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STRONGLY TRANSITIVE GEOMETRIC SPACES: APPLICATIONS TO HYPERRINGS

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ABSTRACT. In this paper, we determine two families $\mathfrak{R}$ and $\mathfrak{G}$ of subsets of a hyperring R and sufficient conditions such that two geometric spaces $(R, \mathfrak{R})$ and $(R, \mathfrak{G})$ are strongly transitive. Moreover, we prove that the relations Γ and α are strongly regular equivalence relations on a hyperfield or a hyperring such that $(R, +)$ has an identity element.

1. Introduction

Hyperstructures represent a natural extension of classical algebraic structures and they were introduced by the French mathematician F. Marty. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. An exhaustive review updated to 1992 of hypergroup theory appears in [3], see also [1, 5, 14, 15, 16]. A recent book [4] contains a wealth of applications.

In this paragraph, we summarize the preliminary definitions and results required in the sequel. Let H be a non-empty set and let $\mathcal{P}^*(H)$ be the family of all non-empty subsets of H. A *hyperoperation* on H is a map $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ and the couple $(H, \circ)$ is called a *hypergroupoid*. If A and B are non-empty subsets of H, then we denote

$$A \circ B = \bigcup_{a \in A,\, b \in B} a \circ b, \quad x \circ A = \{x\} \circ A \quad \text{and} \quad A \circ x = A \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all x, y, z of H we have $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

We say that a semihypergroup $(H, \circ)$ is a *hypergroup* if for all $x \in H$, we have $x \circ H = H \circ x = H$. An element $e \in H$ is called *identity* (*scalar*) if for all $x \in H$ we have $x \in x \circ e \cap e \circ x$ ($x = x \circ e = e \circ x$). We call $(H, \circ)$ is a *canonical hypergroup* if it is commutative, has an scalar identity $e \in H$, and for every $x \in H$ there exists $x^{-1} \in H$ such that $e \in x \circ x^{-1}$.

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There are several types of hyperrings that can be defined on a non-empty set R, see [6]. In what follows we shall consider one of the most general types of hyperrings.

The triple $(R,+,\cdot)$ is a *hyperring* if

(1) $(R,+)$ is a hypergroup;
(2) $(R,\cdot)$ is a semihypergroup;
(3) the hyperoperation " $\cdot$" is distributive over the hyperoperation "$+$", which means that for all x,y,z of R we have:

$$x\cdot(y+z)=x\cdot y+x\cdot z \text{ and } (x+y)\cdot z=x\cdot z+y\cdot z.$$

We call $(R,+,\cdot)$ a *hyperfield* if $(R,+,\cdot)$ is a hyperring and $(R,\cdot)$ is a hypergroup.

A *Krasner hyperring* is a hyperring such that $(R,+)$ is a canonical hypergroup with identity 0 and $\cdot$ is an operation such that 0 is a bilaterally absorbing element. An exhaustive review updated to 2007 of hyperring theory appears in [6].

2. Basic facts about fundamental relation

If H is a hypergroup and $\rho\subseteq H\times H$ is an equivalence relation then for all pairs (A,B) of non-empty subsets of H, we set $A\overline{\overline{\rho}}B$ if and only if $a\rho b$ for all $a\in A$ and $b\in B$. The relation ρ is said to be *strongly regular to the right* if $x\rho y$ implies $x\circ a\ \overline{\overline{\rho}}\ y\circ a$ for all $(x,y,a)\in H^3$. Analogously, we can define *strongly regular to the left*. Moreover ρ is called *strongly regular* if it is strongly regular to the right and to the left. Let H be a hypergroup and ρ an equivalence relation on H. Let $\rho(a)$ be the equivalence class of a with respect to ρ and let $H/\rho=\{\rho(a)\mid a\in H\}$. A hyperoperation $\otimes$ is defined on H/ρ by $\rho(a)\otimes\rho(b)=\{\rho(x)\mid x\in\rho(a)\circ\rho(b)\}$. If ρ is strongly regular then it readily follows that $\rho(a)\otimes\rho(b)=\{\rho(x)\mid x\in a\circ b\}$. It is well known for ρ strongly regular that $<H/\rho,\otimes>$ is a group (see Theorem 31 in [3]), that is $\rho(a)\otimes\rho(b)=\rho(c)$ for all $c\in a\circ b$.

The notion of fundamental relation on hypergroups was introduced by Koskas [13], and then studied by Corsini [3], Freni [9, 10, 11] and Gutan [12], Vougiouklis [20, 21], Davvaz et. al. [1, 7, 8]. In [9], Freni firstly proved that the relation β is transitive in every hypergroup. The relations γ and γ^* were firstly introduced and analyzed by Freni [10]. He proved that the relation γ on hypergroup is transitive and $\gamma=\gamma^*$. Also, Freni [11] determined a family $P_\sigma(H)$ of subsets of a hypergroup H such that the geometric space $(H,P_\sigma(H))$ is strongly transitive.

In 1990, Vougiouklis at the fourth AHA congress [20], introduced the concept of fundamental relation Γ on a hyperring, and then it studied by himself and many authors, for example see [1, 2, 6, 8, 17, 21]. Recently, Mirvakili and Davvaz [18] proved that the relation Γ on every hyperfield is an equivalence relation and $\Gamma=\Gamma^*$. In [8], Davvaz and Vougiouklis introduced a new strongly regular equivalence relation on a hyperring such that the set of quotients is an ordinary commutative ring and later some properties of relation α are studied[17]. Moreover, Pelea [19] introduced the fundamental relation of a multialgebra.

Definition 2.1. [20] Let R be a hyperring. We define the relation Γ as follows: $x\ \Gamma\ y \ \Leftrightarrow\ \exists n \in \mathbb{N}, \exists k_i \in \mathbb{N}, \exists (x_{i1}, \ldots, x_{ik_i}) \in R^{k_i}, 1 \leq i \leq n$, such that $\{x, y\} \subseteq \sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} x_{ij} \right)$.

Definition 2.2. [8] Let R be a hyperring. We consider the relation α as follows : $x\ \alpha\ y \iff \exists n \in \mathbb{N},\ \exists (k_1, \ldots, k_n) \in \mathbb{N}^n,\ \exists \sigma \in \mathbb{S}_n$ and $[\exists (x_{i1}, \ldots, x_{ik_i}) \in R^{k_i}$, $\exists \sigma_i \in \mathbb{S}_{k_i}, (i = 1, \ldots, n)]$ such that

$$x \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} x_{ij} \right) \quad \text{and} \quad y \in \sum_{i=1}^{n} A_{\sigma(i)},$$

where $A_i = \displaystyle\prod_{j=1}^{k_i} x_{i\sigma_i(j)}$.

The relation α and Γ are reflexive and symmetric. Let α^* and Γ^* be the transitive closure of α and Γ. Then we recall the following theorem from [8, 20]:

Theorem 2.3. *Let $(R, +, \cdot)$ be a hyperring.*

(1) *Γ^* is the smallest equivalence relation on R such that the quotient R/Γ^* is a ring. R/Γ^* is called the fundamental ring.*
(2) *α^* is the smallest equivalence relation on R such that the quotient R/α^* is a commutative ring. R/α^* is called the commutative fundamental ring.*

A geometric space is a pair $(S, \mathcal{B})$ such that S is a non-empty set, whose elements we call *points*, and $\mathcal{B}$ is a non-empty family of subsets of S, whose elements we call *blocks*. $\mathcal{B}$ is a covering of S if for every point $y \in S$, there exists a block $B \in \mathcal{B}$ such that $y \in B$. If C is a subset of S, we say that C is a $\mathcal{B}$-part or $\mathcal{B}$-subset of S if for every $B \in \mathcal{B}$,

$$B \cap C \neq \emptyset \Rightarrow B \subseteq C.$$

If $B_1, B_2, \ldots, B_n$ are n blocks of geometric space $(S, \mathcal{B})$ such that $B_i \cap B_{i+1} \neq \emptyset$, for any $i \in \{1, 2, \ldots, n-1\}$, then the n-tuple $(B_1, B_2, \ldots, B_n)$ is called *polygonal* of $(S, \mathcal{B})$. The concept of polygonal allows us to define on S the following relation:

$x \approx y \Leftrightarrow x = y$ or a polygonal $(B_1, B_2, \ldots, B_n)$ exists such that $x \in B_1$ and $y \in B_n$.

The relation $\approx$ is an equivalence relation and it is easy to see that it coincides with the transitive closure of the following relation:

$x \sim y \Leftrightarrow x = y$ or there exists $B \in \mathcal{B}$ such that $\{x, y\} \subseteq B$.

So $\approx$ is equal to $\bigcup_{n \geq 1} \sim^n$, where $\sim^n = \underbrace{\sim \circ \sim \circ \ldots \circ \sim}_{n \text{ times}}$.

Freni [11] proved that the following theorem:

Theorem 2.4. *For every pair (A, B) of blocks of a geometric space $(S, \mathcal{B})$ and for any integer $n \in \mathbb{N}$, the following conditions are equivalent:*

(1) $A \cap B \neq \emptyset, x \in B \Rightarrow \exists C \in \mathcal{B} : (A \cup \{x\}) \subseteq C.$
(2) $A \cap B \neq \emptyset, x \in \Gamma(B) \Rightarrow \exists C \in \mathcal{B} : (A \cup \{x\}) \subseteq C.$

(3) $A \cap \Gamma(B) \neq \emptyset, x \in \Gamma(B) \Rightarrow \exists C \in \mathcal{B} : (A \cup \{x\}) \subseteq C$.

A geometric space $(S, \mathcal{B})$ is *strongly transitive* if the family $\mathcal{B}$ is a covering of S and moreover one of the three equivalence conditions of Theorem 2.4 is satisfied. With this definition it is proved that:

Theorem 2.5. *If $(S, \mathcal{B})$ is a strongly transitive geometric space, then the relation $\sim$ on S is transitive. Hence $\approx = \sim$.*

3. Strongly transitive geometric spaces associated to hyperrings

Freni [11] introduced the notion of strongly transitive geometric space on hypergroups and proved that the relation $\sim = \gamma$ is transitive. Anavariyeh and Davvaz [2] used the notion of strongly transitive geometric space on hypermodules. Now, we use the notion of strongly transitive geometric space on an arbitrary hyperring and obtain new result in this respect.

Let $(R, +, \cdot)$ be a hyperring. We can consider the geometric space $(R, \mathfrak{R})$ whose points are the elements of R and whose blocks are the finite hypersums of hyperproducts of elements of H. Thus, $\mathfrak{R}$ is the family of subsets of R defined as follows:

For every $n \in \mathbb{N} \cup \{0\}$, $k_i \in \mathbb{N} \cup \{0\}$ and $x_{ij} \in R$ where $i = 1, 2, \ldots, n$ and $j = 1, \ldots, k_i$, we set:

$$B_{\mathfrak{R}}\left([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]\right) = \sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij}.$$

Also, we can consider another geometric space $(R, \mathfrak{S})$ whose points are the elements of R and whose blocks are the sets formed by the union of all hypersums of hyperproducts obtained by permuting in the following possible ways:

For every $n \in \mathbb{N} \cup \{0\}$, $k_i \in \mathbb{N} \cup \{0\}$ and $x_{ij} \in R$ where $i = 1, 2, \ldots, n$ and $j = 1, \ldots, k_i$, we set:

$$B_{\mathfrak{S}}\left([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]\right) = \bigcup \left\{ \sum_{i=1}^{n} \prod_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)} | \sigma \in \mathbb{S}_n, \sigma_i \in \mathbb{S}_{k_i} \right\}.$$

With these notations we have:

Lemma 3.1. *Let $(R, +, \cdot)$ be a hyperring. Then*

(1) $B_{\mathfrak{R}}\left([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]\right) \subseteq B_{\mathfrak{S}}\left([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]\right)$.

(2) *Moreover, we have:* $B_{\mathfrak{S}}\left([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]\right) =$
$$= \bigcup \left\{ B_{\mathfrak{R}}\left([x_{\sigma(1)\sigma_{\sigma(1)}(1)}^{\sigma(1)\sigma_{\sigma(1)}(k_{\sigma(1)})}], \ldots, [x_{\sigma(n)\sigma_{\sigma(n)}(1)}^{\sigma(n)\sigma_{\sigma(n)}(k_{\sigma(n)})}]\right) \mid \sigma \in \mathbb{S}_n, \sigma_i \in \mathbb{S}_{k_i} \right\}.$$

Lemma 3.2. *Let $(R, +, \cdot)$ be a hyperring. Then for every $y \in H$ we have:*

(1) $B_{\mathfrak{S}}\left([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]\right) + y = B_{\mathfrak{S}}\left([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}], [y]\right)$.

(2) $y + B_{\mathfrak{S}}\left([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]\right) = B_{\mathfrak{S}}\left([y], [x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]\right)$.

(3) $B_{\mathfrak{S}}\left([x_{11}^{1k_1}],\dots,[x_{n1}^{nk_n}]\right)\cdot y = B_{\mathfrak{S}}\left([x_{11}^{1k_1},y],\dots,[x_{n1}^{nk_n},y]\right)$.

(4) $y\cdot B_{\mathfrak{S}}\left([x_{11}^{1k_1}],\dots,[x_{n1}^{nk_n}]\right) = B_{\mathfrak{S}}\left([y,x_{11}^{1k_1}],\dots,[y,x_{n1}^{nk_n}]\right)$.

(5) $B_{\mathfrak{S}}\left([x_{11}^{1k_1}],\dots,[x_{n1}^{nk_n}]\right) + B_{\mathfrak{S}}\left([y_{11}^{1l_1}],\dots,[y_{m1}^{ml_m}]\right) =$
$= B_{\mathfrak{S}}\left([x_{11}^{1k_1}],\dots,[x_{n1}^{nk_n}],[y_{11}^{1l_1}],\dots,[y_{m1}^{ml_m}]\right)$.

(6) $B_{\mathfrak{S}}\left([x_{11}^{1k_1}],\dots,[x_{n1}^{nk_n}]\right)\cdot B_{\mathfrak{S}}\left([y_{11}^{1l_1}],\dots,[y_{m1}^{ml_m}]\right) =$
$B_{\mathfrak{S}}\left([x_{11}^{1k_1},y_{11}^{1l_1}],\dots,[x_{11}^{1k_1},y_{m1}^{ml_m}],\dots,[x_{n1}^{nk_n},y_{11}^{1l_1}],\dots,[x_{n1}^{nk_n},y_{m1}^{ml_m}]\right)$.

Proof. It is straightforward. □

Lemma 3.3. *Lemma 3.2 is true for the geometric space $(R,\mathfrak{R})$.*

Lemma 3.4. *Let $(R,+,\cdot)$ be a hyperring. Then for every $\sigma\in\mathbb{S}_n$ and $\sigma_i\in\mathbb{S}_{k_i}$ we have*

$$B_{\mathfrak{S}}\left([x_{11}^{1k_1}],\dots,[x_{n1}^{nk_n}]\right) = B_{\mathfrak{S}}\left([x_{\sigma(1)\sigma_{\sigma(1)}(1)}^{\sigma(1)\sigma_{\sigma(1)}(k_{\sigma(1)})}],\dots,[x_{\sigma(n)\sigma_{\sigma(n)}(1)}^{\sigma(n)\sigma_{\sigma(n)}(k_{\sigma(n)})}]\right)$$

Moreover, if $(R,+,\cdot)$ is a commutative hyperring then two geometric spaces $(R,\mathfrak{R})$ and $(R,\mathfrak{S})$ are equal.

Proof. It obtains from definition of geometric spaces $(R,\mathfrak{R})$ and $(R,\mathfrak{S})$. □

Lemma 3.5. *Let $(R,+,\cdot)$ be a hyperring. Then:*

(1) *If $x_{rs}\in a\cdot b$ then* $B_{\mathfrak{S}}\left([x_{11}^{1k_1}],\dots,[x_{n1}^{nk_n}]\right)\subseteq$
$\subseteq B_{\mathfrak{S}}\left([x_{11}^{1k_1}],\dots,[x_{r1}^{r(s-1)},a,b,x_{r(s+1)}^{rk_r}],\dots,[x_{n1}^{nk_n}]\right)$.

(2) *If $x_{rs}\in a+b$ then* $B_{\mathfrak{S}}\left([x_{11}^{1k_1}],\dots,[x_{n1}^{nk_n}]\right)\subseteq$
$\subseteq B_{\mathfrak{S}}\left([x_{11}^{1k_1}],\dots,[x_{(r-1)1}^{(s-1)k_{r-1}}],[x_{r1}^{r(s-1)},a,x_{r(s+1)}^{rk_r}],\right.$
$\left.[x_{r1}^{r(s-1)},b,x_{r(s+1)}^{rk_r}],[x_{(r+1)1}^{(r+1)k_{r+1}}],\dots,[x_{n1}^{nk_n}]\right)$.

(3) *If $x_{rs}\in B = B_{\mathfrak{S}}\left([y_{11}^{1l_1}],\dots,[y_{m1}^{ml_m}]\right)$ then*
$B_{\mathfrak{S}}\left([x_{11}^{1k_1}],\dots,[x_{n1}^{nk_n}]\right)\subseteq B_{\mathfrak{S}}\left([x_{11}^{1k_1}],\dots,[x_{r1}^{r(s-1)},B,x_{r(s+1)}^{rk_r}],\dots,[x_{n1}^{nk_n}]\right)$
$= B_{\mathfrak{S}}\left([x_{11}^{1k_1}],\dots,[x_{(r-1)1}^{(r-1)k_{r-1}}],[\cdot],[x_{(r+1)1}^{(r+1)k_{r+1}}],\dots,[x_{n1}^{nk_n}]\right)$,
where $[\cdot]=[x_{r1}^{r(s-1)},y_{11}^{1l_1},x_{r(s+1)}^{rk_r}],\dots,[x_{r1}^{r(s-1)},y_{m1}^{ml_m},x_{r(s+1)}^{rk_r}]$.

Proof. (1) Let $x_{rs}\in a\cdot b$ and $y\in B_{\mathfrak{S}}([x_{11}^{1k_1}],\dots,[x_{n1}^{nk_n}])$. Then there exists $\sigma\in\mathbb{S}_n$ and $\sigma_r\in\mathbb{S}_{k_r}$ such that $y\in\sum_{i=1}^{n}\prod_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)}$, if $\sigma(u)=r$ and $\sigma_r(v)=s$ then we have $x_{\sigma(u)\sigma_{\sigma(u)}(v)} = x_{rs} \in a \cdot b$ and so we have

$$\begin{aligned}
y &\in \sum_{i=1}^{n} \prod_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)} \\
&= \sum_{i=1}^{u-1} \prod_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)} + \prod_{j=1}^{v-1} x_{r\sigma_r(j)} \cdot x_{rs} \cdot \prod_{j=v+1}^{k_u} x_{r\sigma_r(j)} + \sum_{i=u+1}^{n} \prod_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)} \\
&\subseteq \sum_{i=1}^{u-1} \prod_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)} + \prod_{j=1}^{v-1} x_{r\sigma_r(j)} \cdot a \cdot b \cdot \prod_{j=v+1}^{k_u} x_{r\sigma_r(j)} + \sum_{i=u+1}^{n} \prod_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)} \\
&= B_{\mathfrak{S}}([x_{11}^{1k_1}], \ldots, [x_{r1}^{r(s-1)}, a, b, x_{r(s+1)}^{rk_r}], \ldots, [x_{n1}^{nk_n}]).
\end{aligned}$$

Proof of (2) is similar to (1) and (3) obtains from (1) and (2). $\square$

Lemma 3.6. *Lemma 3.5, is true for the geometric space* $(R, \mathfrak{R})$.

Theorem 3.7. *If* $(R, +, \cdot)$ *is a hyperfield then*

(1) *The geometric space* $(R, \mathfrak{R})$ *is a strongly transitive geometric space.*
(2) *The geometric space* $(R, \mathfrak{S})$ *is a strongly transitive geometric space.*

Proof. (1) Let $B = B_{\mathfrak{R}}\left([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]\right)$ and $B' = B_{\mathfrak{R}}\left([y_{11}^{1l_1}], \ldots, [y_{m1}^{ml_m}]\right)$ be two blocks of $\mathfrak{R}$ such that

$$B \cap B' \neq \emptyset \quad \text{and} \quad x \in B'.$$

Let $b \in B \cap B'$. Since $(R, +, \cdot)$ is a hyperfield, thus there exist $u_1^n \in R$ and $v \in R$ such that $x_{ik_i} \in u_i \cdot x$ and $x \in b \cdot v$. Now, we have:

$$\begin{aligned}
x \in b \cdot v \quad &\subseteq B_{\mathfrak{R}}\left([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]\right) \cdot v \\
&\subseteq B_{\mathfrak{R}}\left([x_{11}^{1(k_1-1)}, u_1, x], \ldots, [x_{n1}^{n(k_n-1)}, u_n, x]\right) \cdot v \\
&= B_{\mathfrak{R}}\left([x_{11}^{1(k_1-1)}, u_1, x, v], \ldots, [x_{n1}^{n(k_n-1)}, u_n, x, v]\right) \\
&\subseteq B_{\mathfrak{R}}\left([x_{11}^{1(k_1-1)}, u_1, B', v], \ldots, [x_{n1}^{n(k_n-1)}, u_n, B', v]\right).
\end{aligned}$$

So $C = B_{\mathfrak{R}}\left([x_{11}^{1(k_1-1)}, u_1, B', v], \ldots, [x_{n1}^{n(k_n-1)}, u_n, B', v]\right)$ is a block of $\mathfrak{R}$ and $x \in C$. Moreover, since $b \in B$, we obtain:

$$\begin{aligned}
B = B_{\mathfrak{R}}\left([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]\right) &\subseteq B_{\mathfrak{R}}\left([x_{11}^{1(k_1-1)}, u_1, x], \ldots, [x_{n1}^{n(k_n-1)}, u_n, x]\right) \\
&\subseteq B_{\mathfrak{R}}\left([x_{11}^{1(k_1-1)}, u_1, b, v], \ldots, [x_{n1}^{n(k_n-1)}, u_n, b, v]\right) \\
&\subseteq B_{\mathfrak{R}}\left([x_{11}^{1(k_1-1)}, u_1, B', v], \ldots, [x_{n1}^{n(k_n-1)}, u_n, B', v]\right) \\
&= C.
\end{aligned}$$

Therefore, $B \cup \{x\} \subseteq C$ and the geometric space $(R, \mathfrak{R})$ is strongly transitive.

In a similar way we obtain (2). □

Theorem 3.8. *Let $(R, +, \cdot)$ be a hyperfield. Then*

(1) *The relation Γ is a strongly regular equivalence relation and $\Gamma = \Gamma^*$, where Γ^* is the fundamental relation on R.*
(2) *The relation α is a strongly regular equivalence relation and $\alpha = \alpha^*$, where α^* is the commutative fundamental relation on R.*

Proof. We prove (2). The proof of (1) is similar. Let $(R, +, \cdot)$ be a hyperring. Then the relation $\sim$ defined on the geometric space $(R, \mathfrak{S})$ coincides with the relation α on the hyperring $(R, +, \cdot)$. Also, the relation $\approx$ defined on the geometric space $(R, \mathfrak{S})$ coincides with the commutative fundamental relation α^* on the hyperring $(R, +, \cdot)$. Now, if $(R, +, \cdot)$ is a hyperfield then the geometric space $(R, \mathfrak{S})$ is strongly transitive by Theorem 3.7. Thus by Theorem 2.5, we have:

$$\alpha \ = \sim \ = \ \approx \ = \ \alpha^*.$$

□

Let $\varphi : R \to R/\Gamma^*$ and $\phi : R \to R/\alpha^*$ be the canonical projection homomorphisms.

Proposition 3.9. *For every non-empty subset A of a hyperring R such that $(R, +)$ has an identity 0, we have:*

(1) $\varphi^{-1}(\varphi(A)) = \omega(R) + A = A + \omega(R)$, *where* $\omega(R) = \varphi^{-1}(0_{R/\Gamma^*})$.
(2) *If A is a $\mathfrak{R}$-part of R, then $\varphi^{-1}(\varphi(A)) = A$.*
(3) $\phi^{-1}(\phi(A)) = D(R) + A = A + D(R)$, *where* $D(R) = \phi^{-1}(0_{R/\alpha^*})$.
(4) *If A is a $\mathfrak{S}$-part of R, then $\phi^{-1}(\phi(A)) = A$.*

Proof. (1) For every $x \in \omega(R) + A$, there exists a pair $(a, b) \in A \times \omega(R)$ such that $x \in b + a$. Then $\varphi(x) = \varphi(b) + \varphi(a) = 0_{R/\Gamma^*} + \varphi(a) = \varphi(a)$. Therefore, $x \in \varphi^{-1}(\varphi(a)) \subseteq \varphi^{-1}(\varphi(A))$.

Conversely, for every $x \in \varphi^{-1}(\varphi(A))$, there exists an element $a \in A$ such that $\varphi(x) = \varphi(a)$. Since $(R, +)$ is a hypergroup, there exists $b \in R$ such that $x \in b + a$, so $\varphi(a) = \varphi(x) = \varphi(b) + \varphi(a)$ and this implies that $\varphi(b) = 0_{R/\Gamma^*}$. Therefore, $b \in \varphi^{-1}(0_{R/\Gamma^*}) = \omega(R)$ and so $x \in b + a \subseteq \omega(R) + A$. This proves that $\varphi^{-1}(\varphi(A)) = \omega(R) + A$.

In the same way, we can prove that $\varphi^{-1}(\varphi(A)) = A + \omega(R)$.

(2) It is obvious that $A \subseteq \varphi^{-1}(\varphi(A))$. Moreover, if $x \in \varphi^{-1}(\varphi(A))$, then there exists an element $a \in A$ such that $\varphi(a) = \varphi(x)$. Hence $x \in \Gamma^*(x) = \Gamma^*(a) \subseteq A$ and $\varphi^{-1}(\varphi(A)) \subseteq A$.

In a similar way, we can prove (3) and (4). □

Theorem 3.10. *Let $(R, +, \cdot)$ be a hyperring such that $(R, +)$ has an identity 0. If A and B are two blocks of $(R, \mathfrak{R})$, $0 \in B$ and $A \cap B \neq \emptyset$ then there exists a block C such that $A \cup \{0\} \subseteq C$. Also, this theorem is true for the geometric space $(R, \mathfrak{S})$.*

Proof. If $x \in A \cap B$ then there exists $x' \in R$ such that $0 \in x + x'$, so we have:

$$0 \in x + x' \subseteq A + x' \subseteq A + 0 + x' \subseteq A + B + x',$$

also,

$$A \subseteq A + 0 \subseteq A + x + x' \subseteq A + B + x'.$$

Set $C = A + B + x'$. Then C is a block and $A \cup \{0\} \subseteq C$. □

Theorem 3.11. *Let $(R, +, \cdot)$ be a hyperring such that $(R, +)$ has an identity 0. Then two geometric spaces $(R, \mathfrak{R})$ and $(R, \mathfrak{S})$ are strongly transitive.*

Proof. Let A and B be two blocks of $(R, \mathfrak{R})$, $A \cap B \neq \emptyset$ and $x \in B$. We prove that there exists a block C of $(R, \mathfrak{R})$ such that $A \cup \{x\} \subseteq C$. Let $b \in A \cap B$ and $a \in A$,. Then $\{a, b\} \subseteq A$ and $\{x, b\} \subseteq B$. Hence, $a\Gamma b$ and $b\Gamma x$ and this implies that $a\Gamma^* x$. Then by part (1) of Proposition 3.9,

$$x \in \varphi^{-1}(\varphi(a)) = a + \omega(R).$$

Thus, there exists $y \in \omega(R)$ such that $x = a + y$. Since $a \in \omega(R)$ so there exists a block A' such that $\{0, y\} \subseteq A'$. Now, we have

$$x \in a + y \subseteq A' + A$$

and

$$A \subseteq 0 + A \subseteq A' + A.$$

Therefore, $C = A' + A$ is a block and $A \cup \{x\} \subseteq C$.

In a similar way we obtain the geometric space $(R, \mathfrak{S})$ is strongly transitive. □

Theorem 3.12. *Let $(R, +, \cdot)$ be a hyperring such that $(R, +)$ has an identity 0. Then*

(1) *The relation Γ is transitive and $\Gamma = \Gamma^*$, where Γ^* is the fundamental relation on R.*
(2) *The relation α is transitive and $\alpha = \alpha^*$, where α^* is the commutative fundamental relation on R.*

Proof. We prove (1), the proof of (2) is similar. Let $(R, +, \cdot)$ be a hyperring. Then the relation $\sim$ defined on the geometric space $(R, \mathfrak{R})$ coincides with the relation Γ on the hyperring $(R, +, \cdot)$. Also, the relation $\approx$ defined on the geometric space $(R, \mathfrak{R})$ coincides with the fundamental relation Γ^* on the hyperring $(R, +, \cdot)$. Now, if $(R, +, \cdot)$ is a hyperring such that $(R, +)$ has an identity 0 then by Theorem 3.11, the geometric space $(R, \mathfrak{R})$ is strongly transitive. Therefore, by Theorem 2.5, we obtain:

$$\Gamma \ = \sim \ = \ \approx \ = \ \Gamma^*.$$

□

Corollary 3.13. Let $(R, +, \cdot)$ be a Krasner hyperring. Then $\Gamma = \Gamma^*$ and $\alpha = \alpha^*$.

Example 1. This example shows that Theorem 3.12 for semihyperrings is not valid.

Set $R = \{a, b, c, d\}$ and hyperoperations $+$ and $\cdot$ are defined as follow:

$+$	a	b	c	d
a	$\{b, c\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$
b	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$
c	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$
d	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$

$\cdot$	a	b	c	d
a	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$
b	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$
c	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$
d	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$

$(R, +, \cdot)$ is a semihyperring. We have $b\Gamma c$ and $b\Gamma d$ but $c\not\Gamma d$ and so Γ is not transitive.

Example 2. Let $R = \{a, b, c, d, e, f, g\}$ and $+$ be a hyperoperation defined as follows:

$+$	a	b	c	d	e	f	g
a	$\{a, b\}$	$\{a, b\}$	c	d	e	f	g
b	$\{a, b\}$	$\{a, b\}$	c	d	e	f	g
c	c	c	$\{a, b\}$	f	g	d	e
d	d	d	g	$\{a, b\}$	f	e	c
e	e	e	f	g	$\{a, b\}$	c	d
f	f	f	e	c	e	g	$\{a, b\}$
g	g	g	d	e	c	$\{a, b\}$	f

$(R, +)$ is a hypergroup. Now, we define a hyperoperation $\cdot$ as follows:

$$x \cdot y = \{a, b\}, \qquad \forall x, y \in R.$$

It is not difficult to see that $(R, +, \cdot)$ is a hyperring and $\Gamma = \Gamma^*$, $\alpha = \alpha^*$ and $\delta_R \neq \Gamma \neq \alpha \neq \Delta_R$. In fact, $\Gamma(a) = \{a, b\}$ and for all $x \in R - \{a, b\}$, $\Gamma(x) = \{x\}$. Also, $\alpha(a) = \{a, b, g, f\}$ and $\alpha(c) = \{c, d, e\}$.

The foregoing examples can be easily extended to larger sets.

Example 3. Let R be a ring as follows:

$$R = TL_2(\mathbb{Z}_2) = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{Z}_2 \right\}.$$

$(R, +, \cdot)$ is a non-commutative ring and so $\Gamma = \Gamma^* = \delta_R$. Moreover,

$$\alpha\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}, \quad \alpha\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right\},$$

$$\alpha\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right\}, \quad \alpha\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right\}.$$

$\alpha = \alpha^*$ and $\Gamma^* \neq \alpha^*$.

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