0 \leq Ch^{r+1} \|(\boldsymbol{u}, \boldsymbol{\theta})\|_0. \quad (3.6)$$

We remind the definition of the *gap* or symmetric distance $\widehat{\delta}_k$ between closed subspaces $\boldsymbol{\mathcal{Y}}$ and $\boldsymbol{\mathcal{Z}}$ of $\boldsymbol{\mathcal{W}}$ in norm $\|\cdot\|_k$, $k = 0, 1$:

$$\widehat{\delta}_k(\boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}) := \max\left\{ \delta_k(\boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}), \delta_k(\boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{Y}}) \right\},$$

with

$$\delta_k(\boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}) := \sup_{\substack{(\boldsymbol{v}, \boldsymbol{\psi}) \in \boldsymbol{\mathcal{Y}} \\ \|(\boldsymbol{v}, \boldsymbol{\psi})\|_k = 1}} \left[\inf_{(\widehat{\boldsymbol{v}}, \widehat{\boldsymbol{\psi}}) \in \boldsymbol{\mathcal{Z}}} \left\| (\boldsymbol{v} - \widehat{\boldsymbol{v}}, \boldsymbol{\psi} - \widehat{\boldsymbol{\psi}}) \right\|_k \right].$$

For the sake of simplicity we state our results for eigenvalues of T converging to a simple eigenvalue of T_0 as $d \to 0$. The following theorem yields d-independent error estimates for the approximate eigenvalues and eigenfunctions. Its proof is a consequence of (3.5)–(3.6), [3, Theorem 7.1 and 7.2] and Lemma 2.1.

Theorem 3.2. *Let μ be an eigenvalue of T converging to a simple eigenvalue μ_0 of T_0 as d tends to zero, Let μ_h be the eigenvalue of T_h that converges to μ as h tends to zero. Let $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{E}}_h$ be the corresponding eigenspaces. Then, for d and h small enough,*

$$\begin{aligned} \widehat{\delta}_1(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}_h) &\leq Ch^r, \\ \widehat{\delta}_0(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}_h) &\leq Ch^{r+1}, \end{aligned}$$

with $C > 0$ independent of d and h.

This theorem yields optimal order error estimates for the approximate eigenfunctions in norms $\|\cdot\|_1$ and $\|\cdot\|_0$. An optimal double order holds for the approximate eigenvalues. In fact, the following theorem has been proved in [9, Theorem 3.4] by adapting to our problem a standard argument for variationally posed eigenvalue problems (see [3, Lemma 9.1], for instance).

Theorem 3.3. *Let $\lambda = \frac{1}{\mu}$ and $\lambda_h = \frac{1}{\mu_h}$, with μ and μ_h as in Theorem 3.2. Then, for d and h small enough,*

$$|\lambda - \lambda_h| \leq Ch^{2r},$$

with $C > 0$ independent of d and h.

4. Numerical results

We report in this section the results of a numerical test computed with a MATLAB code implementing the finite element method described above. We have used the lowest possible order: $r = 1$; namely, piecewise linear continuous elements for the displacements $\boldsymbol{u}_h$ and the rotations $\boldsymbol{\theta}_h$, and piecewise constant discontinuous elements for the shear stresses $\boldsymbol{\gamma}_h$.

We have computed the vibration modes with lowest frequencies $\omega^h := \sqrt{\lambda_h}$ for a helical rod. We have considered a helix with five turns, clamped at both ends. The equation of the helix centroids line parametrized by its arc-length is as follows:

$$\boldsymbol{r}(s) = \left(A \cos\frac{s}{n}, A \sin\frac{s}{n}, C\frac{s}{n} \right), \qquad \text{with} \quad n = \sqrt{A^2 + C^2}; \qquad (4.1)$$

the curvature is $\kappa = A/n^2$, the torsion $\tau = C/n^2$, and the length of the eight-turns helix is $L = 5 \times 2\pi n$. We have taken $A = 100\,\text{cm}$, $C = 25/\pi\,\text{cm}$ and a square of

side-length $b = 20$ cm as the cross section of the rod. Thus, the thickness parameter is in this case $d = 0.0026$. Figure 4.1 shows the undeformed helix.
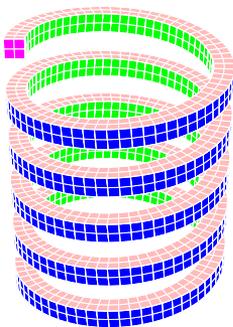
FIG. 4.1. Undeformed helical rod.

We have computed the lowest vibration frequencies $\omega_1^h < \omega_2^h < \omega_3^h < \cdots$ by using uniform meshes of N elements, with different values of N. We have used the following physical parameters, which correspond to steel:

- elastic moduli: $E = 2.1 \times 10^6$ kgf/cm^2 (1 kgf $= 980$ kg cm/s^2);
- Poisson coefficient: $\nu = 0.3$;
- density: $\rho = 7.85 \times 10^{-3}$ kg/cm^3;
- correction factors: $k_1 = k_2 = 1$.

Since no analytical solution is available for this rod, we have estimated the order of convergence by means of a least squares fitting. Table 4.1 shows the lowest vibration frequencies computed on successively refined meshes. It also includes the computed orders of convergence and extrapolated 'exact' vibration frequencies ω_{ex}.

TABLE 4.1. Angular vibration frequencies of a helical rod.

Mode	$N = 320$	$N = 640$	$N = 1280$	$N = 2560$	order	ω_{ex}
ω_1^h	25.3542	25.3437	25.3411	25.3404	2.00	25.3402
ω_2^h	28.9205	28.9120	28.9097	28.9091	1.90	28.9089
ω_3^h	34.7945	34.6718	34.6406	34.6328	1.98	34.6301

To help identifying the different modes, we report three-dimensional plots of the deformed rods. With this purpose, we have used MODULEF to create an auxiliary hexahedral mesh of the actual three-dimensional rod and the displacements at each node of this auxiliary mesh have been computed from $\boldsymbol{u}_h$ and $\boldsymbol{\theta}_h$ as described in Remark 2.2. The resulting deformed rods have been plotted with MODULEF, too.

Figure 4.2 show the lowest-frequency vibration modes. The first one is a typical spring mode, the second one is an extensional mode, and the third one is a kind of 'phone rope' vibration mode.
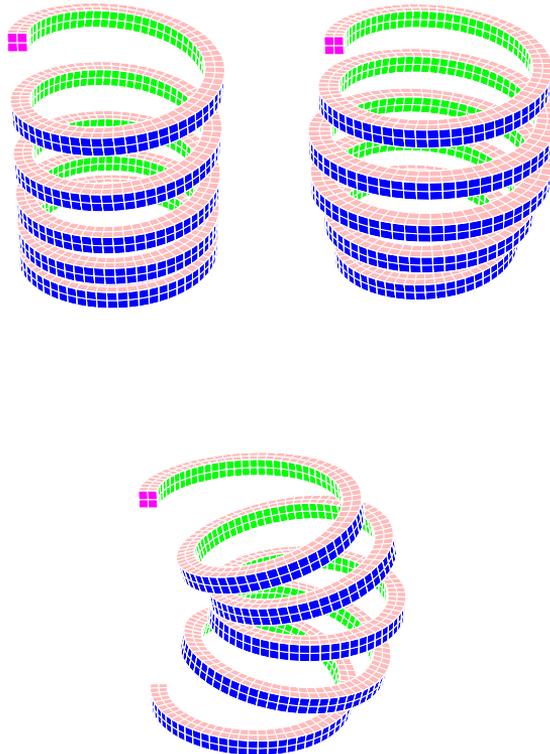
FIG. 4.2. Helical rod. Vibration mode of frequency ω_1 (left), ω_2 (center) and ω_3 (right).

References

[1] Arnold, D.N., *Discretization by finite elements of a model parameter dependent problem*. Numer. Math., 37 (1981) 405–421. 1, 2, 2

[2] Arunakirinathar, K., Reddy, B.D., *Mixed finite element methods for elastic rods of arbitrary geometry*, Numer. Math., 64 (1993) 13–43. 1, 2, 2, 2, 2.4, 2, 3, 3

[3] Babuška. I., Osborn, J., *Eigenvalue problems*, in: Handbook of Numerical Analysis, Vol II. Ciarlet, P.G., Lions, J.L. (eds.), North Holland, Amsterdam, 1991, pp. 641–687. 3, 3, 3

[4] Babuška, I., Suri, M., *On locking and robustness in the finite element method*, SIAM J. Numer. Anal., 29 (1992) 1261–1293. 1

[5] BREZZI, F., FORTIN, M., *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991. 1, 2, 3

[6] CHAPELLE, D., *A locking-free approximation of curved rods by straight beam elements*, Numer. Math., 77 (1997) 299–322. 1, 2, 2.4, 2, 3, 3

[7] DURÁN, R., HERVELLA-NIETO, L., LIBERMAN, E., RODRÍGUEZ, R., SOLOMIN, J., *Approximation of the vibration modes of a plate by Reissner-Mindlin equations*, Math. Comp., 68 (1999) 1447–1463. 1

[8] DURÁN, R., HERNÁNDEZ, E., HERVELLA-NIETO, L., LIBERMAN, E., RODRÍGUEZ, R., *Error estimates for low-order isoparametric quadrilateral finite elements for plates*, SIAM J. Numer. Anal., 41 (2003) 1751–1772. 1

[9] HERNÁNDEZ, E., OTÁROLA, E., RODRÍGUEZ, R., SANHUEZA, F., *Approximation of the vibration modes of a Timoshenko curved rod of arbitrary geometry*, IMA J. Numer. Anal. (to appear). 2, 3, 3

[10] KARAMI, G., FARSHAD, M., YAZDCHI, M., *Free vibrations of spatial rods - a finite-element analysis*. Comm. Appl. Numer. Methods, 6 (1990) 417–428. 1, 2

[11] KATO, T., *Perturbation Theory for Linear Operators*, Springer Verlag, Berlin, 1995. 2, 3

[12] KIKUCHI, F., *Accuracy of some finite element models for arch problems*, Comput. Methods Appl. Mech. Engrg., 35 (1982) 315–345. 1

[13] LITEWKA, P., RAKOWSKI, J., *Free vibrations of shear-flexible and compressible arches by FEM.*, Internat. J. Numer. Methods Eng., 52 (2001) 273–286. 1

[14] LOULA, A.F.D., FRANCA, L.P., HUGHES, T.J.R., MIRANDA, I, *Stability, convergence and accuracy of a new finite element method for the circular arch problem*, Comput. Methods Appl. Mech. Engrg., 63 (1987) 281-303. 1

[15] REDDY, B.D., VOLPI, M.B., *Mixed finite element methods for the circular arch problem*, Comput. Methods Appl. Mech. Engrg., 97 (1992) 125–145. 1

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