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A DESCRIPTION OF HEREDITARY SKEW GROUP ALGEBRAS OF DYNKIN AND EUCLIDEAN TYPE

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Abstract. In this work we study the skew group algebra $\Lambda[G]$ when G is a finite group acting on Λ whose order is invertible in Λ. Here, we assume that Λ is a finite-dimensional algebra over an algebraically closed field k. The aim is to describe all possible actions of a finite abelian group on an hereditary algebra of finite or tame representation type, to give a description of the resulting skew group algebra for each action and finally to determinate their representation type.

1. Introduction

In this work we assume that Λ is a finite-dimensional algebra over an algebraically closed field k. Let G a finite group acting on Λ. The *skew group algebra* $\Lambda[G]$ is the free left Λ-module with basis all the elements in G and multiplication given by $(\lambda g)(\mu h) = \lambda g(\mu)gh$ for all λ, μ in Λ, g, h in G. We study the skew group algebra $\Lambda[G]$ when G is a finite group acting on Λ whose order is invertible in Λ.

There is an extensive literature about skew group algebras $\Lambda[G]$ and crossed product algebras $\Lambda *_\gamma G$, and their relationship with the ring Λ^G, given by elements in Λ that are fixed by G. It is of interest to study which properties of Λ are inherited by $\Lambda[G]$, $\Lambda *_\gamma G$ or Λ^G. Some of these ideas are rooted in trying to develop a Galois Theory for non-commutative rings. See [1, 3, 7, 9, 10, 11, 14, 13] for more details.

It is of interest to find ways to describe $\Lambda[G]$ in terms of Λ because the algebras Λ and $\Lambda[G]$ have many properties in common which are of interest in the representation theory of finite-dimensional algebras, like finite representation type, being hereditary, being an Auslander algebra, being Nakayama, see [2, 16] for more details. However, we must observe that there are properties which are not preserved by this construction, like being a connected algebra, so we are dealing with essentially different algebras.

It is well known [6] that a connected hereditary algebra is of finite representation type if and only if the underlying graph of its quiver is one of the Dynkin diagrams A_n $(n \geq 1)$, D_n $(n \geq 4)$, E_6, E_7 or E_8; some years later it was shown that a a connected hereditary algebra is of tame representation type if and only if the underlying graph of its quiver is one of the euclidean diagrams $\widetilde{A_n}$ $(n \geq 1)$, $\widetilde{D_n}$ $(n \geq 4)$, $\widetilde{E_6}$, $\widetilde{E_7}$ or $\widetilde{E_8}$, see [4, 12, 17].

The aim of this paper is to describe all possible actions of a finite abelian group on an hereditary algebra of finite or tame representation type, to give a description of the resulting skew group algebra for each action and finally to determinate their representation type.

Then, in order to classify the finite and tame representation type hereditary skew group algebras, it suffices to study the group actions on the Dynkin and the euclidean quivers. In order to do this description, we start by considering a short exact sequence of groups $1 \to H \to G \to T \to 1$. We can express $\Lambda[G]$ in terms of the skew group algebra $\Lambda[H][T]$ or the crossed product algebra $\Lambda[H] *_\gamma T$. In this context, we describe when $\Lambda[G]$ is isomorphic to $\Lambda[H][T]$, for G a finite group whose order is invertible in Λ. In section 2 we provide an introduction to the subject, that is, the definition of skew group algebra and crossed product algebra. Finally, in section 3 we consider hereditary algebras of finite representation type and in section 4 we consider hereditary algebras of tame type. In each one of these cases, that is, when the associated quiver is A_n $(n \geq 1)$, D_n $(n \geq 4)$, E_6, E_7, E_8, $\widetilde{A_n}$ $(n \geq 2)$, $\widetilde{D_n}$ $(n \geq 4)$, $\widetilde{E_6}$, $\widetilde{E_7}$ or $\widetilde{E_8}$, we get a connection between $\Lambda[G]$ and the crossed product algebra $\Lambda[H] * G/H$ with a complete description of all the possible groups G/H appearing in each case, where H is the subgroup of G consisting on all the elements acting trivially on a complete set of primitive orthogonal idempotents of the algebra Λ. As a consequence of all these results, we get that if H acts trivially on Λ then the crossed product algebras obtained in each description are skew group algebras. Finally, the case $\widetilde{A}_1$ is considered at the end of section 4.

2. Skew group algebras

This section consists of the preliminaries necessary for the proof of the main results.

Let Λ be a finite-dimensional k-algebra and G a finite group acting on Λ. The skew group algebra $\Lambda[G]$ is the free left Λ-module with basis all the elements in G and multiplication given by $(\lambda g)(\mu h) = \lambda g(\mu) gh$ for all λ, μ in Λ, g, h in G. Clearly $\Lambda[G]$ is a finite-dimensional k-algebra. If we identify each g in G with $1_\Lambda g$ in $\Lambda[G]$ and each λ in Λ with $\lambda 1_G$ in $\Lambda[G]$, we have that G is the group of units of $\Lambda[G]$ and Λ is a k-subalgebra of $\Lambda[G]$.

Let Λ be a basic finite-dimensional algebra (associative with unity) over an algebraically closed field. A *quiver* $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of two sets Q_0 (whose elements are called points, or vertices) and Q_1 (whose elements are called arrows), and two maps $s, t : Q_1 \to Q_0$ which associate to each arrow $\alpha \in Q_1$ its source $s(\alpha) \in Q_o$ and its target $t(\alpha) \in Q_0$. An arrow $\alpha \in Q_1$ of source $a = s(\alpha)$ and target $b = t(\alpha)$ is usually denoted by $\alpha : a \to b$. A quiver $Q = (Q_0, Q_1, s, t)$ is usually denoted briefly by $Q = (Q_0, Q_1)$ or even simply by Q. Thus, a quiver is nothing but an oriented graph without any restriction as to the number of arrows between two points, to the existence of loops or oriented cycles.

We write a path α in Q as a composition of consecutive arrows $\alpha = \alpha_1 \cdots \alpha_r$ where $s(\alpha_i) = t(\alpha_{i+1})$ for all $i = 1, \cdots, r-1$, and we set $t(\alpha) = t(\alpha_1)$, $s(\alpha) = s(\alpha_r)$. The *path algebra* kQ is the k-vector space with basis all the paths in Q, including trivial paths e_x of length zero, one for each vertex $x \in Q_0$. The multiplication of two basis elements is the composition of paths if they are composable, and zero otherwise. A *relation* from x to y is a linear combination $\rho = \sum_{i=1}^{r} \lambda_i u_i$ such that, for each $1 \leq i \leq r$, λ_i is a non-zero scalar and u_i a path of length at least two from x to y. A set of relations on Q generates an ideal I, said to be *admissible*, in the path algebra kQ of Q.

It is well-known that if Λ is basic there exists a quiver Q and a surjective algebra morphism $v : kQ \to \Lambda$ whose kernel I is admissible, where Q, the ordinary quiver of Λ, is defined as follows:

i) If $\{e_1, \cdots, e_n\}$ is a complete set of primitive orthogonal idempotents of Λ, the vertices of Q are the numbers $1, 2, \cdots, n$ which are taken to be in bijective correspondence with the idempotents $e_1, \cdots, e_n$;
ii) Given two points $a, b \in Q_0$ the arrows $\alpha : a \to b$ are in bijective correspondence with the vectors in a basis of the k-vector space $e_b \frac{rad\Lambda}{rad^2\Lambda} e_a$.

Thus we have $\Lambda \simeq kQ/I$. We refer to [2] for more details.

If G is acting on a basic algebra Λ, we can view Λ as kQ/I in such a way that the action of G on Λ is induced by an action of G on kQ which leaves I stable and preserves the natural grading on kQ by the length of paths. Then $\Lambda[G]$ is isomorphic to $(kQ)[G]/I((kQ)[G])$, see [16, Proposition 2.1]. Moreover, if Q contains no multiple arrows, the action of G on kQ is simple: each $g \in G$ permutes the vertices in Q and maps each arrow $\alpha : i \to j$ onto a multiple scalar of the unique arrow from $g(i)$ to $g(j)$. From now on we assume $\Lambda = kQ/I$ with the action of G as described above, Q without double arrows.

Proposition 2.1. *Let G be a finite group acting on Λ, let Q_Λ be the associated quiver of Λ, Q_Λ without double arrows, and $m = |G|$. We consider the action of G on Q induced by an automorphism of algebras which preserves the length of paths of Q. Then*

i) *If $g \in G$, $i \in Q_0$, then $g(e_i) = e_j$ for some $j \in Q_0$;*
ii) *If $g \in G$ and $\alpha \in Q_1$, then $g(\alpha) = \lambda\beta$ for some arrow $\beta \in Q_1$, $\lambda \in k$. In particular, if g fixes the starting and ending point of α then $g(\alpha) = \lambda\alpha$, with $\lambda^m = 1$;*
iii) *If e_i is a source (sink) then $g(e_i)$ is a source (sink);*
iv) *The cardinal of the set of arrows that start (end) in e_i is equal to the cardinal of the set of arrows that start (end) in $g(e_i)$.*

Proof. i) Let e_i be a primitive idempotent in Λ. Since the action of G preserves the vector space generated by arrows, $g(e_i) = \sum_{j=1}^{n} \lambda_j e_j$. Moreover, $g(e_i) = g(e_i^2)$, then we have that $\sum_{j=1}^{n} \lambda_j e_j = \sum_{j=1}^{n} \lambda_j^2 e_j$, and hence $\lambda_j^2 = \lambda_j$, that is, $\lambda_j = 0, 1$. On the other hand, suppose $g(e_i) =$

$e_1 + e_2 + \sum_{j=3}^{n} \lambda_j e_j$. Then

$$e_i = g^{-1}(e_1 + e_2 + \sum_{j=3}^{n} \lambda_j e_j) = g^{-1}(e_1) + g^{-1}(e_2) + g^{-1}(\sum_{j=3}^{n} \lambda_j e_j).$$

But this is a contradiction because e_i es primitive. Then $g(e_i) = e_j$ for some j.

ii) If $\alpha \in Q_1$, $g(\alpha) = \sum \lambda_l \alpha_l$ with $\alpha_l \in Q_1, \lambda_l \in k$. Moreover if $\alpha = e_i \alpha e_j$ then $g(\alpha) = g(e_i)g(\alpha)g(e_j) = \sum \lambda_l \beta$, for some arrow $\beta : g(e_j) \to g(e_i)$, because Q has no double arrows. Then $g(\alpha) = \lambda\beta$.

iii) Let e_i be an idempotent of Λ. Suppose that e_i is a source and $g(e_i)$ is not. Then, there exists an arrow β such that $t(\beta) = g(e_i)$, that is, there exists an index r such that $\beta : g(e_r) \to g(e_i) \in Q_1$. Since the action of G on Q is induced by an automorphism of algebras which preserves the length of paths, there exists an arrow α such that $g(\alpha) = \beta$. Then we have $\beta = g(e_i)\beta g(e_r) = g(e_i)g(\alpha)g(e_r) = g(e_i \alpha e_r) = 0$ because e_i is a source. This contradiction arises from the assumption that $g(e_i)$ is not a source. Similarly we prove that if e_j is a sink then $g(e_j)$ is a sink.

iv) Let $F_{e_i} = \{\alpha \in Q_1 : \quad s(\alpha) = e_i\}$ and V_{e_i} be the k-vector space with basis F_{e_i}. If $g \in G$, by ii) we know that the automorphism $g : \Lambda \to \Lambda$ induces an isomorphism $g : V_{e_i} \to V_{g(e_i)}$. Then the cardinal of F_{e_i} and $F_{g(e_i)}$ are equal.

$\square$

2.1. **Crossed product algebra $\Lambda[H] *_\gamma T$.** The purpose of this section is to present the crossed product algebras in order to study when $\Lambda[G]$ is isomorphic to $\Lambda[H][T]$ where $1 \to H \to G \to T \to 1$ is a short exact sequence of groups. We start with the definition of crossed product algebras and prove, for completeness, Theorem 2.2 that connects the skew group algebras with the crossed product algebras. See [16] for more details.

Let Λ be a ring, G a finite group acting on Λ, $U(\Lambda)$ the units of Λ and $\gamma : G \times G \to U(\Lambda)$, a map satisfying

(1) $\gamma(g, g')\gamma(gg', g'') = g(\gamma(g', g''))\gamma(g, g'g'')$ for g,g',$g'' \in G$;
(2) $\gamma(e, g) = 1 = \gamma(g, e)$ for $g \in G$, e the identity element of G;
(3) $\gamma(g, g')(gg')(\lambda) = g(g'(\lambda))\gamma(g, g')$ for $g, g' \in G$, $\lambda \in \Lambda$.

Then the corresponding *crossed product algebra* $\Lambda *_\gamma G$ has elements $\sum_{g_i \in G} \lambda_i \overline{g_i}$; $\lambda_i \in \Lambda$. Addition is componentwise, and multiplication is given by $\overline{g}\lambda = g(\lambda)\overline{g}$ and $\overline{g_1}\ \overline{g_2} = \gamma(g_1, g_2)\overline{g_1 g_2}$.

Let G be a group and $1 \to H \to G \to T \to 1$ be a short exact sequence of groups. Let $G = Hx_1 \cup Hx_2 \cup \cdots \cup Hx_t$ be a disjoint union of lateral classes. Then $T = \{\overline{x_1}, \cdots, \overline{x_t}\}$ where $\overline{x_i} = Hx_i, \quad \overline{x_1} = 1$.

Theorem 2.2. *If $1 \to H \to G \to T \to 1$ is a short exact sequence of groups, then*

$$\Lambda[G] \simeq \Lambda[H] *_\gamma T$$

where the action of H on Λ is induced by the action of G, the action of T on $\Lambda[H]$ is defined by

$$\overline{x}_j(\lambda h) = x_j(\lambda)\ x_j h x_j^{-1},$$

and $\gamma : T \times T \to U(\Lambda[H])$ *is defined by* $\gamma(\overline{x_i}, \overline{x_j}) = x_i x_j x_r^{-1}$, *with* $\overline{x_i x_j} = \overline{x_r}$.

Proof. We consider the action of H on Λ induced by the action of G and the action of T on $\Lambda[H]$ given by $\overline{x}_j(\lambda h) = x_j(\lambda)\ x_j h x_j^{-1}$. We claim that the action is well defined since H is a normal subgroup of G. If x_i,$x_j \in G$ then $x_i x_j \in H x_r$ for some r, that is $\overline{x_i x_j} = \overline{x_r}$. Let $\gamma : T \times T \to U(\Lambda[H])$ be defined by

$$\gamma(\overline{x_i}, \overline{x_j}) = x_i x_j x_r^{-1}.$$

A direct computation shows that γ is a crossed product. Now let us see that the map $\Phi : \Lambda[G] \longrightarrow \Lambda[H] *_\gamma T$ given by

$$\sum_{i,j} \lambda_{ij} h_i x_j \longmapsto \sum_{j=1}^{t} (\sum_{i=1}^{\frac{m}{t}} \lambda_{ij} h_i) \overline{x}_j$$

is an isomorphism of k-algebras, where $m = |G|$. Clearly Φ is a morphism of k-vector spaces. If $\overline{x_j x_s} = \overline{x}_r$, then

$$\begin{aligned}
\Phi(\lambda h_i x_j\ .\ \lambda' h_t x_s) &= \Phi(\lambda\ (h_i x_j)(\lambda')\ h_i x_j h_t x_s) && (1)\\
&= \Phi(\lambda\ (h_i x_j)(\lambda')\ h_i x_j h_t x_j^{-1}\ x_j x_s x_r^{-1}\ x_r) && (2)\\
&= \lambda\ (h_i x_j)(\lambda')\ h_i x_j h_t x_j^{-1}\ x_j x_s x_r^{-1}\ \overline{x}_r && (3)
\end{aligned}$$

because H is a normal subgroup of G. On the other hand we have

$$\begin{aligned}
\Phi(\lambda h_i x_j)\ .\ \Phi(\lambda' h_t x_s) &= \lambda h_i \overline{x}_j\ .\ \lambda' h_t \overline{x}_s && (4)\\
&= \lambda h_i\ \overline{x}_j(\lambda' h_t)\ \gamma(\overline{x}_j, \overline{x}_s) \overline{x_j x_s} && (5)\\
&= \lambda h_i\ x_j(\lambda')\ x_j h_t x_j^{-1}\ x_j x_s x_r^{-1}\ \overline{x}_r && (6)\\
&= \lambda\ h_i(x_j(\lambda'))\ h_i x_j h_t x_j^{-1}\ x_j x_s x_r^{-1}\ \overline{x}_r && (7)
\end{aligned}$$

and (3) agrees with (7). Furthermore it is clear that Φ is bijective, hence we get $\Lambda[G] \simeq \Lambda[H] *_\gamma T$. □

Corollary 2.3. *If* $1 \to H \to G \xrightarrow{\pi} T \to 1$ *is a short exact sequence of groups that splits on the right, then* $\Lambda[G] = \Lambda[H][T]$.

Proof. We only have to prove that the map γ defined in the theorem above is such that $\gamma(u, v) = 1$ for any u, $v \in T$ where $\gamma(\overline{x}_i, \overline{x}_j) = x_i x_j x_r^{-1}$, with $x_i x_j \in H x_r$ for some r. If the sequence $1 \to H \to G \xrightarrow{\pi} T \to 1$ splits on the right, there exists a map $\delta : T \to G$ such that $\pi \circ \delta = 1_T$. Let $\overline{x_r} = \pi(x_r)$. Since $\overline{x_r} = (\pi \circ \delta)(\overline{x_r})$, then $\overline{\delta(\overline{x_r})} = \overline{x_r}$ and hence we assume that $x_r := \delta(\overline{x_r})$. Since δ is a morphism of groups, $x_i x_j = \delta(\overline{x}_i)\delta(\overline{x}_j) = \delta(\overline{x}_i \overline{x}_j) = \delta(\overline{x_i x_j}) = \delta(\overline{x}_r) = x_r$ and therefore $\gamma(\overline{x}_i, \overline{x}_j) = x_i x_j x_r^{-1} = 1$. Now it is clear that $\Lambda[G] = \Lambda[H][T]$. □

Corollary 2.4. *If $G = H \times T$ or $G = H \ltimes T$ then $\Lambda[G] = \Lambda[H][T]$.*

It is clear that the map $\gamma : T \times T \to U(Z(\Lambda))$ which defines a crossed product is by definition a cocycle with respect to group cohomology, see [8] for more details. We shall prove that if the cocycle γ is a coboundary, then we have $\Lambda *_\gamma T \simeq \Lambda[T]$.

Proposition 2.5. [16, Lemma 5.6] *Let $\delta : T \to U(Z(\Lambda))$ be a map, $\delta(1) = 1$, and let $\gamma : T \times T \to U(\Lambda)$ be given by*

$$\gamma(g,h) = g(\delta(h))\ \delta(gh)^{-1}\ \delta(g).$$

Then $\Lambda *_\gamma T \simeq \Lambda[T]$.

Proof. A direct computation shows that γ defines a crossed product. Let $\Psi : \Lambda *_\gamma T \longrightarrow \Lambda[T]$ be defined by $\Psi(\lambda\bar{t}) = \lambda\delta(t)t$. Then Ψ is a morphism of algebras because

$$\begin{aligned}
\Psi\Big(\lambda\bar{t}.\lambda'\bar{t}'\Big) &= \Psi\Big(\lambda t(\lambda')\ \gamma(t,t')\ \overline{tt'}\Big)\\
&= \Psi\Big(\lambda t(\lambda')\ t(\delta(t'))\ \delta(tt')^{-1}\ \delta(t)\ \overline{tt'}\Big)\\
&= \lambda t(\lambda')\ t(\delta(t'))\ \delta(tt')^{-1}\ \delta(t)\ \delta(tt')\ tt'\\
&= \lambda t(\lambda')\ t(\delta(t'))\ \delta(t)\ tt'\\
&= \lambda\delta(t)\ t(\lambda')\ t(\delta(t'))\ tt'\\
&= \Big(\lambda\delta(t)\ t\Big)\ \Big(\lambda'\delta(t')\ t'\Big)\\
&= \Psi(\lambda\bar{t}).\Psi(\lambda'\bar{t}').
\end{aligned}$$

Therefore Ψ is an isomorphism and hence $\Lambda *_\gamma T \simeq \Lambda[T]$. □

We say that G acts trivially on an element λ if $g(\lambda) = \lambda$ for all $g \in G$. If G is a finite abelian group of order m acting trivially on Λ with m invertible in Λ, then $\Lambda[G] \simeq \prod_{i=1}^m \Lambda$. In fact, by Maschke's theorem we have that $k(G) \simeq \prod_{i=1}^m k$, see [14]. Now the map $\psi : \Lambda \otimes_k k(G) \to \Lambda[G]$ given by $\psi(\lambda \otimes \sum_{i=1}^m \lambda_i g_i) = \sum_{i=1}^m \lambda\lambda_i g_i$ is an isomorphism of k-algebras.

Proposition 2.6. [16, Proposition 5.8] *Let T be a finite cyclic group acting on a commutative local algebra R with the order of T invertible in R. Then $H^2(T, U(R)) = 1$.*

We may infer from the previous proposition that if T is a finite cyclic group with the order of T is invertible in Λ, and $Z(\Lambda)$ (the center of Λ) is a local algebra, $\Lambda *_\gamma T \simeq \Lambda[T]$ because any cocycle is a coboundary. In particular, if Λ is a basic connected algebra without oriented cycles, $Z(\Lambda) = k$.

Corollary 2.7. *Let G be a finite abelian group acting on a basic connected algebra Λ where the associated quiver has no oriented cycles, with the order of G invertible in Λ. Let H be a subgroup of G which acts trivially on Λ, with $T = G/H$ cyclic. Then $\Lambda[G] \simeq \Lambda[H][T]$.*

Proof. It follows from Theorem 2.2 that $\Lambda[G] \simeq \Lambda[H] *_\gamma T$, and $\Lambda[H] \simeq \Lambda \times \cdots \times \Lambda$ by Maschke's theorem. Since H is abelian and acts trivially on Λ, γ takes values in the set of invertible elements of the center of $\Lambda[H]$. But $Z(\Lambda[H]) \simeq k \times \cdots \times k$ and T is cyclic, so Proposition 2.6 implies that γ is a coboundary. From Proposition 2.5 we may deduce that $\Lambda[G] \simeq \Lambda[H][T]$. $\square$

It is known that if G is a finite group of order m acting trivially on the idempotents of Λ and m is invertible in Λ, then G is an abelian group, see [15, Proposition 2.7]. In fact, given $g_1, g_2 \in G$, $g_i(\alpha) = \omega_i\alpha$ with ω_i a m–root of unity. Hence $g_1g_2(\alpha) = \omega_1\omega_2\alpha = g_2g_1(\alpha)$, and this equality holds for any arrow α. Moreover, for every j $g_1g_2(e_j) = e_j = g_2g_1(e_j)$. So $g_1g_2 = g_2g_1$.

Finally, we state a result that will be used in the proof of the main theorem in this work.

Theorem 2.8. [5, Theorem 8]. *Let G be a finite abelian group of order m acting trivially on a complete set of primitive orthogonal idempotents of a simply connected algebra $\Lambda = kQ/I$, Q without double arrows and m invertible in Λ. Then $\Lambda[G] \simeq \prod_{i=1}^m \Lambda$.*

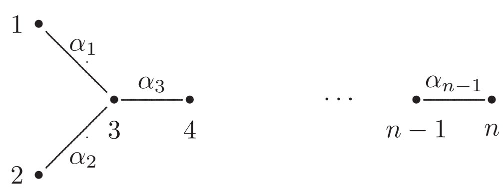
3. $\Lambda[G]$ with G an abelian group and Λ an hereditary algebra of finite representation type

The aim of this section is to describe all possible actions of a finite abelian group on an hereditary algebra of finite representation type and to give a description of the skew group algebra for each action.

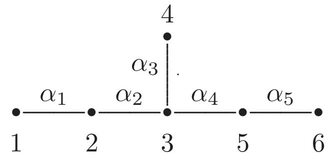
Gabriel has shown in [6] that a connected hereditary algebra is representation-finite if and only if the underlying graph of its quiver is one of the Dynkin diagrams A_n, D_n ($n \geq 4$), E_6, E_7 or E_8, that appear also in Lie theory, where the index in the Dynkin graph always refers to the number of points in the graph. Then, in order to classify the representation-finite hereditary skew group algebras, it suffices to study the group actions on the Dynkin quivers.

A_n ($n \geq 1$)

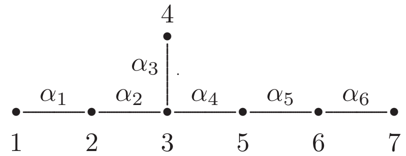
D_n ($n \geq 4$)



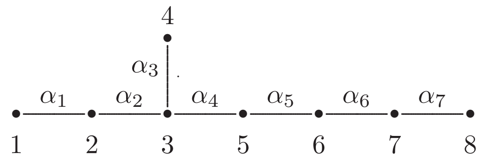
E_6



E_7



E_8



Before we present the results, we need some definitions.

Definition 3.1. *We say that an quiver of type A_{2r+1} has symmetric orientation if it is symmetric with respect to the middle point $r+1$.*

Definition 3.2. *We say that an quiver of type D_n, $n > 4$, has symmetric orientation if $s(\alpha_1) = s(\alpha_2) = e_3$ or $t(\alpha_1) = t(\alpha_2) = e_3$.*

Definition 3.3. *We say that an quiver of type D_4 has*

i) *symmetric orientation of kind (a) if $s(\alpha_1) = s(\alpha_2) = t(\alpha_3) = e_3$ or $t(\alpha_1) = t(\alpha_2) = s(\alpha_3) = e_3$; and,*
ii) *symmetric orientation of kind (b) if $s(\alpha_1) = s(\alpha_2) = s(\alpha_3) = e_3$ or $t(\alpha_1) = t(\alpha_2) = t(\alpha_3) = e_3$.*

Definition 3.4. *We say that the quiver Q of type E_6 has symmetric orientation if it is symmetric with respect to the side $3-4$, that is,*

i) *$s(\alpha_1) = e_1$ and $s(\alpha_5) = e_6$, or $t(\alpha_1) = e_1$ and $t(\alpha_5) = e_6$; and,*
ii) *$s(\alpha_2) = e_3 = s(\alpha_4)$, or $t(\alpha_2) = e_3 = t(\alpha_4)$.*

Remark 3.5. (i) *The quiver A_{2r+1} is symmetric with respect to the middle point $r+1$ if that point is center of symmetry of the quiver.*
(ii) *The quiver E_6 is symmetric with respect to the side $3-4$ if the line obtained with the points $\{3,4\}$ is a symmetry axis of the quiver.*

As we have already mentioned, if G is acting trivially on Λ, we have $\Lambda[G] = \prod_{t=1}^{m} \Lambda$. Hence, from now on, we will assume that G is acting non trivially on Λ. Let $H = \{g : \ g(e_i) = e_i \text{ for all } i \text{ in } Q_0\}$. Clearly H is a normal subgroup of G. Let $T = G/H$, then $1 \to H \to G \to T \to 1$ is a short exact sequence of groups.

Theorem 3.6. *Let $\Lambda = kQ$ be an hereditary algebra, with Q of type A_n $(n \geq 1)$, D_n $(n \geq 4)$, E_6, E_7 or E_8, and G a finite abelian group of order m acting non trivially on Λ, with m invertible in Λ. Let $H = \{g : \ g(e_i) = e_i$ for all i in $Q_0\}$.*

i) *If $H = G$ then $\Lambda[G] = \prod_{t=1}^{m} \Lambda$;*

ii) *If $H \subsetneq G$ and $\Lambda = kQ$ with Q of type A_n then Q is of type A_{2r+1}, with symmetric orientation, the order of G is even and $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda) *_\gamma \mathbb{Z}/2\mathbb{Z}$;*

iii) *If $H \subsetneq G$ and $\Lambda = kQ$ with Q of type D_n, $n > 4$, then Q has symmetric orientation, the order of G is even and $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda) *_\gamma \mathbb{Z}/2\mathbb{Z}$;*

iv) *If $H \subsetneq G$ and $\Lambda = kQ$ with Q of type D_4 then*

iv.1) *Q has symmetric orientation of kind (a), the order of G is even and $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda) *_\gamma \mathbb{Z}/2\mathbb{Z}$,*
or

iv.2) *Q has symmetric orientation of kind (b), the order of G is divisible by 2 or 3, and $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda) *_\gamma \mathbb{Z}/2\mathbb{Z}$ or $\Lambda[G] \simeq (\prod_{t=1}^{m/3} \Lambda) *_\gamma \mathbb{Z}/3\mathbb{Z}$.*

v) *If $H \subsetneq G$ and $\Lambda = kQ$ with Q of type E_6 then Q has symmetric orientation, G is a group of even order and $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda) *_\gamma \mathbb{Z}/2\mathbb{Z}$;*

vi) *If $\Lambda = kQ$ with Q of type E_7 or Q of type E_8 then $H = G$ and $\Lambda[G] = \prod_{t=1}^{m} \Lambda$.*

Proof. In order to prove the theorem, we need a precise description of all the possible actions of G on $\Lambda = kQ$, for each type and orientation of Q. We use Proposition 2.1 to describe all possible actions of G on kQ with Q of type A_n, D_n, E_6, E_7 or E_8.

i) See Theorem 2.8.

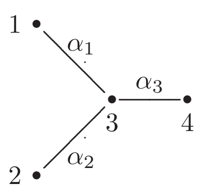
ii) Let $\Lambda = kQ$ with Q of type A_n and let $g \in G$, $g \notin H$. If $g(e_1) = e_1$ then $g(\alpha_1) = \xi_1\alpha_1$. This implies $g(e_2) = e_2$ and $g(\alpha_2) = \xi_2\alpha_2$ with ξ_1, ξ_2 m-roots of unity. Repeating this procedure we have that $g(e_i) = e_i$ implies $g(\alpha_i) = \xi_i\alpha_i$ with ξ_i an m-root of unity, and this for all $i = 1, \cdots, n-1$. Hence the action of g is trivial on the idempotents e_i of Λ. So $g \in H$, a contradiction. So $g(e_1) \neq e_1$. In this case $g(e_1) = e_n$ and e_1, e_n will have to be sinks or sources, see Proposition 2.1. This determines the orientation of α_1 and α_{n-1}. Moreover $g(\alpha_1) = \xi_1\alpha_{n-1}$. So $g(e_2) = e_{n-1}$ and $g(\alpha_2) = \xi_2\alpha_{n-2}$. Inductively, $g(e_i) = e_{n-i+1}$ and $g(\alpha_i) = \xi_i\alpha_{n-i}$, and this for all $i = 1, \cdots, n-1$, with $\xi_i \in k$, $\xi_i \neq 0$. If $n = 2r$ is an even number we have that $g(e_r) = e_{r+1}$, $g(e_{r+1}) = e_r$ and if α_r is the arrow $\alpha_r : r \to r+1$, then $g(\alpha_r) = g(e_{r+1})g(\alpha_r)g(e_r) = e_r g(\alpha_r)e_{r+1} = 0$, contradiction. We also get a contradiction if $\alpha_r : r+1 \to r$. Then if the number of vertices is an even number, the unique possible action on the set of idempotents is the trivial one.

Let $Q = A_{2r+1}$ with g acting non trivially on the set $\{e_1, \ldots, \ e_n\}$ of idempotents of Λ. Then $g(e_i) = e_{2r+2-i}$ and $g(\alpha_i) = \xi_i\alpha_{2r+1-i}$, hence the quiver Q has symmetric orientation. Moreover, $g^2(e_i) = e_i$ for all i, so $g^2 \in H$. Then G has even order $m = 2s$ and $\xi_i^s \xi_{2r+1-i}^s = 1$, for all i. Let $\overline{g}' \in T$, $g' \notin H$. Since $g' \notin H$, g' does not act trivially on the set $\{e_1, \cdots, e_n\}$ of idempotents of Λ. By the previous reasoning, the unique non trivial action is given by $g'(e_i) = e_{2r+2-i}$. Then $gg'(e_i) = g(e_{2r+2-i}) = e_{2r+2-(2r+2-i)} = e_i$. As a consequence $gg' \in H$, that is $\overline{gg'} = 1$. Then $\overline{g}' = (\overline{g})^{-1} = \overline{g}$ because $g^2 \in H$, and hence $T \simeq \mathbb{Z}/2\mathbb{Z}$. Hence, if the group G does not act trivially on the set $\{e_1, \cdots, e_n\}$ of idempotents of Λ, in accordance with the previous analysis we have that $m = 2s$ is an even number and Q is of type A_{2r+1} with symmetric orientation. In this case we have $T \simeq \mathbb{Z}/2\mathbb{Z}$. Hence $\Lambda[G] \simeq \Lambda[H] *_\gamma \mathbb{Z}/2\mathbb{Z} \simeq (\prod_{t=1}^{s} \Lambda) *_\gamma \mathbb{Z}/2\mathbb{Z}$, see Theorem 2.2 and Theorem 2.8.

iii) Let $\Lambda = kQ$ with Q of type D_n, $n > 4$. Assume that the group G is not acting trivially on the set $\{e_1, \cdots, e_n\}$ of idempotents of Λ. We observe that all $g \in G$ must satisfy $g(e_3) = e_3$, see Proposition 2.1. If $g \notin H$ then $g(e_1) = e_2$, $g(e_2) = e_1$ and $g(e_i) = e_i$ for all $i = 3, \cdots, n$. This determines the orientation of the arrows, that is, Q has symmetric orientation, and $g(\alpha_1) = \xi_1\alpha_2$, $g(\alpha_2) = \xi_2\alpha_1$, $g(\alpha_i) = \xi_i\alpha_i$ for all $i = 3, \cdots, n-1$, with $\xi_3, \cdots, \xi_{n-1}$ m-roots of unity, $\xi_1, \xi_2 \in k$ non zero. Then $g^2(e_i) = e_i$ for all i, that is, $g^2 \in H$. So G has even order $m = 2s$ and $\xi_1^s\xi_2^s = 1$. Let $g' \in G$, $g' \notin H$. By the previous reasoning, g' and g act in the unique possible non trivial way on the complete set of idempotents of Λ. Then $gg'(e_1) = g(e_2) = e_1$, $gg'(e_2) = g(e_1) = e_2$ and $gg'(e_i) = e_i$ for all $i = 3, \cdots, n$. Hence $gg' \in H$, that is $\overline{gg'} = 1$, then $\overline{g}' = (\overline{g})^{-1} = \overline{g}$ because $g^2 \in H$. Hence $T \simeq \mathbb{Z}/2\mathbb{Z}$ and $|G|$ is an even number.

It follows from the previous analysis that $|G| = m = 2s$ is an even number and the quiver Q has symmetric orientation. Hence we have $\Lambda[G] \simeq \Lambda[H] *_\gamma \mathbb{Z}/2\mathbb{Z} \simeq (\prod_{t=1}^{s} \Lambda) *_\gamma \mathbb{Z}/2\mathbb{Z}$.

iv) Let $\Lambda = kQ$ with Q of type D_4.



Let $g \in G$, $g \notin H$. Necessarily $g(e_3) = e_3$, by Proposition 2.1, and all possible cases are:

i) $g_1(e_1) = e_2$, $g_1(e_2) = e_1$, $g_1(e_4) = e_4$;
ii) $g_2(e_1) = e_4$, $g_2(e_2) = e_2$, $g_2(e_4) = e_1$;
iii) $g_3(e_1) = e_1$, $g_3(e_2) = e_4$, $g_3(e_4) = e_2$;
iv) $g_4(e_1) = e_2$, $g_4(e_2) = e_4$, $g_4(e_4) = e_1$;
v) $g_5(e_1) = e_4$, $g_5(e_2) = e_1$, $g_5(e_4) = e_2$.

In fact $g_1^2, g_2^2, g_3^2 \in H$, $g_4^3, g_5^3 \in H$ and $g_4g_5 \in H$, so $\overline{g}_4 = (\overline{g}_5)^{-1}$ in T. On the other hand, since $g_ig_j \neq g_jg_i$ for all i, j with $i \neq j$ and $1 \leq i, j \leq 4$ and

G is abelian, we have that G cannot contain simultaneously elements acting as g_i, g_j for all i, j with $i \neq j$ and $1 \leq i, j \leq 4$. Consequently $T \simeq \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$. The cases i), ii) and iii) determine the orientation of the arrows α_1 and α_2, that is, Q has symmetric orientation of kind (a) or (b), and the cases iv) and v) determine the orientation of all the arrows, that is, Q has symmetric orientation of kind (b).

In accordance with Definition 3.3 and with the previous analysis for Q of type D_4, we have that the quiver Q has symmetric orientation of kind (a) and $m = 2s$, or has symmetric orientation of kind (b) and $m = 2s$ or $m = 3s$. From Theorem 2.2 and Theorem 2.8 we have that, in the first case, $T \simeq \mathbb{Z}/2\mathbb{Z}$ and $\Lambda[G] \simeq \Lambda[H] *_\gamma \mathbb{Z}/2\mathbb{Z} \simeq (\prod_{t=1}^{s} \Lambda) *_\gamma \mathbb{Z}/2\mathbb{Z}$. In the second case, the order of G is divisible by 2 or 3, $T \simeq \mathbb{Z}/2\mathbb{Z}$ or $T \simeq \mathbb{Z}/3\mathbb{Z}$, and $\Lambda[G] \simeq \Lambda[H] *_\gamma \mathbb{Z}/2\mathbb{Z} \simeq (\prod_{t=1}^{s} \Lambda) *_\gamma \mathbb{Z}/2\mathbb{Z}$ or $\Lambda[G] \simeq \Lambda[H] *_\gamma \mathbb{Z}/3\mathbb{Z} \simeq (\prod_{t=1}^{s} \Lambda) *_\gamma \mathbb{Z}/3\mathbb{Z}$.

v) We need again a precise description of all the possible actions of G on $\Lambda = kQ$ with Q of type E_6. Let $g \in G$, $g \notin H$. By Proposition 2.1 , $g(e_3) = e_3$, and this implies that $g(e_4) = e_4$. On the other hand $g(e_1) = e_1$ or e_6. If $g(e_1) = e_1$, then $g(e_2) = e_2$ and $g(e_5) = e_5$. This is a contradiction, because $g \notin H$. Then $g(e_1) = e_6$, and this implies that $g(e_2) = e_5$ and $g(e_6) = e_1$. This determines the orientation of the arrows, and we have $g(\alpha_1) = \xi_1\alpha_5$, $g(\alpha_2) = \xi_2\alpha_4$, $g(\alpha_3) = \xi_3\alpha_3$, $g(\alpha_5) = \xi_5\alpha_1$ and $g(\alpha_4) = \xi_4\alpha_2$ with $\xi_1, \xi_2, \xi_4, \xi_5 \in k$ non zero and ξ_3 an m-root of unity. Since $g^2(e_i) = e_i$ for all i, then $g^2 \in H$. So G has even order $m = 2s$ and $\xi_1^s\xi_5^s = 1$, $\xi_2^s\xi_4^s = 1$. Let $g' \in G$ be such that $g' \notin H$. Hence, $gg'(e_i) = e_i$ for all i. Therefore $\overline{gg'} = 1$, $\overline{g}^2 = 1$ and $\overline{g'}^2 = 1$ that is, $\overline{g'} = \overline{g}$, and then $T \simeq \mathbb{Z}/2\mathbb{Z}$. Hence, if the group G does not act trivially on the set $\{e_1, \cdots, e_6\}$ of idempotents of Λ, in accordance with the previous analysis, we have that $|G| = m = 2s$ is an even number, Q has symmetric orientation and $T \simeq \mathbb{Z}/2\mathbb{Z}$. Hence $\Lambda[G] \simeq \Lambda[H] *_\gamma \mathbb{Z}/2\mathbb{Z} \simeq (\prod_{t=1}^{s} \Lambda) *_\gamma \mathbb{Z}/2\mathbb{Z}$, see Theorem 2.2 and Theorem 2.8.

vi) If we consider the cases Q of type E_7 or E_8, the unique possible action on the set of idempotents is the trivial one. Hence $G = H$ and $T = 1$ and the result follows from i).

□

Corollary 3.7. *Let $\Lambda = kQ$ be an hereditary algebra, with Q of type A_n $(n \geq 1)$, D_n $(n \geq 4)$, E_6, E_7 or E_8, and G an abelian group of order m acting on Λ, with m invertible in Λ. Suppose that G does not act trivially on the set $\{e_1, \cdots, e_n\}$ of idempotents of Λ and H acts trivially on Λ.*

i) *If $\Lambda = kQ$, with Q of type A_{2r+1} with symmetric orientation, then $\Lambda[G] \simeq (\prod_{t=1}^{s} \Lambda)[\mathbb{Z}/2\mathbb{Z}]$;*

ii) *If $\Lambda = kQ$, with Q of type D_n, $n > 4$, with symmetric orientation, then $\Lambda[G] \simeq (\prod_{t=1}^{s} \Lambda)[\mathbb{Z}/2\mathbb{Z}]$;*

iii) *If* $\Lambda = kQ$, *with* Q *of type* D_4 *with symmetric orientation of kind (a), then* $\Lambda[G] \simeq (\prod_{t=1}^{s} \Lambda)[\mathbb{Z}/2\mathbb{Z}]$;
iv) *If* $\Lambda = kQ$, *with* Q *of type* D_4 *with symmetric orientation of kind (b), then* $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda)[\mathbb{Z}/2\mathbb{Z}]$ *or* $\Lambda[G] \simeq (\prod_{t=1}^{m/3} \Lambda)[\mathbb{Z}/3\mathbb{Z}]$.
v) *If* $\Lambda = kQ$, *with* Q *of type* E_6 *with symmetric orientation, then* $\Lambda[G] \simeq (\prod_{t=1}^{s} \Lambda)[\mathbb{Z}/2\mathbb{Z}]$.

Proof. It follows from Theorem 3.6 and Corollary 2.7. □

The following Corollary follows easily from [16, (2.3), (2.4)].

Corollary 3.8. *Let* $\Lambda = kQ$ *be an hereditary algebra, with* Q *of type* A_n $(n \geq 1)$, D_n $(n \geq 4)$, E_6, E_7 *or* E_8, *and* G *a cyclic group of order* m *acting on* Λ, *with* m *invertible in* Λ.

(i) *Let* k *be a field such that* $\operatorname{char} k \neq 2, 3$. *If* $\Lambda = kQ$ *is an hereditary algebra with* Q *of type* D_4 *and* $G = \mathbb{Z}/3\mathbb{Z}$ *is acting non trivially on the set* $\{e_1, \cdots, e_4\}$ *of idempotents of* Λ, *then the skew group algebra* $\Lambda[\mathbb{Z}/3\mathbb{Z}]$ *is Morita equivalent to an algebra* kQ' *with* Q' *of type* D_4. *If* $G = \mathbb{Z}/2\mathbb{Z}$ *is acting non trivially on the set* $\{e_1, \cdots, e_n\}$ *of idempotents of* Λ, *then the skew group algebra* $\Lambda[\mathbb{Z}/2\mathbb{Z}]$ *is Morita equivalent to an algebra* kQ' *with* Q' *of type* A_5.
(ii) *Let* k *be a field such that* $\operatorname{char} k \neq 2$. *If* $\Lambda = kQ$ *is an hereditary algebra, with* Q *of type* A_{2r+1} *and* $G = \mathbb{Z}/2\mathbb{Z}$ *is acting non trivially on the set* $\{e_1, \cdots, e_{2r+1}\}$ *of idempotents of* Λ, *then the skew group algebra* $\Lambda[\mathbb{Z}/2\mathbb{Z}]$ *is Morita equivalent to an algebra* kQ' *with* Q' *of type* D_{r+2} *if* $r \geq 2$ *and of type* A_3 *if* $r = 1$.
(iii) *Let* k *be a field such that* $\operatorname{char} k \neq 2$. *If* $\Lambda = kQ$ *is an hereditary algebra, with* Q *of type* D_n, $n > 4$, *and* $G = \mathbb{Z}/2\mathbb{Z}$ *is acting non trivially on the set* $\{e_1, \cdots, e_n\}$ *of idempotents of* Λ, *then the skew group algebra* $\Lambda[\mathbb{Z}/2\mathbb{Z}]$ *is Morita equivalent to an algebra* kQ' *with* Q' *of type* A_{2n-3}.
(iv) *Let* k *be a field such that* $\operatorname{char} k \neq 2$. *If* $\Lambda = kQ$ *is an hereditary algebra, with* Q *of type* E_6, *and* $G = \mathbb{Z}/2\mathbb{Z}$ *is acting non trivially on the set* $\{e_1, \cdots, e_n\}$ *of idempotents of* Λ, *then the skew group algebra* $\Lambda[\mathbb{Z}/2\mathbb{Z}]$ *is Morita equivalent to an algebra* kQ' *with* Q' *of type* E_6.

For an example of Corollary 3.8 see [16, (2.3), (2.4)].

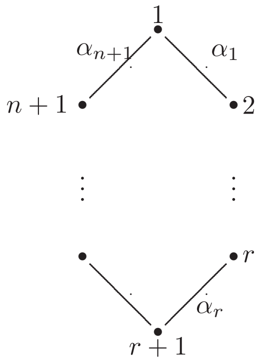
4. $\Lambda[G]$ with G an abelian group and Λ an hereditary algebra of tame representation type

The aim of this section is to describe all possible actions of a finite abelian group on an hereditary algebra of tame representation type, to give a description of the skew group algebra for each action and finally to determinate their representation type.

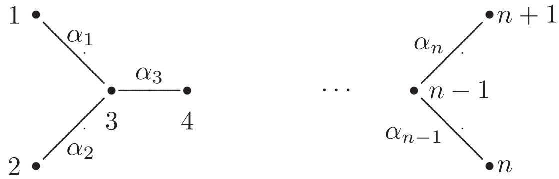
It is well known that a connected hereditary algebra is of tame representation type if and only if the underlying graph of its quiver is one of the euclidean diagrams

$\widetilde{A_n}$ $(n \geq 1)$, $\widetilde{D_n}$ $(n \geq 4)$, $\widetilde{E_6}$, $\widetilde{E_7}$ or $\widetilde{E_8}$ where an euclidean diagram $\widetilde{\Delta_n}$ has $n+1$ points. Then, in order to classify the tame representation type hereditary skew group algebras, it suffices to study the group actions on the euclidean quivers. It is necessary to clarify that the case $\widetilde{A_1}$ will be considered separately later on.

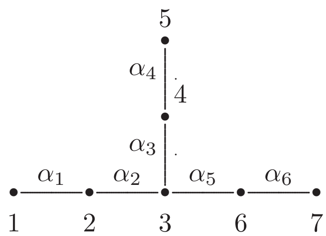
$\widetilde{A_n}$ $(n \geq 1)$



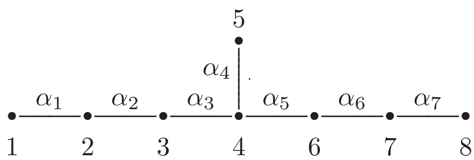
$\widetilde{D_n}$ $(n \geq 4)$



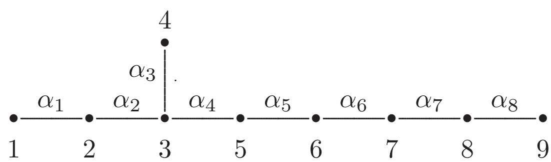
$\widetilde{E_6}$



$\widetilde{E_7}$



$\widetilde{E_8}$



Before we present the results, we need some definitions.

Definition 4.1. *We say that an quiver of type* $\widetilde{A_n}$ *(*$n \geq 2$*) has*

i) *symmetric orientation if* $n = 2r - 1$ *is odd and the quiver is symmetric with respect to an axis* $i - -i + r$*,*
ii) *cyclic orientation of order* s *if the full subquivers with vertices* $\{j(s-1)+1, j(s-1)+2, \cdots, (j+1)(s-1)+1\}$ *are all equal, and* s *is minimal with respect to this property (*$1 < s \leq n+1$*).*

Remark 4.2. *Supose you have* $\widetilde{A_n}$ *with a fixed oritentation. Choose* s *such that* $g(1) = s$*, for any action* g*. This set, a non-empty set of the natural numbers, has a first element and this is the* s *of the definition.*

Definition 4.3. *We say that an quiver of type* $\widetilde{D_n}$*,* $n > 4$*, has*

i) *symmetric orientation of kind (a) if*
$t(\alpha_1) = t(\alpha_2) = e_3$*, or*
$s(\alpha_1) = s(\alpha_2) = e_3$*, or*
$t(\alpha_n) = t(\alpha_{n-1}) = e_{n-1}$*, or*
$s(\alpha_n) = s(\alpha_{n-1}) = e_{n-1}$*,*
ii) *symmetric orientation of kind (b) if* $n = 2r$ *is even and the quiver is symmetric with respect to the middle point* $r + 1$*;*

Definition 4.4. *We say that an quiver of type* $\widetilde{D_4}$ *has symmetric orientation of order* t *if the number of arrows starting at the vertex* 3 *is equal to* t*, for* $t = 1, 2, 3, 4$*.*

Definition 4.5. *We say that an quiver of type* $\widetilde{E_6}$ *has*

i) *symmetric orientation of kind (a) if* $s(\alpha_1) = e_1$*,* $s(\alpha_4) = e_5$*,* $s(\alpha_6) = e_7$ *or* $t(\alpha_1) = e_1$*,* $t(\alpha_4) = e_5$ $t(\alpha_6) = e_7$ *and* e_3 *is a source or a sink;*
ii) *symmetric orientation of kind (b) if it is not symmetric of kind (a) and it is symmetric with respect to the side* $3 - 4 - 5$*.*

Definition 4.6. *We say that an quiver of type* $\widetilde{E_7}$ *has symmetric orientation if it is symmetric with respect to the side* $5 - 4$*.*

Remark 4.7. (i) *We say that the quiver* Q *is symmetric with respect to the middle point* $r + 1$ *if that point is center of symmetry of the quiver* Q*.*
(ii) *We say that the quiver* Q *is symmetric with respect to an axis* $i - -i + r$*, if the line obtained with the points* $\{i, i + r\}$ *is symmetry axis of the quiver.*

Let G be a group and we will assume that G is acting trivially on Λ, we have $\Lambda[G] = \prod_{t=1}^{m} \Lambda$. Hence, from now on, we will assume that G is acting non trivially on Λ. Let $H = \{g : g(e_i) = e_i \text{ for all } i \text{ in } Q_0\}$. Clearly H is a normal subgroup of G. Let $T = G/H$, then $1 \to H \to G \to T \to 1$ is a short exact sequence of groups.

Theorem 4.8. *Let* $\Lambda = kQ$ *be a tame hereditary algebra, with* Q *of type* $\widetilde{A_n}$ *(*$n > 1$*),* $\widetilde{D_n}$ *(*$n \geq 4$*),* $\widetilde{E_6}$*,* $\widetilde{E_7}$ *or* $\widetilde{E_8}$*, and* G *a finite abelian group of order* m *acting non trivially on* Λ*, with* m *invertible in* Λ*.*

i) *If* $H = G$ *then* $\Lambda[G] = \prod_{t=1}^{m} \Lambda$*;*

ii) *If* $H \subsetneq G$ *and* $\Lambda = kQ$ *with* Q *of type* $\widetilde{A_n}$ *then*

ii.1) Q *is symmetric not cyclic,* $n = 2r - 1$ *if it is symmetric with respect to one axis, or* $n = 4r' - 1$ *if it is symmetric with respect to a pair of perpendicular axes, the order of* G *is divisible by* 2 *or* 4 *respectively, and* $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda) *_\gamma \mathbb{Z}/2\mathbb{Z}$ *or* $\Lambda[G] \simeq (\prod_{t=1}^{m/4} \Lambda) *_\gamma (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$*;*

ii.2) Q *is cyclic of order* s*, not symmetric,* M *is the smallest natural number such that* $M(s-1)$ *is divisible by* $n+1$*, the order of* G *is divisible by* M *and* $\Lambda[G] \simeq (\prod_{t=1}^{m/M} \Lambda) *_\gamma \mathbb{Z}/M\mathbb{Z}$*;*
or

ii.3) Q *is symmetric and cyclic of order* $r+1$*,* $n = 2r - 1$*, the order of* G *is divisible by* 2 *or* 4 *and* $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda) *_\gamma \mathbb{Z}/2\mathbb{Z}$ *or* $\Lambda[G] \simeq (\prod_{t=1}^{m/4} \Lambda) *_\gamma (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$.

iii) *If* $H \subsetneq G$ *and* $\Lambda = kQ$ *with* Q *of type* $\widetilde{D_n}$*,* $n > 4$ *then*

iii.1) Q *has symmetric orientation of kind* (*b*)*, not* (*a*)*, the order of* G *is even and* $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda) *_\gamma \mathbb{Z}/2\mathbb{Z}$*,*

iii.2) Q *has symmetric orientation of kind* (*a*)*, not* (*b*)*, the order of* G *is divisible by* 2 *or* 4*, and* $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda) *_\gamma \mathbb{Z}/2\mathbb{Z}$ *or* $\Lambda[G] \simeq (\prod_{t=1}^{m/4} \Lambda) *_\gamma (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$*,*
or

iii.3) Q *has symmetric orientation of kind* (*a*) *and* (*b*)*, the order of* G *is divisible by* 2 *or* 4 *and* $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda) *_\gamma \mathbb{Z}/2\mathbb{Z}$*,* $\Lambda[G] \simeq (\prod_{t=1}^{m/4} \Lambda) *_\gamma (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ *or* $\Lambda[G] \simeq (\prod_{t=1}^{m/4} \Lambda) *_\gamma \mathbb{Z}/4\mathbb{Z}$.

iv) *If* $H \subsetneq G$ *and* $\Lambda = kQ$ *with* Q *of type* $\widetilde{D_4}$ *then*

iv.1) Q *is symmetric of order* 1 *or* 3*, the order of* G *is divisible by* 2 *or* 3 *and* $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda) *_\gamma \mathbb{Z}/2\mathbb{Z}$ *or* $\Lambda[G] \simeq (\prod_{t=1}^{m/3} \Lambda) *_\gamma \mathbb{Z}/3\mathbb{Z}$*;*

iv.2) Q *is symmetric of order* 2*, the order of* G *is divisible by* 2 *or* 4 *and* $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda) *_\gamma \mathbb{Z}/2\mathbb{Z}$ *or* $\Lambda[G] \simeq (\prod_{t=1}^{m/4} \Lambda) *_\gamma (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$*;*
or

iv.3) Q *is symmetric of order* 4*, the order of* G *is divisible by* 2*,* 3 *or* 4 *and* $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda) *_\gamma \mathbb{Z}/2\mathbb{Z}$*,* $\Lambda[G] \simeq (\prod_{t=1}^{m/3} \Lambda) *_\gamma \mathbb{Z}/3\mathbb{Z}$*,* $\Lambda[G] \simeq (\prod_{t=1}^{m/4} \Lambda) *_\gamma (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ *or* $\Lambda[G] \simeq (\prod_{t=1}^{m/4} \Lambda) *_\gamma \mathbb{Z}/4\mathbb{Z}$ *;*

v) *If* $H \subsetneq G$ *and* $\Lambda = kQ$ *with* Q *of type* $\widetilde{E_6}$ *then*

v.1) Q *has symmetric orientation of kind* (*a*)*, the order of* G *is divisible by* 2 *or* 3 *and* $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda) *_\gamma \mathbb{Z}/2\mathbb{Z}$ *or* $\Lambda[G] \simeq (\prod_{t=1}^{m/3} \Lambda) *_\gamma \mathbb{Z}/3\mathbb{Z}$*;*
or

v.2) Q *has symmetric orientation of kind* (*b*)*, the order of* G *is divisible by* 2 *and* $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda) *_\gamma \mathbb{Z}/2\mathbb{Z}$*;*

vi) *If* $H \subsetneq G$ *and* $\Lambda = kQ$ *with* Q *of type* $\widetilde{E_7}$ *then* Q *has symmetric orientation,* G *is a group of even order and* $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda) *_\gamma \mathbb{Z}/2\mathbb{Z}$*;*

vii) *If* $\Lambda = kQ$ *with* Q *of type* $\widetilde{E_8}$ *then* $H = G$ *and* $\Lambda[G] = \prod_{t=1}^{m} \Lambda$.

Proof. In order to prove the theorem, we need a precise description of all the possible actions of G on $\Lambda = kQ$, for each type and orientation of Q. We use Proposition 2.1 to describe all possible actions of G on kQ with Q of type $\widetilde{A_n}$, $\widetilde{D_n}$, $\widetilde{E_6}$, $\widetilde{E_7}$ or $\widetilde{E_8}$. We observe that we identify the elements of $\mathbb{Z}/(n+1)\mathbb{Z}$ with the natural numbers $1, 2, \cdots, n+1$ in the indexes of the idempotents e_i.

i) Theorem 2.8 cannot be applied because Λ is not simply connected. Using [16, (2.3), (2.4)] for this case we have $\Lambda[G] = \prod_{t=1}^{m} \Lambda$ (where $s = 1$, $|G| = n$, $m = n/s = n$, $\mu = 0. \cdots, n-1$ under the conditions of [16, (2.3)]) .

ii) Let $\Lambda = kQ$ with Q of type $\widetilde{A_n}$ and let $g \in G$, $g \notin H$. Assume that g fixes at least one point, say $g(e_j) = e_j$. Then $g(e_{j+1}) = e_{j+1}$ or $g(e_{j+1}) = e_{j-1}$. In the first case, repeating this procedure we have that $g(e_i) = e_i$ for all i, and so $g \in H$, a contradiction. In the second case, we get that $g(e_i) = e_{2j+n+1-i}$, for all i, and this determines the orientation of the arrows. If $n = 2r$, we have $g(e_{r+j}) = e_{r+j+1}$ and $g(e_{r+j+1}) = e_{r+j}$, a contradiction since there is only one arrow joining e_{r+j} and e_{r+j+1}. So $n = 2r-1$ and Q has symmetric orientation. Moreover, $g^2(e_i) = e_i$ for all i, so $g^2 \in H$.

Now let $g \in G$, $g \notin H$, $g(e_i) \neq e_i$ for all i. Let $g(e_1) = e_j$. If $g(e_2) = e_{j-1}$, the previous reasoning says that there must exist a middle point between 2 and $j-1$ that will be fixed by g, a contradiction. So $g(e_2) = e_{j+1}$, and inductively we get that $g(e_i) = e_{j-1+i}$. This determines the orientation of the arrows, and so Q is cyclic of order s, where s is the first element in the set $\{j \in \mathbb{N} : \text{there exists } g \in G \text{ such that } g(e_i) = e_{j-1+i}\}$. Let $g_0 \in G$ be such that $g_0(e_i) = e_{s-1+i}$. Let $j-1 = q(s-1)+t$, with $0 \leq t < s-1$. Then $gg_0^{-q}(e_i) = g(e_{-q(s-1)+i}) = e_{(j-1)-q(s-1)+i} = e_{t+i}$. If $t \neq 0$, we get a contradiction to the minimality of s. So $j-1 = q(s-1)$ and $gg_0^{-q} \in H$ and $\overline{g} = \overline{g_0}^q$ in T.

We denote by

$$G_1 = \{g \in G : g \notin H, g(e_j) = e_j \text{ for some } j\},$$

$$G_2 = \{g \in G : g \notin H, g(e_i) = e_{j-1+i} \text{ for some } j,\ 1 < j \leq n+1, \forall i\}.$$

We have already proved that $G_1 \neq \emptyset$ if and only if Q has symmetric orientation, and $G_2 \neq \emptyset$ if and only if Q is cyclic of order s.

Assume first that Q is cyclic of order s and is not symmetric. We have seen that $\overline{g} = \overline{g_0}^q$ in T for any $g \in G_2$. Moreover, $g_0^h(e_i) = e_{h(s-1)+i}$, so $g_0^h \in H$ if and only if $h(s-1)$ is divisible by $n+1$. Let M be the smallest natural number such that $M(s-1)$ is divisible by $n+1$. We conclude that $T \simeq \mathbb{Z}/M\mathbb{Z}$ in this case.

Assume now that Q is symmetric but not cyclic, and let $g, g' \in G_1$, that is, $g(e_i) = e_{2j+n+1-i}$ and $g'(e_i) = e_{2t+n+1-i}$ for some j and t, $n = 2r-1$. We assume, without loss of generality, that $t > j$. If $gg' = g'g$, then $e_{2(j-t)+i} = gg'(e_i) = g'g(e_i) = e_{2(t-j)+i}$, so $2(t-j)$ is divisible by r. Since $1 \leq t, j \leq n+1$, we have that $2(t-j) = qr$ for $q = 0, 1, 2, 3$. If $q = 0, 2$ then $g(e_i) = g'(e_i)$ and hence $g^{-1}g' \in H$, that is, $\overline{g} = \overline{g'}$ in T, and hence

$T \simeq \mathbb{Z}/2\mathbb{Z}$ in this case. If $q = 1, 3$ then $n = 4r' - 1$ and Q is symmetric with respect to the axes $j - -j + 2r'$ and $j + r' - -j + 3r'$ and in this case $T \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Finally, assume that Q is symmetric ($n = 2r - 1$) and cyclic of order s, and let $g \in G_1$ and $g_0 \in G_2$, that is, $g(e_i) = e_{2j+n+1-i}$ and $g_0(e_i) = e_{s-1+i}$. If $gg_0 = g_0g$, then $e_{2j+n+1-s+1-i} = gg_0(e_i) = g_0g(e_i) = e_{s-1+2j+n+1-i}$, so $s - 1$ is divisible by r. Since $1 < s \leq n + 1$, we have that $s - 1 = r$ and in this case $M = 2$ and $T \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Finally, from Theorem 2.2 and [16, (2.3)] we have the conclusions, that is $\Lambda[G] \simeq \Lambda[H] *_\gamma T$ or $\Lambda[G] \simeq \Lambda[H] *_\gamma T \simeq (\prod_{t=1}^{s} \Lambda) *_\gamma T$.

iii) Let $\Lambda = kQ$ with Q of type D_n, $n > 4$. Let $g \in G$, $g \notin H$. We observe first that $\{g(e_3), g(e_{n-1})\} = \{e_3, e_{n-1}\}$, see Proposition 2.1.

Assume first that $g(e_3) = e_3$ and $g(e_{n-1}) = e_{n-1}$. Then $g(e_i) = e_i$ for all $i = 4, \cdots, n-2$. Since $g \notin H$, we must have $g(e_1) = e_2$ or $g(e_n) = g(e_{n+1})$. This implies that Q has symmetric orientation of kind (a) and all possible actions are given by:

1) $g_1(e_1) = e_2$, $g_1(e_2) = e_1$, $g_1(e_n) = e_{n+1}$, $g_1(e_{n+1}) = e_n$;
2) $g_2(e_1) = e_2$, $g_2(e_2) = e_1$, $g_2(e_n) = e_n$, $g_2(e_{n+1}) = e_{n+1}$;
3) $g_3(e_1) = e_1$, $g_3(e_2) = e_2$, $g_3(e_n) = e_{n+1}$, $g_3(e_{n+1}) = e_n$.

Since $g_1^2, g_2^2, g_3^2 \in H$ and $g_2g_3 = g_1 = g_3g_2$, we conclude that $T \simeq \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Assume now that $g(e_3) = e_{n-1}$ and $g(e_{n-1}) = e_3$. Then, using the same argument as in the proof of Theorem 3.6 in the case of A_n, we conclude that Q is symmetric of kind (b). If Q is not of kind (a), the unique possible non trivial action on the complete set of idempotents is given by $g(e_1) = e_{n+1}$, $g(e_{n+1} = g(e_1)$, $g(e_2) = e_n$, $g(e_n) = e_2$ and $g(e_i) = e_{n-i+2}$ for all $i = 3, \cdots, n-1$. In this case, $T \simeq \mathbb{Z}/2\mathbb{Z}$.

To finish with this case, we have to assume that Q is symmetric of kind (a) and (b). Then all the possible non trivial actions are given by

1) $g_1(e_1) = e_{n+1}$, $g_1(e_2) = e_n$, $g_1(e_n) = e_2$, $g_1(e_{n+1}) = e_1$, $g_1(e_i) = e_{n-i+3}$ for all $i = 3, \cdots, n-1$;
2) $g_2(e_1) = e_n$, $g_2(e_2) = e_{n+1}$, $g_2(e_n) = e_1$, $g_1(e_{n+1}) = e_2$, $g_2(e_i) = e_{n-i+3}$ for all $i = 3, \cdots, n-1$;
3) $g_3(e_1) = e_{n+1}$, $g_3(e_2) = e_n$, $g_3(e_n) = e_1$, $g_3(e_{n+1}) = e_2$, $g_3(e_i) = e_{n-i+3}$ for all $i = 3, \cdots, n-1$;
4) $g_4(e_1) = e_n$, $g_4(e_2) = e_{n+1}$, $g_4(e_n) = e_2$, $g_4(e_{n+1}) = e_1$, $g_4(e_i) = e_{n-i+3}$ for all $i = 3, \cdots, n-1$;
5) $g_5(e_1) = e_2$, $g_5(e_2) = e_1$, $g_5(e_n) = e_n$, $g_5(e_{n+1}) = e_{n+1}$, $g_5(e_i) = e_i$ for all $i = 3, \cdots, n-1$;
6) $g_6(e_1) = e_1$, $g_6(e_2) = e_2$, $g_6(e_n) = e_{n+1}$, $g_6(e_{n+1}) = e_n$, $g_6(e_i) = e_i$ for all $i = 3, \cdots, n-1$;
7) $g_7(e_1) = e_2$, $g_7(e_2) = e_1$, $g_7(e_n) = e_{n+1}$, $g_7(e_{n+1}) = e_n$, $g_7(e_i) = e_i$ for all $i = 3, \cdots, n-1$.

Now $g_1^2, g_2^2, g_5^2, g_6^2, g_7^2 \in H$, $g_3^4 \in H$, $g_4^4 \in H$ and $g_4g_3 \in H$. Moreover, if $i \neq j$, then $g_ig_j = g_jg_i$ implies that (i,j) is equal to $(1,2)$, $(1,7)$, $(2,7)$, $(3,4)$, $(3,7)$ or $(4,7)$. Moreover $g_1g_2g_7, g_4g_7 \in H$. Hence $T \simeq \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$.

Finally, from Theorem 2.2 and Theorem 2.8 we have that $\Lambda[G] \simeq \Lambda[H] *_\gamma T \simeq (\prod_{t=1}^s \Lambda) *_\gamma T$.

iv) Let $\Lambda = kQ$ with Q of type $\widetilde{D_4}$. Let $g \in G$, $g \notin H$. Necessarily, by Proposition 2.1, $g(e_3) = e_3$. If Q is symmetric of order 1 or 3, the same reasoning made for D_4 in the proof of Theorem 3.6 holds, and hence $T \simeq \mathbb{Z}/2\mathbb{Z}$ or $T \simeq \mathbb{Z}/3\mathbb{Z}$.

If Q is symmetric of order 2, assume that $s(\alpha_1) = s(\alpha_2) = 3 = t(\alpha_3) = t(\alpha_4)$. Hence all the possible cases for $g \notin H$ are:
- i) $g_1(e_1) = e_2$, $g_1(e_2) = e_1$, $g_1(e_4) = e_5$, $g_1(e_5) = e_4$;
- ii) $g_2(e_1) = e_1$, $g_2(e_2) = e_2$, $g_2(e_4) = e_5$, $g_2(e_5) = e_4$;
- iii) $g_3(e_1) = e_2$, $g_3(e_2) = e_1$, $g_3(e_4) = e_4$, $g_3(e_5) = e_5$.

In fact $g_1^2, g_2^2, g_3^2 \in H$, $g_2g_3(e_i) = g_1(e_i)$ and $g_sg_j(e_i) = g_jg_s(e_i)$ for all s, j with $1 \leq i, j \leq 3$ and for all $i = 1, \cdots, n$. Consequently $T \simeq \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

If Q is symmetric of order 4, all the possible cases for $g \notin H$ are in one to one correspondence with the non trivial permutations of e_1, e_2, e_4, e_5. Hence $T \simeq \mathbb{Z}/2\mathbb{Z}$, $Z/3\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $Z/4\mathbb{Z}$ (all the possible abelian subgroups of S_4).

Finally, from Theorem 2.2 and Theorem 2.8 we have that $\Lambda[G] \simeq \Lambda[H] *_\gamma T \simeq (\prod_{t=1}^s \Lambda) *_\gamma T$.

v) This case follows from an argument similar to what has been done in the proof of Theorem 3.6 for the case D_4 (for any $g \in G$, $g(e_3) = e_3$ and the action of g on e_2, e_4, e_6 is uniquely determined by the action of g in e_1, e_5 and e_7).

vi) Let $\Lambda = kQ$ with Q of type $\widetilde{E_7}$, and let $g \notin H$. By Proposition 2.1, $g(e_4) = e_4$ and then $g(e_5) = e_5$. Now $g(e_1) = e_1$ or e_8. In the first case we get that $g(e_i) = e_i$ for all i, and so $g \in H$, a contradiction. Then $g(e_1) = e_8$ and this determines completely the orientation of the arrows, that is, Q has symmetric orientation, and the action of g on the complete set of idempotents of Λ. Since $g^2 \in H$, we can deduce that $|G| = m = 2s$ is an even number. Let $g' \in G$, $g' \notin H$. By the previous reasoning, g' and g act in the unique possible way on the complete set of idempotents of Λ. Then $gg'(e_i) = e_i$ for all i, hence $gg' \in H$, that is, $\overline{g'} = \overline{g}^{-1} = \overline{g}$ in T. So $T \simeq \mathbb{Z}/2\mathbb{Z}$ and $\Lambda[G] \simeq \Lambda[H] *_\gamma \mathbb{Z}/2\mathbb{Z} \simeq (\prod_{t=1}^s \Lambda) *_\gamma \mathbb{Z}/2\mathbb{Z}$, see Theorem 2.2 and Theorem 2.8.

vii) If we consider the case Q of type $\widetilde{E_8}$, the unique possible action on the set of idempotents is the trivial one. Hence $G = H$, $T = 1$ and the result follows from i).

□

Corollary 4.9. *Let $\Lambda = kQ$ be an hereditary algebra, with Q of type $\widetilde{A_n}$ $(n > 1)$, $\widetilde{D_n}$ $(n \geq 4)$, $\widetilde{E_6}$, $\widetilde{E_7}$ or $\widetilde{E_8}$, and G an abelian group of order m acting on Λ, with m invertible in Λ. Suppose that G does not act trivially on the set $\{e_1, \cdots, e_{n+1}\}$ of idempotents of Λ and H acts trivially on Λ.*

- i) *If $\Lambda = kQ$ with Q of type $\widetilde{A_n}$ $(n > 1)$ and*
  - i.1) *Q is symmetric not cyclic, $n = 2r-1$ then $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda)[\mathbb{Z}/2\mathbb{Z}]$ or $\Lambda[G] \simeq (\prod_{t=1}^{m/4} \Lambda)[\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}]$;*
  - i.2) *Q is cyclic of order s, not symmetric, then $\Lambda[G] \simeq (\prod_{t=1}^{m/M} \Lambda)[\mathbb{Z}/M\mathbb{Z}]$;*
    or
  - i.3) *Q is symmetric and cyclic of order $r+1$, $n = 2r-1$, then $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda)[\mathbb{Z}/2\mathbb{Z}]$ or $\Lambda[G] \simeq (\prod_{t=1}^{m/4} \Lambda)[\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}]$.*
- ii) *If $\Lambda = kQ$ with Q of type $\widetilde{D_n}$, $n > 4$, and*
  - ii.1) *Q with symmetric orientation of kind (b), not (a), then*
    $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda)[\mathbb{Z}/2\mathbb{Z}]$,
  - ii.2) *Q with symmetric orientation of kind (a), not (b), then $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda)[\mathbb{Z}/2\mathbb{Z}]$ or $\Lambda[G] \simeq (\prod_{t=1}^{m/4} \Lambda)[\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}]$,*
    or
  - ii.3) *Q with symmetric orientation of kind (a) and (b) then*
    $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda)[\mathbb{Z}/2\mathbb{Z}]$, $\Lambda[G] \simeq (\prod_{t=1}^{m/4} \Lambda)[\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}]$ or $\Lambda[G] \simeq (\prod_{t=1}^{m/4} \Lambda)[\mathbb{Z}/4\mathbb{Z}]$.
- iii) *If $\Lambda = kQ$ with Q of type $\widetilde{D_4}$ and*
  - iii.1) *Q is symmetric of order 1 or 3 then $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda)[\mathbb{Z}/2\mathbb{Z}]$ or $\Lambda[G] \simeq (\prod_{t=1}^{m/3} \Lambda)[\mathbb{Z}/3\mathbb{Z}]$;*
  - iii.2) *Q is symmetric of order 2 then $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda)[\mathbb{Z}/2\mathbb{Z}]$ or $\Lambda[G] \simeq (\prod_{t=1}^{m/4} \Lambda)[\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}]$;*
    or
  - iii.3) *Q is symmetric of order 4 then $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda)[\mathbb{Z}/2\mathbb{Z}]$, $\Lambda[G] \simeq (\prod_{t=1}^{m/3} \Lambda)[\mathbb{Z}/3\mathbb{Z}]$ or $\Lambda[G] \simeq (\prod_{t=1}^{m/4} \Lambda)[\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}]$;*
- iv) *If $\Lambda = kQ$ with Q of type $\widetilde{E_6}$ and*
  - v.1) *Q with symmetric orientation of kind (a) then $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda)[\mathbb{Z}/2\mathbb{Z}]$ or $\Lambda[G] \simeq (\prod_{t=1}^{m/3} \Lambda)[\mathbb{Z}/3\mathbb{Z}]$;*
    or
  - iv.2) *Q with symmetric orientation of kind (b) then $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda)[\mathbb{Z}/2\mathbb{Z}]$;*
- v) *If $\Lambda = kQ$ with Q of type $\widetilde{E_7}$ and Q with symmetric orientation then $\Lambda[G] \simeq (\prod_{t=1}^{m/2} \Lambda)[\mathbb{Z}/2\mathbb{Z}]$.*

Proof. It follows from Theorem 4.8 and Corollary 2.7. □

The following corollary follows easily from [16, (2.3), (2.4)].

Corollary 4.10. *Let $\Lambda = kQ$ be an hereditary algebra, with Q of type $\widetilde{A_n}$ $(n > 1)$, $\widetilde{D_n}$ $(n \geq 4)$, $\widetilde{E_6}$, $\widetilde{E_7}$ or $\widetilde{E_8}$, and G an cyclic group of order m acting on Λ, with m invertible in Λ.*

i) *If $\Lambda = kQ$ with Q of type $\widetilde{A_3}$ and $G = \mathbb{Z}/2\mathbb{Z}$ is acting non trivially on the set $\{e_1, \cdots, e_4\}$ of idempotents of Λ then the skew group algebra $\Lambda[\mathbb{Z}/2\mathbb{Z}]$ is Morita equivalent to an algebra kQ' with Q' of type $\widetilde{D_4}$.*

ii) *If $\Lambda = kQ$ with Q of type $\widetilde{A}_{5+2r}$, $r \geq 0$ and $G = \mathbb{Z}/2\mathbb{Z}$ is acting non trivially on the set $\{e_1, \cdots, e_n\}$ of idempotents of Λ then the skew group algebra $\Lambda[\mathbb{Z}/2\mathbb{Z}]$ is Morita equivalent to an algebra kQ' with Q' of type $\widetilde{D}_{5+r}$.*

iii) *If $\Lambda = kQ$ with Q of type $\widetilde{D_4}$, and $G = \mathbb{Z}/2\mathbb{Z}$ is acting non trivially on the set $\{e_1, \cdots, e_5\}$ of idempotents of Λ then the skew group algebra $\Lambda[\mathbb{Z}/2\mathbb{Z}]$ is Morita equivalent to an algebra kQ' with Q' of type $\widetilde{A_3}$. If $G = \mathbb{Z}/3\mathbb{Z}$ is acting non trivially on the set $\{e_1, \cdots, e_5\}$ of idempotents of Λ then the skew group algebra $\Lambda[\mathbb{Z}/3\mathbb{Z}]$ is Morita equivalent to an algebra kQ' with Q' of type $\widetilde{E_6}$. If $G = \mathbb{Z}/4\mathbb{Z}$ is acting non trivially on the set $\{e_1, \cdots, e_5\}$ of idempotents of Λ then the skew group algebra $\Lambda[\mathbb{Z}/4\mathbb{Z}]$ is Morita equivalent to an algebra kQ' with Q' of type $\widetilde{D_4}$.*

iv.1) *If $\Lambda = kQ$ with Q of type $\widetilde{D}_{5+r}$, $r \geq 0$ and $G = \mathbb{Z}/2\mathbb{Z}$ is acting non trivially on the set $\{e_1, \cdots, e_n\}$ of idempotents of Λ and the action of $g \in G$ on Λ is induced by a reflection in the quiver, then the skew group algebra $\Lambda[\mathbb{Z}/2\mathbb{Z}]$ is Morita equivalent to an algebra kQ' with Q' of type $\widetilde{A}_{5+2r}$.*

iv.2) *If $\Lambda = kQ$ with Q of type $\widetilde{D}_{2r}$, $r \geq 3$ and $G = \mathbb{Z}/2\mathbb{Z}$ is acting non trivially on the set $\{e_1, \cdots, e_n\}$ of idempotents of Λ and the action of $g \in G$ on Λ is induced by a reflection with respect the middle point e_{r+1} in the quiver,*

   a) *if Q is of type $\widetilde{D_6}$ then the skew group algebra $\Lambda[\mathbb{Z}/2\mathbb{Z}]$ is Morita equivalent to an algebra kQ' with Q' of type $\widetilde{D_4}$;*

   b) *if Q is of type $\widetilde{D}_{2r}$, $r \geq 4$ then the skew group algebra $\Lambda[\mathbb{Z}/2\mathbb{Z}]$ is Morita equivalent to an algebra kQ' with Q' of type $\widetilde{D}_{2k-3}$.*

v) *If $\Lambda = kQ$ with Q of type $\widetilde{E}_6$ and $G = \mathbb{Z}/2\mathbb{Z}$ is acting non trivially on the set $\{e_1, \cdots, e_7\}$ of idempotents of Λ then the skew group algebra $\Lambda[\mathbb{Z}/2\mathbb{Z}]$ is Morita equivalent to an algebra kQ' with Q' of type $\widetilde{E}_7$. If $G = \mathbb{Z}/3\mathbb{Z}$ is acting non trivially on the set $\{e_1, \cdots, e_7\}$ of idempotents of Λ then the skew group algebra $\Lambda[\mathbb{Z}/2\mathbb{Z}]$ is Morita equivalent to an algebra kQ' with Q' of type $\widetilde{D}_4$.*

vi) *If $\Lambda = kQ$ with Q of type $\widetilde{E}_7$ and $G = \mathbb{Z}/2\mathbb{Z}$ is acting non trivially on the set $\{e_1, \cdots, e_8\}$ of idempotents of Λ then the skew group algebra $\Lambda[\mathbb{Z}/2\mathbb{Z}]$ is Morita equivalent to an algebra kQ' with Q' of type $\widetilde{E}_7$.*

The case $\widetilde{A_1}$ is not considered in Theorem 4.8 because the techniques we use do not hold in this case. In fact, Λ is the Kronecker algebra and Proposition 2.1 does not hold since this algebra has double arrows. Moreover, Theorem 2.8 cannot be applied because Λ is not simply connected. We will only consider the case of a cyclic group acting on the Kronecker algebra, then it is possible to apply directly [16, (2.3)].

If G is a cyclic group acting on Λ, the Kronecker algebra, $|G| = m$ with m invertible in Λ then all possible actions are given by:

$\widetilde{A}_1$
$$\begin{tikzcd} \overset{\bullet}{1} \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & \overset{\bullet}{2} \end{tikzcd}$$

1) $g(e_i) = e_i$, $i = 1, 2$, $g(\alpha) = \alpha$, $g(\beta) = \beta$ and in this case the skew group algebra $\Lambda[G] \simeq (\prod_{t=1}^{m} \Lambda)$. [16, (2.3)]
2) $g(e_i) = e_i$, $i = 1, 2$, $g(\alpha) = \lambda\alpha$, $g(\beta) = \mu\beta$ with $\lambda^m = \mu^m = 1$,
   2.1) if $\lambda = 1$ and $\mu \neq 1$ then the skew group algebra $\Lambda[G]$ is hereditary of type $\widetilde{A}_{2m}$. [16, (2.3)]
   2.2) If $\lambda \neq 1$ and $\mu \neq 1$ then the skew group algebra $\Lambda[G] \simeq \prod_{t=1}^{m} \Lambda$. [16, (2.3)]
3) $g(e_i) = e_i$, $i = 1, 2$, $g(\alpha) = \lambda\beta$, $g(\beta) = \mu\alpha$ with $\lambda^m = \mu^m = 1$, and in this case the skew group algebra $\Lambda[G]$ is hereditary of type $\prod_{t=1}^{2m} A_1$. [16, (2.3)]

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