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WEAK TYPE $(1, 1)$ OF MAXIMAL OPERATORS ON METRIC MEASURE SPACES

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Abstract. A discretization method for the study of the weak type $(1, 1)$ for the maximal of a sequence of convolution operators on $\mathbb{R}^n$ has been introduced by Miguel de Guzmán and Teresa Carrillo, by replacing the integrable functions by finite sums of Dirac deltas. Trying to extend the above mentioned result to integral operators defined on metric measure spaces, a general setting containing at once continuous, discrete and mixed contexts, a caveat comes from the result in *On restricted weak type* $(1, 1)$*; the discrete case* (Akcoglu M.; Baxter J.; Bellow A.; Jones R., Israel J. Math. 124 (2001), 285–297). There a sequence of convolution operators in $\ell^1(\mathbb{Z})$ is constructed such that the maximal operator is of restricted weak type $(1, 1)$, or equivalently of weak type $(1, 1)$ over finite sums of Dirac deltas, but not of weak type $(1, 1)$. The purpose of this note is twofold. First we prove that, in a general metric measure space with a measure that is absolutely continuous with respect to some doubling measure, the weak type $(1, 1)$ of the maximal operator associated to a given sequence of integral operators is equivalent to the weak type $(1, 1)$ over linear combinations of Dirac deltas with positive integer coefficients. Second, for the non-atomic case we obtain as a corollary that any of these weak type properties is equivalent to the weak type $(1, 1)$ over finite sums of Dirac deltas supported at different points.

1. Introduction

The problem of determination of the weak type $(1, 1)$ of maximal operators associated to a sequence of convolution kernels from its behavior on classes of special functions or distributions has as starting point the results of Moon in [14]. There, the weak type $(1, q)$ of the maximal operator associated to the convolution operators induced by a sequence of integrable kernels in $\mathbb{R}^n$, is proved to be equivalent to the restricted weak type $(1, q)$, with $q \geq 1$. This means that to guarantee the weak type $(1, q)$ of such operator, is enough to test its action over the collection of all characteristic functions of measurable sets in $\mathbb{R}^n$ with finite measure.

The next relevant step was introduced by T. Carrillo y M. de Guzmán (see [8] and [5]), where characteristic functions are substituted by Dirac deltas, again in the Euclidean space. A generalization of these results concerning the structure of the class of special functions providing the weak type $(1, q)$ $(1 \leq q < \infty)$ of such

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a maximal operator on $\mathbb{R}^n$, is proved by F. Chiarenza and A. Villani in [6]. Later on, in [13], T. Menárguez y F. Soria showed how to applied the discrete approach to obtain the best constants for the weak type of maximal operator, which for the Hardy-Littlewood maximal operator is finally achieved by Melas in [10]. Extensions to weighted inequalities and for non convolution integral operators in $\mathbb{R}^n$ are proved by T. Menárguez (see [11] and [12]). Let us also mention some recent results by J. Aldaz and J. Varona in [4], where Dirac deltas are substituted by more general measures for convolution type operators.

A natural question, taking into account the recent developments of real and harmonic analysis on metric spaces, is whether or not these results can be extended to a metric measure space, for example to a space of homogeneous type or even to non doubling settings.

Being the integers with the restriction of the usual distance and with the counting measure a space of homogeneous type, the remarkable example given by Akcoglu, Baxter, Bellow and Jones in [3] gives us the answer to our general aim: no, it is not posible to deduce the weak type of a maximal of a sequence of convolution operators on $\mathbb{Z}$ from its weak type on Dirac deltas.

These facts together leads us to at least two problems. First, if we consider non atomic metric measure spaces and sequences of integral operators with continuous kernels, we ask for the natural extension of the result in [5]. Second, in a general context containing at once discrete, continuous and mixed situations, look for small classes of functions which are enough in order to test the weak type $(1,1)$ of such a maximal operator. Actually we shall solve the first problem as a corollary of our approach to the second one.

We would like to mention that the main tool for our proof is the dyadic analysis on spaces of homogeneous type started by Christ in [7].

The paper is organized as follows. In Section 2 we introduce the geometric setting and the basic properties of the dyadic families introduced by M. Christ in [7]. In Section 3 we introduce the basic properties of the kernels defining the sequence of integral operators and we state and prove the main results of this paper.

2. Dyadic sets on spaces of homogeneous type

In this section we introduce the geometric setting and we remind some properties of the "dyadic cubes" constructed by Christ. Even when the results hold on quasi-metric spaces, a theorem due to Macías an Segovia (see [9]) allows us to work on a metric setting. Let (X,d) be a metric space and let μ be a positive Borel measure on X. We shall say that μ is *regular* on X if

$$\mu(E) = \inf\{\mu(U) : E \subseteq U,\, U \text{ open}\} = \sup\{\mu(K) : K \subseteq E,\, K \text{ compact}\},$$

for every Borel subset E of X. The measure μ satisfies the *doubling property* on X if there exists a constant $A \geq 1$ such that the inequalities

$$0 < \mu(B_d(x,2r)) \leq A\mu(B_d(x,r)) < \infty$$

hold for every $x \in X$ and every $r > 0$, where $B_d(x,s) = \{y \in X : d(x,y) < s\}$. We shall say that a metric measure space (X,d,μ) is a *space of homogeneous type* if μ is a regular measure satisfying the doubling property on X. Then if (X,d,μ) is a space of homogeneous type, the set of all the continuous functions on X with compact support is dense in $L^1(X,\mu)$. Notice that if (X,d) is a complete metric space and μ is a finite doubling measure on X, then (X,d,μ) is a space of homogeneous type (see [1]).

Given (X,d,μ) a space of homogeneous type, let us state as a theorem the main properties of the dyadic families constructed by M. Christ in [7]. For $0 < \delta < 1$, and for each $n \in \mathbb{Z}$ let $\mathcal{N}_n = \{x_{n,k} : k \in K(n)\}$ be a maximal δ^n-disperse subset of X, where $K(n)$ is an initial interval of natural numbers that may coincide with $\mathbb{N}$, and $K(n)$ is finite for every n if and only if X is bounded. Set $\mathcal{A} = \{(n,k) : n \in \mathbb{Z},\ k \in K(n)\}$.

Theorem 1 (Christ)**.** *Let (X,d,μ) be a space of homogeneous type. Then there exist $a > 0$, $c > 0$, $0 < \delta < 1$, and a family $\{Q_k^n : (n,k) \in \mathcal{A}\}$ of subsets of X satisfying the following properties.*

(1) *Q_k^n is an open subset of X, for every $(n,k) \in \mathcal{A}$;*
(2) *$B_d(x_{n,k}, a\delta^n) \subseteq Q_k^n$ for every $(n,k) \in \mathcal{A}$;*
(3) *$Q_k^n \subseteq B_d(x_{n,k}, c\delta^n)$ for every $(n,k) \in \mathcal{A}$;*
(4) *for each $n \in \mathbb{Z}$, $Q_k^n \cap Q_i^n \neq \emptyset$ implies $k = i$;*
(5) *for every $(n,k) \in \mathcal{A}$ and every $\ell < n$ there exists a unique $i \in K(n)$ such that $Q_k^n \subseteq Q_i^\ell$;*
(6) *if $n \geq \ell$, then either $Q_k^n \subseteq Q_i^\ell$, or $Q_k^n \cap Q_i^\ell = \emptyset$, for each $k \in K(n)$, $i \in K(\ell)$;*
(7) *$\mu\left(X \setminus \bigcup_{k \in K(n)} Q_k^n\right) = 0$, for every $n \in \mathbb{Z}$;*
(8) *$\mu(\partial Q_k^n) = 0$, for every $(n,k) \in \mathcal{A}$, where ∂Q_k^n denotes the boundary of Q_k^n.*
(9) *$\mu(Q_k^n) = \sum_{i : Q_i^\ell \subset Q_k^n} \mu(Q_i^\ell)$, for each $n \in \mathbb{Z}$, $\ell \geq n+1$ and $k \in K(n)$.*
(10) *X is bounded if and only if there exists $(n,k) \in \mathcal{A}$ such that $X = Q_k^n$.*

For the proof see [7] and [2]. Let us write $\mathcal{D}$ to denote the class of all "dyadic sets" Q_k^n in the above theorem, i.e.

$$\mathcal{D} = \bigcup_{n \in \mathbb{Z}} \{Q_k^n : k \in K(n)\}.$$

As already mentioned, in a space of homogeneous type the set of all the continuous functions with compact support is dense in L^1. This fact allows to prove that in this case the set of all linear combinations of characteristic functions of dyadic sets is also dense in L^1, which will be essential in the proof of the main result of this paper.

3. The main results

Let us start by introducing some terminology and notation. Let (X,d,ν) be a metric measure space, where ν is a σ-finite positive Borel measure on X. Let us consider a sequence $\{k_\ell\}$ of kernels, where each $k_\ell : X \times X \to \mathbb{R}$ is a measurable function such that $k_\ell(\cdot, y) \in L^1(X,\nu)$ *uniformly* in $y \in X$. This means that for each ℓ there exists $C_\ell < \infty$ such that

$$\|k_\ell(\cdot, y)\|_{L^1(X,\nu)} \leq C_\ell, \quad \text{for every } y \in X.$$

Given $f \in L^1(X)$ we define

$$K_\ell f(x) = \int_X k_\ell(x,y) f(y)\, d\nu(y),$$

$$K^* f(x) = \sup_\ell |K_\ell f(x)| .$$

Notice that by Fubini-Tonelli's theorem, $K_\ell f(x) \in \mathbb{R}$ for almost every $x \in X$, and then $K^* f$ is a measurable function defined on X.

If $k : X \times X \to \mathbb{R}$ is a continuous function, $x_1, x_2, \ldots, x_H \in X$ are **different** points and $\varepsilon > 0$, taking

$$f_\varepsilon(x) = \sum_{i=1}^{H} \frac{\mathcal{X}_{B_d(x_i,\varepsilon)}(x)}{\nu(B_d(x_i,\varepsilon))}$$

we have that

$$K f_\varepsilon(x) = \sum_{i=1}^{H} \frac{1}{\nu(B_d(x_i,\varepsilon))} \int_{B_d(x_i,\varepsilon)} k(x,y)\, d\nu(y).$$

On the continuity of k we have that $Kf_\varepsilon(x)$ converges to $\sum_{i=1}^H k(x,x_i)$ when ε tends to zero. By the other hand, $f_\varepsilon \to f = \sum_{i=1}^H \delta_{x_i}$ in the weak sense when ε tends to zero, where δ_{x_i} denotes the Dirac delta concentrated at the point x_i. In this sense we can consider the operator K acting over this kind of measures f, given by

$$Kf(x) = \sum_{i=1}^{H} k(x, x_i).$$

We shall say that K is of *weak type (1,1) over finite sums of Dirac deltas* (in (X,ν)) if there exists a constant $C > 0$ such that for each $\lambda > 0$ the inequality

$$\nu\big(\{|Kf| > \lambda\}\big) \leq C\, \frac{H}{\lambda}$$

holds for every $f = \sum_{i=1}^H \delta_{x_i}$, where $x_1, x_2, \ldots, x_H$ are H different points in X.

Notice that H is the total variation of the measure f.

Also we shall say that the maximal operator K^* is of weak type $(1,1)$ over finite sums of Dirac deltas if there exists $C > 0$ such that for every $\lambda > 0$ and every $f = \sum_{i=1}^H \delta_{x_i}$ we have

$$\nu\big(\{K^* f > \lambda\}\big) \leq C\, \frac{H}{\lambda}\, .$$

Let us observe that both definitions given above can be written forgetting about Dirac deltas. In particular, for the case of the maximal operator K^* we have that the condition $\nu\big(\{K^*f > \lambda\}\big) \leq C\,\frac{H}{\lambda}$ for every $f = \sum_{i=1}^{H} \delta_{x_i}$ and every $\lambda > 0$, is equivalent to say that the inequality

$$\nu\left(\left\{x \in X : \sup_{\ell}\left|\sum_{i=1}^{H} k_\ell(x, x_i)\right| > \lambda\right\}\right) \leq C\,\frac{H}{\lambda}$$

holds for every collection $x_1, x_2, \ldots, x_H$ of different points in X, for every $H \in \mathbb{N}$ and for every $\lambda > 0$.

Notice finally that if each $k_\ell : X \times X \to \mathbb{R}$ is a continuos function with compact support, Fubini-Tonelli's theorem implies that each $K_\ell f(x)$ is well defined for every $x \in X$ and for every $f \in L^1(X, \nu)$, and it is an integrable function. Moreover, $K_\ell f$ is bounded and with compact support. Then K^*f is a measurable function defined on every point of X, provided that $f \in L^1(X, \nu)$.

With the above definitions we are in position to state and prove the extensions of the above mentioned theorem of Miguel de Guzmán to metric measure spaces. As we already noticed, the characterization of the weak type $(1,1)$ contained in that theorem is not true in general measure spaces. Actually this is the case of spaces with isolated points, even for convolution operators. In fact, K. H. Moon proves in [14] that the maximal operator associated to a sequence of convolution operators in $L^1(\mathbb{R}^n)$ is of weak type $(1,q)$, $q \geq 1$, if and only if is of *restricted* weak type $(1,q)$, i.e., if the weak type inequality holds for characteristic functions of sets with finite measure. A somehow surprising situation occurs when the extension of Moon's result is considered in such a simple discrete setting as is $\mathbb{Z}$. In fact, in [3] the authors construct a sequence of convolution operators on $\mathbb{Z}$ whose maximal operator is of restricted weak type $(1,1)$ but not of weak type $(1,1)$. Notice that if E is any finite subset of $\mathbb{Z}$, let us say $E = \{x_1, x_2, \ldots, x_H\}$ with $x_1, x_2 \ldots, x_H$ different integer numbers, we have that

$$K^*\mathcal{X}_E(x) = \sup_{n\in\mathbb{N}}\left|\sum_{j\in\mathbb{Z}} k_n(x-j)\mathcal{X}_E(j)\right| = \sup_{n\in\mathbb{N}}\left|\sum_{i=1}^{H} k_n(x-x_i)\right|.$$

The above considerations show that a direct extension of the result of M. de Guzmán y T. Carrillo to general metric measure spaces is impossible. Nevertheless the weak type $(1,1)$ for the maximal of a given sequence of operators is equivalent to its weak type $(1,1)$ of the class of all *linear combinations of Dirac deltas with positive integer coefficients*. In fact our result in this direction is the following.

Theorem 2. *Let (X, d, ν) be a metric measure space, where ν is a measure such that $d\nu = g\,d\mu$, with $g \in L^1_{loc}(X, d, \mu)$ and (X, d, μ) a space of homogeneous type. Let $\{k_\ell\}$ be a sequence of continuous kernels with compact support on $X \times X$. Then K^* is of weak type $(1,1)$ if and only if there exists a constant $C > 0$ such that for every $\lambda > 0$ and every finite collection $x_1, x_2, \ldots, x_H \in X$ of* ***not necessarily***

different *points, we have*

$$\nu\left(\left\{x\in X:\sup_{\ell}\left|\sum_{i=1}^{H}k_\ell(x,x_i)\right|>\lambda\right\}\right)\le C\,\frac{H}{\lambda}\,.$$

Proof. Let us start by proving that the weak type $(1,1)$ over linear combinations of Dirac deltas with positive integer coefficients of K^* implies the weak type $(1,1)$ of K^* on (X,ν). If for a fixed natural number N we call $K_N^*f(x)=\max_{1\le\ell\le N}|K_\ell f(x)|$, then it is clear that

$$\bigcup_{N=1}^{\infty}\{x\in X:K_N^*f(x)>\lambda\}=\{x\in X:K^*f(x)>\lambda\}\,,$$

and that $K_N^*\le K_{N+1}^*$. Hence it is enough to prove that for each fixed N the inequality

$$\nu\left(\{x\in X:K_N^*f(x)>\lambda\}\right)\le C\,\frac{\sum_{i=1}^{H}c_i}{\lambda}\,,$$

holds with C independent of N. So we take a fix N and we will show the weak type $(1,1)$ of K_N^* in three steps.

Step 1. We first prove that if $f=\sum_{i=1}^{H}c_i\delta_{x_i}$ with $c_i\in\mathbb{R}^+$, then for every $\lambda>0$ we have that

$$\nu\left(\{x\in X:K^*f(x)>\lambda\}\right)\le C\,\frac{\sum_{i=1}^{H}c_i}{\lambda}\,.$$

If $c_i\in\mathbb{Q}^+$, we write $c_i=n_i/m_i$, with $n_i,m_i\in\mathbb{N}$, and

$$\sum_{i=1}^{H}c_ik_\ell(x,x_i)=\frac{1}{\prod_{j=1}^{H}m_j}\sum_{i=1}^{H}\widetilde{c}_ik_\ell(x,x_i)\,,$$

where $\widetilde{c}_i=n_i\prod_{\substack{j=1\\ j\ne i}}^{H}m_j\in\mathbb{N}$. Then, if $\widetilde{f}=\sum_{i=1}^{H}\widetilde{c}_i\delta_i$ we have

$$\begin{aligned}\nu\left(\{x\in X:K^*f(x)>\lambda\}\right) &= \nu\left(\left\{x\in X:K^*\widetilde{f}(x)>\lambda\prod_{j=1}^{H}m_j\right\}\right)\\ &\le C\,\frac{\sum_{i=1}^{H}\widetilde{c}_i}{\lambda\prod_{j=1}^{H}m_j}=C\,\frac{\sum_{i=1}^{H}c_i}{\lambda}\,.\end{aligned}$$

Now take $c_i\in\mathbb{R}^+$ and write $c_i=d_i+r_i$, with $d_i\in\mathbb{Q}^+$ and $r_i\ge 0$ will be conveniently chosen later, so small as needed. Then taking $\overline{f}=\sum_{i=1}^{H}d_i\delta_{x_i}$, for every $0<\alpha<\lambda$ we have

$$\begin{aligned}\nu\left(\{x\in X:K_N^*f(x)>\lambda\}\right) &\le \nu\left(\left\{x\in X:K_N^*\overline{f}(x)>\lambda-\alpha\right\}\right)\\ &+ \nu\left(\left\{x\in X:K_N^*(f-\overline{f})(x)>\alpha\right\}\right)\\ &\le C\,\frac{\sum_{i=1}^{H}d_i}{\lambda-\alpha}\end{aligned}$$

$$
\begin{aligned}
&+\quad \nu\left(\left\{x\in X:\max_{1\le\ell\le N}\left|\sum_{i=1}^{H} r_i k_\ell(x,x_i)\right|>\alpha\right\}\right)\\
&\le\quad C\,\frac{\sum_{i=1}^{H} c_i}{\lambda-\alpha}+\frac{1}{\alpha}\sum_{\ell=1}^{N}\sum_{i=1}^{H} r_i\int_X |k_\ell(x,x_i)|\,d\nu(x)\\
&=\quad C\,\frac{\sum_{i=1}^{H} c_i}{\lambda-\alpha}+\frac{1}{\alpha}\sum_{i=1}^{H} r_i\sum_{\ell=1}^{N}\|k_\ell(\cdot,x_i)\|_1 .
\end{aligned}
$$

Since each r_i can be chosen arbitrarily small, we have

$$\nu\left(\{x\in X: K_N^* f(x)>\lambda\}\right)\le C\,\frac{\sum_{i=1}^{H} c_i}{\lambda-\alpha}$$

for every $0<\alpha<\lambda$. The desired inequality follows taking limit for $\alpha\to 0$.

Step 2. We want to prove now that K_N^* is of weak type $(1,1)$ over linear combinations of characteristic functions of the dyadic sets constructed by Christ (see Section 2). Let $h=\sum_{i=1}^{H} c_i\mathcal{X}_{Q_i}$, with $Q_i\in\mathcal{D}$. Notice that we may assume that $c_i>0$ and that the sets Q_i are disjoint. We want to see that for every $\lambda>0$ and for every function h as above,

$$\nu\left(\{x\in X: K_N^* h(x)>\lambda\}\right)\le C\,\frac{\|h\|_1}{\lambda}=C\,\frac{\sum_{i=1}^{H} c_i\nu(Q_i)}{\lambda}.$$

Let us observe first that if $h=\sum_{i=1}^{H} c_i\mathcal{X}_{Q_i}$ is the given simple function, and if η is a given positive real number, then we can write, except on a set with ν-measure equal to zero, $h=\sum_{j=1}^{M} d_j\mathcal{X}_{\widetilde{Q}_j}$ with $\widetilde{Q}_j$ disjoint dyadic sets in $\mathcal{D}$ such that $\operatorname{diam}(\widetilde{Q}_j)<\eta$ and $d_j>0$ for every $j=1,2,\dots,M$ (see properties (3), (4), (6) and (8) in Theorem 1). Then we will keep writing $h=\sum_{i=1}^{H} c_i\mathcal{X}_{Q_i}$ and when necessary we shall assume that the diameter of each Q_i is as small as we need.

Let $f=\sum_{i=1}^{H} c_i\nu(Q_i)\delta_{x_i}$, where δ_{x_i} denotes the Dirac delta concentrated at x_i, the "center" of Q_i (see properties 2 and 3 in Theorem 1). For the fixed N and for $0<\alpha<\lambda$ we write

$$
\begin{aligned}
\nu\left(\{x\in X: K_N^* h(x)>\lambda\}\right) &\le \nu\left(\{x\in X: K_N^* f(x)>\lambda-\alpha\}\right)\\
&\quad+\ \nu\left(\{x\in X: K_N^* (h-f)(x)>\alpha\}\right)\\
&\le C\,\frac{\|f\|_1}{\lambda-\alpha}+\nu\left(\{x\in X: K_N^* (h-f)(x)>\alpha\}\right)\\
&= C\,\frac{\|h\|_1}{\lambda-\alpha}+\nu\left(\{x\in X: K_N^* (h-f)(x)>\alpha\}\right).
\end{aligned}
$$

Then all we have to do is to show that the second term in the last member of the above inequalities can be made arbitrarily small by an adequate choice of the size of the dyadic sets Q_i in the definition of the function h. In fact, since

$$\begin{aligned} |K_\ell(h-f)(x)| &= \left|\sum_{i=1}^{H} c_i \int_{Q_i} k_\ell(x,y)\, d\nu(y) - \sum_{i=1}^{H} c_i \nu(Q_i) k_\ell(x,x_i)\right| \\ &= \left|\sum_{i=1}^{H} c_i \left[\int_{Q_i} k_\ell(x,y)\, d\nu(y) - \int_{Q_i} k_\ell(x,x_i)\, d\nu(y)\right]\right| \\ &\le \sum_{i=1}^{H} c_i \int_{Q_i} |k_\ell(x,y) - k_\ell(x,x_i)|\, d\nu(y)\,, \end{aligned}$$

we have

$$\begin{aligned} \sum_{\ell=1}^{N} \nu(\{|K_\ell(h-f)| > \alpha\}) &\le \sum_{\ell=1}^{N} \frac{1}{\alpha} \int_X |K_\ell(h-f)(x)|\, d\nu(x) \\ &\le \sum_{\ell=1}^{N} \frac{1}{\alpha} \int_X \left(\sum_{i=1}^{H} c_i \int_{Q_i} |k_\ell(x,y) - k_\ell(x,x_i)|\, d\nu(y)\right) d\nu(x) \\ &= \frac{1}{\alpha} \sum_{\ell=1}^{N} \sum_{i=1}^{H} c_i \int_{Q_i} \left(\int_{F_\ell} |k_\ell(x,y) - k_\ell(x,x_i)|\, d\nu(x)\right) d\nu(y)\,. \end{aligned}$$

where F_ℓ denotes the projection in the first variable of the support of k_ℓ, so it is a bounded set and with finite measure. Since each k_ℓ is a continuous function with compact support, given $\varepsilon > 0$ there exists $\delta = \delta(\ell, \varepsilon) > 0$ such that $|k_\ell(x,y) - k_\ell(x,z)| < \varepsilon$ *for every* $x \in X$, provided that $d(y,z) < \delta$. Since we can take the diameter of each Q_i small, we conclude the proof of the Step 2.

Step 3. From the technique of reduction to a dense subspace (see for example [8], Thm. 3.1.1) and the previous step prove we obtain the theorem in one direction.

For the converse, let us assume now that K^* is of weak type $(1,1)$. We want to prove that K^* is of weak type $(1,1)$ over linear combinations of Dirac deltas with positive integer coefficients. In fact, let $x_1, x_2, \ldots, x_H$ a set of different points in X, and let $f = \sum_{i=1}^{H} n_i \delta_{x_i}$ with n_i a positive integer for every i. Defining

$$\beta = \min\{d(x_i, x_h) : 1 \le i, h \le H,\ i \ne h\}\,,$$

we have that $d(x_i, x_h) \ge \beta > 0$ when $i \ne h$. Fix real numbers δ and c as in Christ's Theorem (Thm. 1), and let n be a positive integer satisfying $c\delta^n < \beta/4$. For each $i = 1, 2, \ldots, H$, there exists $Q^n_{j(i)} \in \mathcal{D}$ such that $x_i \in \overline{Q^n_{j(i)}}$. Notice that if $i \ne h$ then $Q^n_{j(i)} \cap Q^n_{j(h)} = \emptyset$. In fact, let us suppose that there exists $x \in Q^n_{j(i)} \cap Q^n_{j(h)} \subseteq B(x_{n,j(i)}, c\delta^n) \cap B(x_{n,j(h)}, c\delta^n)$. Then

$$\begin{aligned} d(x_i, x_h) &\le d(x_i, x_{n,j(i)}) + d(x_{n,j(i)}, x) + d(x, x_{n,j(h)}) + d(x_{n,j(h)}, x_h) \\ &< 4c\delta^n \\ &< \beta\,, \end{aligned}$$

which is absurd if $i \neq h$. Let us define the function $\overline{f}$ as

$$\overline{f}(y) = \sum_{i=1}^{H} \frac{n_i}{\nu\left(Q^n_{j(i)}\right)} \mathcal{X}_{Q^n_{j(i)}}(y)\,.$$

As before, fix N, $\lambda > 0$ and α such that $0 < \alpha < \lambda$, and write

$$\begin{aligned}
\nu(\{K^*_N f > \lambda\}) &\leq \nu(\{K^*_N \overline{f} > \lambda - \alpha\}) + \nu(\{K^*_N(\overline{f} - f) > \alpha\}) \\
&\leq C\,\frac{\|\overline{f}\|_1}{\lambda - \alpha} + \sum_{\ell=1}^{N} \nu(\{|K_\ell(\overline{f} - f)| > \alpha\}) \\
&= C\,\frac{\sum_{i=1}^{H} n_i}{\lambda - \alpha} + \sum_{\ell=1}^{N} \nu(\{|K_\ell(\overline{f} - f)| > \alpha\}),
\end{aligned}$$

where $K_\ell(\overline{f} - f)(x)$ means

$$K_\ell(\overline{f} - f)(x) \;=\; \sum_{i=1}^{H} \frac{n_i}{\nu\left(Q^n_{j(i)}\right)} \int_{Q^n_{j(i)}} [k_\ell(x,y) - k_\ell(x,x_i)]\, d\nu(y)\,.$$

Hence

$$\begin{aligned}
\sum_{\ell=1}^{N} \nu(\{|K_\ell(\overline{f} - f)| > \alpha\}) &\leq \sum_{i=1}^{H} \frac{n_i}{\alpha\nu\left(Q^n_{j(i)}\right)} \sum_{\ell=1}^{N} \int_X \left(\int_{Q^n_{j(i)}} |k_\ell(x,y) - k_\ell(x,x_i)|\, d\nu(y) \right) d\nu(x) \\
&\leq \sum_{i=1}^{H} \frac{n_i}{\alpha\nu\left(Q^n_{j(i)}\right)} \sum_{\ell=1}^{N} \int_{F_\ell} \left(\int_{Q^n_{j(i)}} |k_\ell(x,y) - k_\ell(x,x_i)|\, d\nu(y) \right) d\nu(x).
\end{aligned}$$

As in the Step 2 given $\varepsilon > 0$ we get

$$\sum_{\ell=1}^{N} \nu(\{|K_\ell(\overline{f} - f)| > \alpha\}) < \varepsilon$$

by an adequate choice for the diameter of the dyadic sets, since the each kernel k_ℓ is a continuous function and we have a finite number of them. Then we have shown that

$$\nu(\{K^*_N f > \lambda\}) \leq C\,\frac{\sum_{i=1}^{H} n_i}{\lambda}\,,$$

as desired. □

As we already mentioned, for the non-atomic case we obtain as a corollary that the theorem of de Guzmán and Carrillo can be extended to certain metric measure spaces. More precisely, the following result state that the weak type $(1,1)$ for the maximal operator of a sequence of integral operators with continuous kernels with compact support, is equivalent to the weak type $(1,1)$ over finite sums of Dirac deltas supported at different points.

Theorem 3. *Let (X,d,ν) be a metric measure space without isolated points, where ν is a measure such that $d\nu = g\,d\mu$, with $g \in L^1_{loc}(X,d,\mu)$ and (X,d,μ) a space of homogeneous type. Let $\{k_\ell\}$ be a sequence of continuous kernels with compact* *support on $X \times X$. Then K^* is of weak type $(1,1)$ if and only if K^* is of weak type $(1,1)$ over finite sums of Dirac deltas. In other words, K^* is of weak type $(1,1)$ if and only if there exists a constant $C > 0$ such that for every $\lambda > 0$ and every finite set $x_1, x_2, \ldots, x_H$ of different points in X, we have*

$$\nu\left(\left\{x \in X : \sup_{\ell}\left|\sum_{i=1}^{H} k_\ell(x, x_i)\right| > \lambda\right\}\right) \leq C\,\frac{H}{\lambda}.$$

Proof. Notice that after Theorem 2, we only have to prove that if K^* is of weak type $(1,1)$ over finite sums of Dirac deltas, then it is of weak type $(1,1)$ over linear combinations of Dirac deltas with positive integer coefficients. In fact, we know that there exists a constant $C > 0$ such that for every finite set $x_1, x_2, \ldots, x_H \in X$ of different points and for every $\lambda > 0$, we have

$$\nu\left(\left\{x \in X : \sup_{\ell}\left|\sum_{i=1}^{H} k_\ell(x, x_i)\right| > \lambda\right\}\right) \leq C\,\frac{H}{\lambda}.$$

We want to see that for every finite set $x_1, x_2, \ldots, x_H \in X$ of different points and for every $\lambda > 0$, if $f = \sum_{i=1}^{H} n_i \delta_{x_i}$ with $n_i \in \mathbb{N}$, then

$$\begin{aligned} \nu\left(\{x \in X : K^* f(x) > \lambda\}\right) &= \nu\left(\left\{x \in X : \sup_{\ell}\left|\sum_{i=1}^{H} n_i k_\ell(x, x_i)\right| > \lambda\right\}\right) \\ &\leq C\,\frac{\sum_{i=1}^{H} n_i}{\lambda}. \end{aligned}$$

As in the proof of Theorem 2, it will be sufficient to prove that if N is a fixed natural number, then

$$\nu\left(\{x \in X : K_N^* f(x) > \lambda\}\right) \leq C\,\frac{\sum_{i=1}^{H} n_i}{\lambda},$$

where C is independent of N. Then let us fix N. Since X does not have isolated points, for each $x_i \in X$ we can chose n_i different points $b_i^1, b_i^2, \ldots, b_i^{n_i}$ in X sufficiently close to x_i, and such that the set $\{b_i^r : 1 \leq i \leq H,\, 1 \leq r \leq n_i\}$ is also a collection of different points. For each fixed ℓ we write

$$\begin{aligned} K_\ell f(x) &= \sum_{i=1}^{H} n_i k_\ell(x, x_i) \\ &= \sum_{i=1}^{H}\sum_{r=1}^{n_i}[k_\ell(x, x_i) - k_\ell(x, b_i^r)] + \sum_{i=1}^{H}\sum_{r=1}^{n_i} k_\ell(x, b_i^r). \end{aligned}$$

Hence for every α such that $0 < \alpha < \lambda$, we have

$$\begin{aligned} \nu\left(\{x \in X : K_N^* f(x) > \lambda\}\right) &\leq \nu\left(\left\{x \in X : \max_{1\leq \ell \leq N}\left|\sum_{i=1}^{H}\sum_{r=1}^{n_i}[k_\ell(x, x_i) - k_\ell(x, b_i^r)]\right| > \alpha\right\}\right) \\ &+ \nu\left(\left\{x \in X : \max_{1\leq \ell \leq N}\left|\sum_{i=1}^{H}\sum_{r=1}^{n_i} k_\ell(x, b_i^r)\right| > \lambda - \alpha\right\}\right) \end{aligned}$$

$$= \quad I_1 + I_2.$$

We know that

$$I_2 \leq C \, \frac{\sum_{i=1}^{H} n_i}{\lambda - \alpha} \, ,$$

so all we have to do is to show that given $\varepsilon > 0$ and α satisfying $0 < \alpha < \lambda$, we can chose the elements b_i^r such that $I_1 < \varepsilon$. In fact, let

$$A^\ell(x) = \sum_{i=1}^{H} \sum_{r=1}^{n_i} [k_\ell(x, x_i) - k_\ell(x, b_i^r)] \, .$$

Then

$$I_1 = \nu \left(\left\{ x \in X : \max_{1 \leq \ell \leq N} |A^\ell(x)| > \alpha \right\} \right) \leq \sum_{\ell=1}^{N} \nu \left(\left\{ x \in X : |A^\ell(x)| > \alpha \right\} \right) \, .$$

For a fixed ℓ, from Chebyshev's inequality we have

$$\begin{aligned} \nu \left(\left\{ x \in X : |A^\ell(x)| > \alpha \right\} \right) \quad &\leq \quad \frac{1}{\alpha} \sum_{i=1}^{H} \sum_{r=1}^{n_i} \int_X |k_\ell(x, x_i) - k_\ell(x, b_i^r)| \, d\nu(x) \\ &= \quad \frac{1}{\alpha} \sum_{i=1}^{H} \sum_{r=1}^{n_i} \int_{F_\ell} |k_\ell(x, x_i) - k_\ell(x, b_i^r)| \, d\nu(x), \end{aligned}$$

where as before F_ℓ denotes the projection in the first variable of the support of k_ℓ, so it is a bounded set and with finite measure. Since each k_ℓ is a continuous function with compact support, given $\varepsilon > 0$ there exists $\delta = \delta(\ell, \varepsilon) > 0$ such that $|k_\ell(x, y) - k_\ell(x, z)| < \varepsilon$ *for every* $x \in X$, provided that $d(y, z) < \delta$. Notice also that only a finite number of kernels k_ℓ are involved, so that I_1 becomes small after an appropriate choice of b_i^r. Hence

$$I_1 + I_2 \leq C \, \frac{\sum_{i=1}^{H} n_i}{\lambda - \alpha} \, ,$$

and taking $\alpha \to 0$ we obtain the result.

□

The next result of this section is devoted to relax the regularity hypothesis on k_ℓ. Its proof is obtained by inspection of the proof of Theorem 2.

Theorem 4. *Let (X, d, ν) be a metric measure space, where ν is a measure such that $d\nu = g \, d\mu$, with $g \in L^1_{loc}(X, d, \mu)$ and (X, d, μ) a space of homogeneous type. Let $\{k_\ell\}$ be a sequence of kernels such that each $k_\ell : X \times X \to \mathbb{R}$ is a measurable function satisfying*

(1) *$k_\ell(\cdot, y) \in L^1(X, \nu)$ uniformly in $y \in X$,*
(2) *$\int_X |k_\ell(x, y) - k_\ell(x, z)| \, d\nu(x) \to 0$ when $d(y, z) \to 0$.*

Then K^* is of weak type $(1,1)$ if and only if there exists a constant $C>0$ such that for every $\lambda>0$ and every finite collection $x_1,x_2,\dots,x_H\in X$ of points not necessarily different, we have

$$\nu\left(\left\{x\in X:\sup_\ell\left|\sum_{i=1}^H k_\ell(x,x_i)\right|>\lambda\right\}\right)\le C\,\frac{H}{\lambda}\,.$$

It is clear that an analogous extension of Theorem 3 can be proved.

Notice that it is possible to obtain a refined result for spaces wich are neither discrete nor purely continuous. For example, for the set

$$X=\bigcup_{n\in\mathbb{Z}}(2n,2n+1)\cup\bigcup_{n\in\mathbb{Z}}\{2n+3/2\}=:X_1\cup X_2$$

endowed with the usual distance on $\mathbb{R}$ and the measure that counts on X_2 and measures lengths on X_1, is a space of homogeneous type (see [15]).

Moreover, Macías and Segovia prove in [9] that in spaces of homogeneous type the set of points with positive measure (atoms) is countable and coincides with the set of isolated points. With this characterization for the atoms we have that K^* if of weak type $(1,1)$ if and only if there exists a constant C such that for every finite set $\{x_1,x_2,\dots,x_H\}$ of different points in X and for every choice of natural numbers $n_1,n_2,\dots,n_H$ satisfying $n_i=1$ when $\nu(\{x_i\})=0$, we have that

$$\nu\left(\left\{x\in X:\sup_\ell\left|\sum_{i=1}^H n_ik_\ell(x,x_i)\right|>\lambda\right\}\right)\le C\,\frac{\sum_{i=1}^H n_i}{\lambda}$$

for every $\lambda>0$.

We shall finally mention that the hypotheses of the above theorems concerning continuity sometimes can be relaxed. For the basic case of Hardy-Littlewood type operator defined on a space of homogeneous type (X,d,ν) by

$$\begin{aligned}Mf(x)&=\sup_{\ell\in\mathbb{Z}}\frac{1}{\nu\left(B_d(x,2^{-\ell})\right)}\int_{B_d(x,2^{-\ell})}|f(y)|\,d\nu(y)\\&=\sup_{\ell\in\mathbb{Z}}\int_X k_\ell(x,y)|f(y)|\,d\nu(y),\end{aligned}$$

for $f\in L^1(\nu)$, where

$$k_\ell(x,y)=\frac{1}{\nu\left(B_d(x,2^{-\ell})\right)}\mathcal{X}_{B_d(x,2^{-\ell})}(y),$$

the continuity required in Theorem 2 does not hold even in Euclidean situations. On the other hand, the L^1 continuity required in Theorem 4 does not hold in typical spaces of homogeneous type. In fact,

$$\begin{aligned}\int |k_\ell(x,y)-k_\ell(x,z)|\,d\nu(x) &= \int \frac{1}{\nu(B_d(x,2^{-\ell}))}\left|\mathcal{X}_{B_d(x,2^{-\ell})}(y)-\mathcal{X}_{B_d(x,2^{-\ell})}(z)\right|\,d\nu(x)\\ &= \int_X \frac{1}{\nu(B_d(x,2^{-\ell}))}\left|\mathcal{X}_{B_d(y,2^{-\ell})}(x)-\mathcal{X}_{B_d(z,2^{-\ell})}(x)\right|\,d\nu(x)\\ &= \int_{B_d(y,2^{-\ell})\triangle B_d(z,2^{-\ell})} \frac{1}{\nu(B_d(x,2^{-\ell}))}\,d\nu(x)\end{aligned}$$

where $E\triangle F$ denotes the symmetric difference of the sets E and F, i.e. $E\triangle F = (E-F)\cup(F-E)$. The convergence to zero of the last integral when $d(y,z)\to 0$ is equivalent to the convergence to zero of $\nu\left(B_d(y,2^{-\ell})-B_d(z,2^{-\ell})\right)$ for each ℓ. The next example shows a non-atomic space of homogeneous type for which this property does not hold. In $\mathbb{R}^2$ endowed with the distance $\bar{d}((x_1,y_1),(x_2,y_2)) = \max\{|x_2-x_1|,|y_2-y_1|\}$, let X be the subset defined as

$$X=\{(x,y)\in\mathbb{R}^2:\bar{d}((x,y),(0,0))=2\}\cup\{(x,0):-1\le x\le 1\}$$

(see Figure 1) with the arc length measure λ.

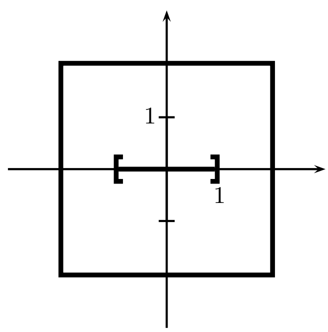

FIGURE 1. $X=\{(x,y)\in\mathbb{R}^2:\bar{d}((x,y),(0,0))=2\}\cup\{(x,0):-1\le x\le 1\}$

It is not difficult to see that $(X,\bar{d},\lambda)$ is a space of homogeneous type. Take the sequence $\{z_n\}$ in X defined as $z_n=(1/n,0)$. This sequence converges to the point $z=(0,0)$, and for each n (see Figure 2), we have

$$B_{\bar{d}}(z_n,2)-B_{\bar{d}}(z,2)=\{(2,y):-2<y<2\}.$$

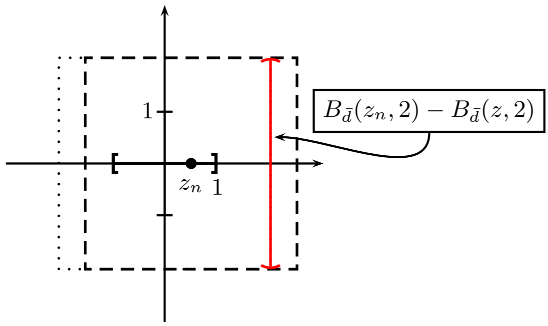

FIGURE 2. $B_{\bar{d}}(z_n,2) - B_{\bar{d}}(z,2) = \{(2,y) : -2 < y < 2\}$

Then $\lambda(B_{\bar{d}}(z_n,2) - B_{\bar{d}}(z,2)) = \lambda(\{(2,y) : -2 < y < 2\}) = 4$ for each n, so that $\lambda(B_{\bar{d}}(z_n,2) - B_{\bar{d}}(z,2))$ does not tend to zero when n tends to infinity.

Nevertheless, the kernels $k_\ell(x,y)$ in such a general situation can be controlled by a sequence of continuous kernels. For instance consider

$$\widetilde{k}_\ell(x,y) = \frac{\varphi\left(2^\ell d(x,y)\right)}{\int \varphi\left(2^\ell d(x,z)\right)\, d\nu(z)}\,,$$

where φ is the continuous function defined on the non-negative real numbers by $\varphi(t) = 1$ for every t in the interval $[0,1]$, $\varphi(t) = 0$ if $t \geq 2$, and linear on $[1,2]$. It is not difficult to show that each $\widetilde{k}_\ell$ is continuous and that

$$\frac{1}{A} k_\ell(x,y) \leq \widetilde{k}_\ell(x,y) \leq A k_{\ell-1}(x,y),$$

where A denotes the doubling constant for ν. Then the weak type for the maximal operator M associated with the kernels k_ℓ is equivalent to the weak type for the maximal operator associated with the kernels $\widetilde{k}_\ell$.

The new sequence $\{\widetilde{k}_\ell\}$ falls under the scope of Theorem 2, so that the next result holds even when the kernels are not smooth.

Corollary 5. *Let (X,d,ν) be a space of homogeneous type. Then the Hardy-Littlewood maximal function is of weak type $(1,1)$ if and only if there exists a constant $C > 0$ such that for every $\lambda > 0$ and every finite collection $x_1, x_2, \ldots, x_H \in X$ of not necessarily different points,*

$$\nu\left(\left\{x \in X : \sup_{\ell\in\mathbb{Z}} \sum_{i=1}^{H} k_\ell(x,x_i) > \lambda\right\}\right) \leq C\,\frac{H}{\lambda}\,,$$

where

$$k_\ell(x,y) = \frac{1}{\nu\left(B_d(x,2^{-\ell})\right)} \mathcal{X}_{B_d(x,2^{-\ell})}(y).$$

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