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FUNCTIONAL VERSIONS OF THE CARISTI-KIRK THEOREM

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ABSTRACT. Many functional versions of the Caristi-Kirk fixed point theorem are nothing but logical equivalents of the result in question.

1. INTRODUCTION

Let (M, d) be a complete metric space; and $x \vdash \varphi(x)$, some function from M to $R_+ := [0, \infty[$ with

(1a) φ is d-lsc over M ($\liminf_n \varphi(x_n) \geq \varphi(x)$, whenever $x_n \to x$).

Let also $T : M \to M$ be a selfmap of M; and put $\text{Fix}(T) = \{x \in M; x = Tx\}$, $\text{Per}(T) = \cup\{\text{Fix}(T^n); n \geq 1\}$; each point of the former (latter) will be called *fixed* (*periodic*) under T. The following 1975 fixed point result in Caristi and Kirk [10] is basic for us.

Theorem 1. *Suppose that*

(1b) $d(x, Tx) \leq \varphi(x) - \varphi(Tx)$, *for all* $x \in M$.

Then, necessarily,

$$\text{Fix}(T) = \text{Per}(T) \neq \emptyset \quad (T \textit{ is strongly fp-admissible}); \tag{1.1}$$

hence, in particular,

$$\text{Fix}(T) \neq \emptyset \quad (T \textit{ is fp-admissible}). \tag{1.2}$$

The original proof of this result is by transfinite induction; see also Wong [40]. (It works as well for highly specialized versions of Theorem 1; cf. Kirk and Saliga [19]). Note that, in terms of the associated (to φ) order

(a1) $(x, y \in M)$ $x \leq y$ iff $d(x, y) \leq \varphi(x) - \varphi(y)$

the contractivity condition (1b) reads

(1c) $x \leq Tx$, for all $x \in M$ (i.e.: T is *progressive*).

So, by the Bourbaki meta-theorem [7], the underlying result is logically equivalent with the Zorn maximality principle subsumed to the precise order; i.e., with Ekeland's variational principle [12]. This tells us that the sequential type argument used in its proof (cf. Section 6) is also working in our framework; see also the paper

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of Pasicki [25]. A proof of Theorem 1 involving the chains of the structure $(M, \leq)$ may be found in Turinici [33]; and its sequential translation has been developed in Dancs, Hegedus and Medvegyev [11]. Further aspects involving the general case may be found in Brunner [9] and Manka [21]; see also Taskovic [32], Valyi [39], Nemeth [23] and Isac [14].

Now, the Caristi-Kirk fixed point theorem found (especially via Ekeland's approach) some basic applications to control and optimization, generalized differential calculus, critical point theory and normal solvability; see the above references for details. So, it must be not surprising that, soon after its formulation, many extensions of Theorem 1 were proposed. (These involve its standard version related to (1.2); and referred to as Theorem 1(st). But, only a few are concerned with the extended version of the same, related to (1.1); and referred to as Theorem 1(ex)). For example, in the 1982 paper by Ray and Walker [28], the following result of this type was obtained. Call the function $b : R_+ \to R_+$, *semi-normal* when

(b1) b is decreasing and $b(R_+) \subseteq R_+^0 :=]0, \infty[$;

and *normal* provided (in addition)

(c1) $B(\infty) = \infty$; where $B(t) = \int_0^t b(\tau)d\tau,\ t \geq 0$.

(Further aspects involving these notions will be developed in Section 2).

Theorem 2. *Assume that there exists a normal function $b : R_+ \to R_+$ and some $a \in M$ with*

(1d) $b(d(a,x))d(x,Tx) \leq \varphi(x) - \varphi(Tx)$, *for all $x \in M$.*

Then, T has at least one fixed point in M.

Clearly, Theorem 2 includes Theorem 1(st), to which it reduces when $b = 1$. The reciprocal inclusion also holds (cf. Park and Bae [24]). Summing up, Theorem 2 is but a logical equivalent of Theorem 1(st). (For a different proof of this, we refer to Section 4 below).

On the other hand, in the 2005 paper by Turinici [38] the following fixed point result was established. Call the function $c : R_+ \to R_+$, *right locally bounded from above* at $\alpha \in R_+$ when

(d1) $\sup c([\alpha, \beta]) < \infty$, for some $\beta = \beta(\alpha) > \alpha$.

If this holds for each $\alpha \in R_+$, we say that c is *right locally bounded from above* (on R_+). Further, let us say that $H : R_+^2 \to R_+$ is *locally bounded (above)* in case: the image of each bounded part in R_+^2 is bounded (in R_+).

Theorem 3. *Let the right locally bounded from above (on R_+) $c : R_+ \to R_+$ and the locally bounded (above) $H : R_+^2 \to R_+$ be such that*

(1e) $d(x,Tx) \leq H(c(\varphi(x)), c(\varphi(Tx)))[\varphi(x) - \varphi(Tx)]$, $\forall x \in M$.

Then, conclusions of Theorem 2 are retainable.

This result extends the one in Bae, Cho and Yeom [4]; see also Bae [3]. But, as precise there, all these extend Theorem 1(st); hence, so does Theorem 3. (In fact, a direct verification is available, via $c = 1, H = 1$). The converse inclusion is also true, by the provided proof; wherefrom Theorem 3 is a logical equivalent of Theorem 1(st). For a technical extension of such facts we refer to Section 4.

Further, in Section 5, we show that our developments include as well the fixed point statement (comparable with Theorem 1(ex)) due to Feng and Liu [13]. And in Section 6, some extensions of Theorem 1 are given, in terms of maximality principles over metrical structures comparable with the 1976 Brezis-Browder's [8]. In fact, the obtained statements are extendable beyond the metrizable context; we shall discuss them elsewhere.

2. Semi-normal functions

(A) Let $b : R_+ \to R_+$ be a semi-normal function (cf. (b1)). In particular, it is Riemann integrable on each compact interval of R_+ and

$$\int_p^q b(\xi)d\xi = (q-p)\int_0^1 b(p+\tau(q-p))d\tau, \quad 0 \le p < q < \infty. \tag{2.1}$$

Some basic facts involving the couple (b, B) (where the primitive $B : R_+ \to R_+$ is the one of (c1)) are being collected in

Lemma 1. *The following are valid*

(i) $sb(t+s) \le B(t+s) - B(t) \le sb(t)$, $\forall t, s \in R_+$

(ii) B *is a topological order isomorphism from* $[0, \infty[$ *to* $[0, B(\infty)[$*; hence so is* B^{-1} *(between* $[0, B(\infty)[$ *and* $[0, \infty[$*)*

(iii) B *is almost concave:* $t \vdash [B(t+s) - B(t)]$ *is decreasing on* R_+, $\forall s \in R_+$

(iv) B *is concave:* $B(t + \lambda(s-t)) \ge (1-\lambda)B(t) + \lambda B(s)$*, for all* $t, s \in R_+$ *with* $t < s$ *and all* $\lambda \in [0, 1]$

(v) B *is sub-additive and* B^{-1} *is super-additive.*

The proof is evident, by (2.1) above; so, we do not give details. Note that iii) and iv) are equivalent to each other, under ii). This follows from the (non-differential) mean value theorem in Bantaş and Turinici [5].

(B) Now, let $b : R_+ \to R_+$ be a normal function (cf. (b1)+(c1)); note that, in such a case

$$B \text{ and } B^{-1} \text{ are topological order isomorphisms of } R_+. \tag{2.2}$$

Further, let (M, d) be a metric space; and $\Gamma : M \to R_+$, some function with

(2a) $|\Gamma(x) - \Gamma(y)| \le d(x, y)$, $\forall x, y \in M$ (non-expansiveness).

Given the function $\varphi : M \to R_+$ let us attach it the function $\psi = \psi(B, \Gamma; \varphi)$ from M to R_+ as

(a2) $\psi(x) = B^{-1}[B(\Gamma(x)) + (\varphi(x) - \varphi_*)] - \Gamma(x)$, $\quad x \in M$.

(Here, by convention, $\varphi_* = \inf[\varphi(M)]$). This may be viewed as an "explicit" formula; its "implicit" version is given as

(b2) $\varphi(x) = \varphi_* + [B(\Gamma(x) + \psi(x)) - B(\Gamma(x))]$, $\quad x \in M$.

The definition is consistent, via (2.2); moreover, φ is d-lsc if and only if ψ is d-lsc. The following variational completion of these is available.

Lemma 2. *Under these conventions,*

$$b(\Gamma(x))d(x,y) \leq \varphi(x) - \varphi(y) \Longrightarrow d(x,y) \leq \psi(x) - \psi(y). \tag{2.3}$$

Proof. Let the points $x, y \in M$ be as in the premise of this implication. By Lemma 1(i) and the implicit formula (b2), this gives $B(\Gamma(x)+d(x,y)) - B(\Gamma(x)) \leq [B(\Gamma(x)+\psi(x)) - B(\Gamma(x))] - [B(\Gamma(y)+\psi(y)) - B(\Gamma(y))]$; or equivalently (by a simple re-arrangement) $B(\Gamma(x) + d(x,y)) + B(\Gamma(y) + \psi(y)) - B(\Gamma(y)) \leq B(\Gamma(x) + \psi(x))$. On the other hand, (2a) yields $\Gamma(x)+d(x,y) \geq \Gamma(y)$; so, by Lemma 1(iii): $B(\Gamma(x)+ d(x,y) + \psi(y)) - B(\Gamma(x) + d(x,y)) \leq B(\Gamma(y) + \psi(y)) - B(\Gamma(y))$. A simple combination with the previous relation gives $B(\Gamma(x) + d(x,y) + \psi(y)) \leq B(\Gamma(x) + \psi(x))$. It suffices now taking (2.2) into account to get the desired conclusion. □

In particular, the nonexpansivity condition (2a) holds under

(c2) $\Gamma(x) = d(a,x), \quad x \in M, \quad$ for some $a \in M$.

In such a case, Lemma 2 includes directly the statement in Park and Bae [24]. Further aspects may be found in Suzuki [30]; see also Turinici [37].

3. Ray-Walker statements

Let (M,d) be a complete metric space; and $\varphi : M \to R_+$ be a d-lsc function (cf. (1a)). Further, let $b : R_+ \to R_+$ be semi-normal (in the sense of (b1)); and $\Gamma : M \to R_+$ be nonexpansive (cf. (2a)). Given the self-map $T : M \to M$, we are interested in establishing sufficient conditions under which (1.1) or (1.2) be retainable. There are two possible answers, according to the normality condition being or not fulfilled.

(A) The former of these requires the normality setting (of (c1)).

Theorem 4. *Assume that b is normal and*

(3a) $b(\Gamma(x))d(x,Tx) \leq \varphi(x) - \varphi(Tx), \quad$ *for each* $x \in M$.

Then, T is strongly fp-admissible ($\mathrm{Fix}(T) = \mathrm{Per}(T) \neq \emptyset$)*; hence fp-admissible* ($\mathrm{Fix}(T) \neq \emptyset$)*.*

Proof. Let $\psi = \psi(B, \Gamma; \varphi)$ stand for the associated (to φ) function. By the remarks in Section 2, ψ is d-lsc too. On the other hand, (3a) and Lemma 2 give (1b) (modulo ψ). This shows that Theorem 1 applies to (M,d) and ψ; wherefrom, the desired conclusion is clear. □

As in Theorem 1, we have a couple of fixed point statements under this formulation; referred to as Theorem 4(ex) and Theorem 4(st). Clearly, Theorem 4 includes Theorem 1 (for $b = 1$). The reciprocal is also valid, by the argument above. Hence, this result is nothing but a logical equivalent of Theorem 1. On the other hand, when Γ is taken as in (c2), Theorem 4(st) is just Theorem 2 above. Further aspects may be found in Zhong, Zhu and Zhao [41]; see also Lin and Du [20].

(B) Another answer to the same is to be stated in the original semi-normality setting (with (c1) not accepted). Roughly speaking, this is to be obtained via "surrogates" of (c1); like, e.g.,

(3b) there exists $a \in M$ with $\Gamma(a) = 0$, $\varphi(a) - \varphi_* < B(\infty)$.
(Here, as already precise, $\varphi_* := \inf[\varphi(M)]$).

Theorem 5. *Assume that b is semi-normal and (3a)+(3b) hold. Then T has at least one fixed point in M.*

Proof. Without loss, one may assume $\varphi(a) > \varphi_*$; because $\varphi(a) = \varphi_*$ gives (by (3a)) $a \in \mathrm{Fix}(T)$. Denote $M_a = \{x \in M; B(\Gamma(x)) \leq \varphi(a) - \varphi(x)\}$. Clearly, M_a is nonempty closed (by the assumptions about φ and the continuity of $\{B, \Gamma\}$). Further, take some arbitrary fixed $x \in M_a$. By Lemma 1(i) and the choice of Γ one gets (via (3a))

$$B(\Gamma(Tx)) \leq B(\Gamma(x) + d(x,Tx)) - B(\Gamma(x)) + B(\Gamma(x)) \leq$$

$$b(\Gamma(x))d(x,Tx) + B(\Gamma(x)) \leq \varphi(x) - \varphi(Tx) + \varphi(a) - \varphi(x) = \varphi(a) - \varphi(Tx).$$

This shows that $Tx \in M_a$; hence M_a is T-invariant. In addition (for each $x \in M_a$) $B(\Gamma(x)) \leq \varphi(a) - \varphi_*$; wherefrom (cf. (3b)) $\Gamma(x) \leq \rho := B^{-1}(\varphi(a) - \varphi_*)$. This (by (b1)) gives $b(\Gamma(x)) \geq b(\rho) > 0$; hence (again by (3a)) we finally derive (1b) over M_a (modulo $\chi(.) := \varphi(.)/b(\rho)$). Summing up, Theorem 1(st) applies to (M_a, d) and χ; wherefrom, the desired conclusion is clear. □

Now, Theorem 5 includes Theorem 1(st) (for $b = 1$ and Γ as in (c2)). The reciprocal is also true, by the argument above; hence, Theorem 5 is a logical equivalent of Theorem 1(st). Combining with a preceding fact, one therefore derives that Theorem 4(st) includes Theorem 5. Concerning this aspect, note that the function $b : R_+ \to R_+$ given as (for some $\lambda < -1$)

$$b(t) = 1, t \in [0,1[; \quad b(t) = t^\lambda, t \in [1,\infty[$$

is semi-normal; but not normal. Hence, this inclusion is not obtainable in a direct way. Further technical aspects may be found in Turinici [36]; see also Jachymski [16].

4. BCY approaches

Let again (M,d) be a complete metric space; and $\varphi : M \to R_+$ be some d-lsc function. Given the self-map $T : M \to M$, the "dual" way of getting (1.2) is by using contractivity condition like in Theorem 3. A natural extension of these is as below. Let $H : R_+^2 \to R_+$ be a mapping. We say that the function $c : R_+ \to R_+$ is *right locally H-proper* at $\alpha \in R_+$ if there exists $\beta = \beta(\alpha) > \alpha$ and a strictly increasing continuous function $h : [\alpha, \beta] \to R_+$ with

(a4) $H(c(t), c(s)) \leq \dfrac{h(t) - h(s)}{t - s}$, when $\alpha < s < t < \beta$.

If $\alpha \in R_+$ is generic in such a convention, we say that c is *right locally H-proper* (on R_+). Assume that (H, c) is endowed with this last property. The following fixed point result is available.

Theorem 6. *Let the contractivity condition (1e) be true. Then, T has at least one fixed point in M.*

Proof. If $\varphi(y) \leq \varphi(Ty)$ for some $y \in M$ then (1e) gives $d(y, Ty) = 0$; wherefrom $y = Ty$. So, without loss, one may assume that

(4a) $\varphi(x) > \varphi(Tx) > \varphi_*$, for each $x \in M$;

where, as usually, $\varphi_* = \inf[\varphi(M)]$. By the right local H-properness of c at φ_*, there must be some $\beta > \varphi_*$ and a strictly increasing continuous function $h : [\varphi_*, \beta] \to R_+$ in such a way that (a4) is retainable (with $\alpha = \varphi_*$). Take some $x_0 \in M$ with $\varphi(x_0) \leq \beta$ (possibly, by the choice of β); and put $M_0 = \{x \in M : \varphi(x) \leq \varphi(x_0)\}$. This is a nonempty part of M (since it contains x_0); which, in addition, is closed (by the choice of φ) and T-invariant (from (4a)). Further, define the function (from M_0 to R_+) $\psi(x) = h(\varphi(x)), x \in M_0$; it is d-lsc over M_0, by the d-lsc property of φ and the choice of h. Finally, (1e) yields (via (a4)+(4a)) the evaluation (1b) on M_0 (modulo ψ). Summing up, Theorem 1(st) is applicable to (M_0, d) and ψ; wherefrom, the conclusion is clear. □

Now, Theorem 6 includes Theorem 1(st), to which it reduces when $H = 1$, $c = 1$. The reciprocal inclusion also holds, by the argument above. Summing up, Theorem 6 is logically equivalent with Theorem 1(st); hence to Theorem 3 as well. Concerning this last aspect, note that Theorem 6 includes directly Theorem 3 (by taking h as a linear function). However, the reciprocal inclusion is not "easily" obtainable. In fact, let us consider the choice $H(t, s) = \min\{t, s\}$, $t, s \in R_+$; as well as (for $0 < \lambda < 1$)

$$c(0) = 1; \quad c(t) = t^{-\lambda}, \text{ for each } t > 0.$$

Clearly, c is right locally H-proper at origin [just take (as attached function) $h(t) = (1/(1-\lambda))t^{1-\lambda}, t \in R_+$]; hence, right locally H-proper (on R_+); wherefrom, Theorem 6 works here. On the other hand, Theorem 3 is not (directly) applicable when $\varphi_* = 0$; hence the claim.

Finally, combining this with the construction of Theorem 4, we may state a "hybrid" fixed point result as follows. In addition to (H, c) (subject to the precise conditions) take a couple (b, Γ); where $b : R_+ \to R_+$ is normal (cf. (b1)+(c1)) and $\Gamma : M \to R_+$ is nonexpansive (according to (2a)).

Theorem 7. *Let the above data be such that*

(4b) $b(\Gamma(x))d(x, Tx) \leq H(c(\varphi(x)), c(\varphi(Tx)))[\varphi(x) - \varphi(Tx)], \quad \forall x \in M.$

Then, T has at least one fixed point in M.

Proof. **(Sketch)** By the same way as in Theorem 6, we get (3a) over M_0 (modulo ψ); wherefrom, all is clear (via Theorem 4(st)). □

This result extends Theorem 6; hence, Theorem 1(st) as well. On the other hand, Theorem 7 follows from Theorem 1(st); because, so does Theorem 4(st). Summing up, Theorem 7 is but a logical equivalent of Theorem 1(st); note that this conclusion also includes the fixed point statement in Suzuki [31]. Further, we may ask whether the construction in Theorem 5 may be used as well in deriving a fixed point result extending Theorem 6. The answer is positive; further aspects will be delineated elsewhere. Some extensions of the obtained facts to multivalued maps

$T : M \to \mathcal{P}(M)$ are immediate; note that, in such a way, one extends the fixed point results in Mizoguchi and Takahashi [22]; see also Petruşel and Sîntămărian [27].

5. FL results

Let (M, d) be a complete metric space; and $\varphi : M \to R_+$, some d-lsc function (cf. (1a)). Further, let $T : M \to M$ be a selfmap of M. In the 2006 paper by Feng and Liu [13], an interesting fixed point result (comparable with Theorem 1) was established. Let $G : R_+ \to R_+$ be a function with

(a5) G is continuous, increasing, subadditive and $G^{-1}(0) = \{0\}$;

it will be referred to as a *Feng-Liu* function. Clearly, $G(\infty) > 0$, in view of (a5) (the last part). We also note the useful property

$$\text{for each } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that } G(\tau) < \delta \Longrightarrow \tau < \varepsilon. \tag{5.1}$$

For, if this fails, there must be an $\varepsilon > 0$ such that: for each $\delta > 0$, there exists $\tau = \tau(\delta) \geq \varepsilon$ with $G(\tau) < \delta$. But then (as G is increasing) we necessarily have $G(\varepsilon) = 0$, contradiction; hence the claim.

Theorem 8. *Suppose that some Feng-Liu function G may be found with*

(5a) $G(d(x, Tx)) \leq \varphi(x) - \varphi(Tx)$, *for all* $x \in M$.

Then, conclusions of Theorem 1 are retainable.

This result includes Theorem 1 for $G = 1$. (In fact, it rephrases the 1998 one in Jachymski [15]; we do not give details). But, the reciprocal is also true; this will follow from the

Proof. (**of Theorem 8**) Denote for simplicity $e(x, y) = G(d(x, y)), x, y \in M$. This is a metric over M, by the imposed (upon G) properties. Moreover, by (5.1), we have the generic equivalencies

$$\text{(for each sequence } (x_n)\text{): } d\text{-Cauchy} \Longleftrightarrow e\text{-Cauchy} \tag{5.2}$$

$$\text{(for each } (x_n) \text{ and } x\text{): } x_n \xrightarrow{d} x \text{ if and only if } x_n \xrightarrow{e} x. \tag{5.3}$$

In particular, this shows that (M, e) is complete and φ is e-lsc, if we take into account the conditions of Theorem 8. This, added to (5a), tells us that Theorem 1 is applicable to (M, e) and φ; hence the conclusion. □

Summing up, Theorem 8 is but a logical equivalent of Theorem 1. Further technical aspects may be found in Jachymski [15], Petruşel [26], Bîrsan [6] and Rozoveanu [29]; see also Kada, Suzuki and Takahashi [17].

6. Almost metrical extensions

(A) Let M be a nonempty set; and $(\leq)$, some *quasi-order* (i.e.: reflexive and transitive relation) over it. By a *pseudometric* on M we shall mean any map $e : M \times M \to R_+$. If, in addition, e is *reflexive* $[e(x,x) = 0, \forall x \in M]$, *triangular* $[e(x,z) \leq e(x,y) + e(y,z), \forall x,y,z \in M]$ and *sufficient* $[e(x,y) = 0$ implies $x = y]$, we say that it is an *almost metric* (over M). Suppose that we fixed such an object. Call $z \in M$, $(\leq)$-*maximal*, in case: $w \in M$ and $z \leq w$ imply $z = w$. Existence results involving such points may be viewed as (almost) “metrical” versions of the Zorn-Bourbaki maximality principle [7]. To state one of these, one may proceed as below. Call the ascending sequence (x_n) in M, *e-Cauchy* when: $\forall \delta > 0, \exists n(\delta)$, such that $n(\delta) \leq p \leq q \Longrightarrow e(x_p, x_q) \leq \delta$; and *e-asymptotic*, provided: $e(x_n, x_{n+1}) \to 0$, as $n \to \infty$. Clearly, each (ascending) e-Cauchy sequence is e-asymptotic too. The reciprocal is also true when all such sequences are involved; i.e., the global conditions below are equivalent each other:

(6a) each ascending sequence is e-Cauchy
(6b) each ascending sequence is e-asymptotic.

Either of these will be referred to as: $(M, \leq)$ is *regular* (modulo e). The following answer to the posed question obtained in Turinici [34] is available.

Proposition 1. *Assume that $(M, \leq)$ is regular (modulo e) and*

(6c) *$(M, \leq)$ is sequentially inductive:*
each ascending sequence is bounded above (modulo $(\leq)$).

Then, for each $u \in M$ there exists a $(\leq)$-maximal $v \in M$ with $u \leq v$.

Note that the non-sufficient version of this result extends the Brezis-Browder ordering principle [8]. Further statements in the area were obtained in Altman [1] and Anisiu [2]; see also Kang and Park [18]. However, all these are (mutually) equivalent; see Turinici [35] for details.

(B) A basic application of these facts may be given along the following lines. Let (M, e) be an almost metric space; and $\varphi : M \to R_+$ be a function. The regularity condition below is considered

(6d) (e, φ) is descending complete: for each e-Cauchy sequence (x_n) with $(\varphi(x_n))$ descending there exists $x \in M$ in such a way that $x_n \xrightarrow{e} x$ and $\varphi(x_n) \geq \varphi(x), \forall n$.

Theorem 9. *Let the precise condition be admitted; and let $T : M \to M$ be a selfmap with the property (1b). Then, (1.1) is retainable in the stronger sense: for each $u \in M$ there exists $v = v(u)$ in* $\mathrm{Fix}(T) = \mathrm{Per}(T)$ *with*

$$e(u,v) \leq \varphi(u) - \varphi(v) \quad \text{(hence } \varphi(u) \geq \varphi(v)\text{)} \tag{6.1}$$

$$x \in M, e(v,x) \leq \varphi(v) - \varphi(x) \Longrightarrow v = x. \tag{6.2}$$

Proof. Let $(\leq)$ stand for the order (i.e.: antisymmetric quasi-order) given by (a1) (modulo e). We claim that Proposition 1 applies for $(M, \leq; e)$; wherefrom, all is clear. Let (x_n) be an ascending sequence in M

(6e) $e(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m)$, whenever $n \leq m$.

The sequence $(\varphi(x_n))$ is descending in R_+; hence a Cauchy one. As a consequence, (x_n) is e-Cauchy; wherefrom $(M, \leq)$ is regular (modulo e). Putting these together there must be (via (6d)) some $y \in M$ with $x_n \xrightarrow{e} y$ and $\varphi(x_n) \geq \varphi(y)$, for all n. Fix some rank n. By (6e) and the triangular property of e,

$$e(x_n, y) \leq e(x_n, x_m) + e(x_m, y) \leq \varphi(x_n) - \varphi(x_m) + e(x_m, y), \ \forall m \geq n.$$

This, by the relation above, yields (passing to limit as $m \to \infty$)

$$e(x_n, y) \leq \varphi(x_n) - \lim_m \varphi(x_m) \leq \varphi(x_n) - \varphi(y) \ \ \text{(i.e.: } x_n \leq y\text{)}.$$

As n was arbitrarily chosen, y is an upper bound of (x_n) in M; which tells us that $(M, \leq)$ is sequentially inductive; hence the claim. □

Now, the regularity condition (6d) holds whenever (M, e) is complete and

(6f) φ is descending e-lsc: $\varphi(x_n) \geq \varphi(x)$ for all n, whenever $(\varphi(x_n))$ is descending and $x_n \xrightarrow{e} x$.

In particular, (6f) holds when φ is e-lsc. Note that, in such a case, Theorem 9 is just the fixed point statement in Caristi and Kirk [10] (Theorem 1). Some related aspects may be found in Isac [14] and Nemeth [23].

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References

[1] Altman M., *A generalization of the Brezis-Browder principle on ordered sets,* Nonlinear Analysis, 6 (1981), 157-165.

[2] Anisiu M. C., *On maximality principles related to Ekeland's theorem,* Seminar Funct. Analysis Numer. Meth. (Faculty of Math. Research Seminars), Preprint No. 1 (8 pp), "Babeş-Bolyai" Univ., Cluj-Napoca (România), 1987.

[3] Bae J. S., *Fixed point theorems for weakly contractive multivalued maps,* J. Math. Analysis Appl., 284 (2003), 690-697.

[4] Bae J. S., Cho E. W. and Yeom S. H., *A generalization of the Caristi-Kirk fixed point theorem and its applications to mapping theorems,* J. Korean Math. Soc., 31 (1994), 29-48.

[5] Bantaş G. and Turinici M., *Mean value theorems via division methods,* An. Şt. Univ. "A. I. Cuza" Iaşi (S. I-a, Mat.), 40 (1994), 135-150.

[6] Bîrsan T., *New generalizations of Caristi's fixed point theorem via Brezis-Browder principle,* Math. Moravica, 8 (2004), 1-5.

[7] Bourbaki N., *Sur le theoreme de Zorn,* Archiv der Math., 2 (1949/1950), 434-437.

[8] Brezis H. and Browder F. E., *A general principle on ordered sets in nonlinear functional analysis,* Advances in Math., 21 (1976), 355-364.

[9] Brunner N., *Topologische Maximalprinzipien,* Zeitschrift Math. Logik Grundlagen Math., 33 (1987), 135-139.

[10] Caristi J. and Kirk W. A., *Geometric fixed point theory and inwardness conditions,* in "The Geometry of Metric and Linear Spaces" (Michigan State Univ., 1974), pp. 74-83, Lecture Notes Math. vol. 490, Springer, Berlin, 1975.

[11] Dancs S., Hegedus M. and Medvegyev P., *A general ordering and fixed-point principle in complete metric space,* Acta Sci. Math. (Szeged), 46 (1983), 381-388.

[12] Ekeland I., *Nonconvex minimization problems,* Bull. Amer. Math. Soc. (New Series), 1 (1979), 443-474.

[13] Feng Y. and Liu S., *Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings,* J. Math. Anal. Appl., 317 (2006), 103-112.

[14] Isac G., *Un theoreme de point fixe de type Caristi dans les espaces localement convexes,* Review Res. Fac. Sci. Univ. Novi Sad (Math. Series), 15 (1985), 31-42.

[15] Jachymski J. R., *Caristi's fixed point theorem and selections of set-valued contractions,* J. Math. Anal. Appl., 227 (1998), 55-67.

[16] Jachymski J. R., *Converses to the fixed point theorems of Zermelo and Caristi,* Nonlinear Analysis, 52 (2003), 1455-1463.

[17] Kada O., Suzuki T. and Takahashi W., *Nonconvex minimization theorems and fixed point theorems in complete metric spaces,* Math. Japonica, 44 (1996), 381-391.

[18] Kang B. G. and Park S., *On generalized ordering principles in nonlinear analysis,* Nonlinear Analysis, 14 (1990), 159-165.

[19] Kirk W. A. and Saliga M., *The Brezis-Browder order principle and extensions of Caristi's theorem,* Nonlinear Analysis, 47 (2001), 2765-2778.

[20] Lin L. J. and Du W. S., *Ekeland's variational principle, minimax theorems and existence of nonconvex equilibria in complete metric spaces,* J. Math. Anal. Appl., 323 (2006), 360-370.

[21] Manka R., *Turinici's fixed point theorem and the Axiom of Choice,* Reports Math. Logic, 22 (1988), 15-19.

[22] Mizoguchi N. and Takahashi W., *Fixed point theorems for multivalued mappings on complete metric spaces,* J. Math. Anal. Appl., 141 (1989), 177-188.

[23] Nemeth A. B., *A nonconvex vector minimization principle,* Nonlinear Analysis, 10 (1986), 669-678.

[24] Park S. and Bae J. S., *On the Ray-Walker extension of the Caristi-Kirk fixed point theorem,* Nonlinear Analysis, 9 (1985), 1135-1136.

[25] Pasicki L., *A short proof of the Caristi-Kirk fixed point theorem,* Comment. Math. (Prace Mat.), 20 (1977/1978), 427-428.

[26] Petruşel A., *Caristi type operators and applications,* Studia Universitatis "Babeş-Bolyai" (Mathematica), 48 (2003), 115-123.

[27] Petruşel A. and Sîntămărian A., *Single-valued amd multi-valued Caristi type operators,* Publ. Math. Debrecen, 60 (2002), 167-177.

[28] Ray W. O. and Walker A., *Mapping theorems for Gateaux differentiable and accretive operators,* Nonlinear Analysis, 6 (1982), 423-433.

[29] Rozoveanu P., *Ekeland's variational principle for vector valued functions,* Math. Reports (St. Cerc. Mat.), 2(52) (2000), 351-366.

[30] Suzuki T., *Generalized distance and existence theorems in complete metric spaces,* J. Math. Anal. Appl., 253 (2001), 440-458.

[31] Suzuki T., *Generalized fixed point theorems by Bae and others,* J. Math. Anal. Appl., 302 (2005), 502-508.

[32] Taskovic M. R., *The Axiom of Choice, fixed point theorems and inductive ordered sets,* Proc. Amer. Math. Soc., 116 (1992), 897-904.

[33] Turinici M., *Maximal elements in a class of order complete metric spaces,* Math. Japonica, 5 (1980), 511-517.

[34] Turinici M., *Pseudometric extensions of the Brezis-Browder ordering principle,* Math. Nachr. 130 (1987), 91-103.

[35] Turinici M., *Metric variants of the Brezis-Browder ordering principle,* Demonstr. Math., 22 (1989), 213-228.

[36] Turinici M., *Mapping theorems and general Newton-Kantorovich processes,* Ann. Sci. Math. Quebec, 14 (1990), 85-95.

[37] Turinici M., *Pseudometric versions of the Caristi-Kirk fixed point theorem,* Fixed Point Theory (Cluj-Napoca), 5 (2004), 147-161.

[38] Turinici M., *Functional type Caristi-Kirk theorems,* Libertas Math., 25 (2005), 1-12.

[39] Valyi I., *A general maximality principle and a fixed point theorem in uniform space,* Periodica Math. Hungarica, 16 (1985), 127-134.

[40] Wong C. S., *On a fixed point theorem of contractive type,* Proc. Amer. Math. Soc., 57 (1976), 283-284.

[41] Zhong C. K., Zhu J. and Zhao P. H., *An extension of multi-valued contraction mappings and fixed points,* Proc. Amer. Math. Soc., 128 (1999), 2439-2444.

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