REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 53, No. 1, 2012, 61–70

SEMI-CONVERGENCE ANALYSIS OF THE INEXACT UZAWA METHOD FOR SINGULAR SADDLE POINT PROBLEMS*

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ABSTRACT. Recently, various Uzawa methods were proposed based on different matrix splitting for solving nonsingular saddle point problems, and the necessary and sufficient condition of the convergence for those Uzawa methods were derived. Motivated by their results, in this paper we give the semi-convergence analysis of the inexact Uzawa method which is applied to solve singular saddle point problems under certain conditions.

1. INTRODUCTION

We consider the following 2×2 block linear systems of the form:

$$\mathcal{A}u = \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} = b, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix, $B \in \mathbb{R}^{m\times n}$ with rank $B = r$ and $m > n$, $f \in \mathbb{R}^m$ and $g \in \mathbb{R}^n$ are two given vectors, denote B^T as the transpose of the matrix B. When $r = n$, note that the coefficient matrix $\mathcal{A}$ is nonsingular and the linear systems (1.1) have a unique solution. When $r < n$, the coefficient matrix $\mathcal{A}$ is singular, and at the moment, we suppose that the linear systems (1.1) are consistent, i.e., $b \in \mathcal{R}(\mathcal{A})$, the range of $\mathcal{A}$. The linear systems (1.1) are referred to as saddle point problems. Other popular name of saddle point systems are Karush-Kuhn-Tucker (KKT) systems or augmented systems, especially in the optimization literature and the least squares problem, respectively. The saddle point problems (1.1) are important and arise in a large number of scientific and engineering applications, for example the field of computational fluid dynamics [17], constrained and weighted least squares [8], interior point methods in constrained optimization [6], mixed finite element approximations of elliptic partial differential equations (PDEs) [11]. Especially, see [5] for a comprehensive survey.

In recent years, a large amount of work have been developed to solve the linear systems (1.1). As is known, there exist two kinds of methods to solve the

2000 *Mathematics Subject Classification.* 65F10; 65F50.
Key words and phrases. preconditioner; saddle point problems; inexact Uzawa method; semi-convergence; iterative method.
*This research is supported by NSFC (60973015, 61170311), Chinese Universities Specialized Research Fund for the Doctoral Program (20110185110020), Sichuan Province Sci. & Tech. Research Project (12ZC1802) and NCWU(201119).
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linear systems: direct methods and iterative methods. Direct methods are widely employed when the size of the coefficient matrix is not too large, and are usually regarded as robust methods. However, frequently, the matrices A and B are large and sparse, so iterative methods become more attractive than direct methods for solving the saddle point problems (1.1). The frequently used iterative methods in scientific computing are the Uzawa methods, because they are simple and have minimal computer memory requirements. The study of the Uzawa method for nonsingular systems (1.1) and its convergence can be traced back to Uzawa [1] and there have been many investigations by many researchers since then, see [1–3, 13, 15, 16, 18, 19, 21, 23, 25, 28] and also many references cited therein for more details. In fact, most Uzawa type methods can be indicated in terms of splitting of the coefficient matrix. Different matrix splitting can lead to different Uzawa method. Certainly, there also exist many other methods for solving saddle point problems, such as, the preconditioned iterative methods [9, 24]. For a broad overview of the numerical solution of saddle point systems, one can see [5].

The best known Uzawa method is the inexact Uzawa method which is studied in [10, 15, 16] by different consideration. The inexact Uzawa method is defined as follows [10, 15]:

Method 1.1 (Inexact Uzawa method). *Assume that the preconditioners Q_A and Q_B are symmetric and positive definite, for initial vectors $x_0 \in \mathbb{R}^m$ and $y_0 \in \mathbb{R}^n$, the sequence $\{x_i, y_i\}$ is defined for $i = 1, 2 \ldots$ by*

$$\begin{cases} x_{i+1} = x_i + Q_A^{-1}(f - (Ax_i + By_i)), \\ y_{i+1} = y_i + Q_B^{-1}(B^T x_{i+1} - g). \end{cases} \tag{1.2}$$

Evidently, different preconditioners lead to different Uzawa methods from the inexact Uzawa method (1.2), see Table 1. By introducing a relaxation parameter, the inexact Uzawa method (1.2) is extended to the general inexact Uzawa method in [13]. Those Uzawa methods are studied when the matrix B is of full column rank. On the computation effectiveness of those Uzawa methods, one can see related references. In most cases, the matrix B is full column rank in scientific computing and engineering applications, but not always. If $r < n$, the linear systems (1.1) become the singular saddle point problems. When the linear systems (1.1) are consistent, Zheng, Bai and Yang [30] show that the GSOR method proposed in [2] can be used to solve the singular saddle point problems (1.1) and the iterative method is semi-convergent. Li and Huang [20] give the semi-convergent analysis of the GSSOR method presented in [29] for the singular saddle point problems (1.1).

In this paper, we show that the inexact Uzawa method (1.2) can be used to solve the linear systems (1.1) when $r < n$. In Section 2, we will illustrate the semi-convergence of the inexact Uzawa method, and furthermore demonstrate the semi-convergence of the general inexact Uzawa method proposed in [13]. The conclusions are given in Section 3.

TABLE 1. The different Uzawa methods for different preconditioners

The different preconditioners Q_A, Q_B	The corresponding different Uzawa method
$Q_A = A$, $Q_B = \frac{1}{\tau} I$	Uzawa algorithm [1]
$Q_A = A$, $Q_B = Q$	Preconditioned Uzawa algorithm [16]
$Q_A = \frac{1}{\omega} A$, $Q_B = \frac{1}{\omega} Q$	SOR-like method [18]
$Q_A = \frac{1}{\omega} A$, $Q_B = \frac{1}{\tau} Q$	GSOR method [2]
$Q_A = \frac{1}{\omega} P$, $Q_B = \frac{1}{\tau} Q$	PIU method [3]
$Q_A = \frac{1}{\omega\alpha} A$, $Q_B = \frac{1}{\omega} Q$	GSOR-like method [19]
$Q_A = \frac{1}{\omega} A$, $Q_B = \frac{w}{\gamma} Q$	Two-parameter GSOR method [21]
$Q_A = \frac{1}{\omega} A$, $Q_B = \frac{1-\omega\alpha}{\omega} Q$	MSOR-like method [25]

2. The semi-convergence of the inexact Uzawa method and general inexact Uzawa method

Before the semi-convergence of the inexact Uzawa method and the general inexact Uzawa method are given, we first give some basic concepts and lemmas for latter use.

For a matrix $A \in \mathbb{R}^{m\times m}$, the splitting $A = M - N$ is a nonsingular splitting if M is nonsingular. Denote $\sigma(A)$ and $\rho(A)$ as the spectrum and spectral radius of a square matrix A, respectively. I is the identity matrix with appropriate dimension. Let $T = M^{-1}N$, then solving linear systems $Ax = c$ is equivalent to considering the following iterative scheme

$$x_{i+1} = Tx_i + M^{-1}c, \qquad k = 0, 1, 2\ldots \tag{2.1}$$

It is well known that for nonsingular systems the iterative scheme (2.1) is convergent if and only if $\rho(T) < 1$. But for the singular systems, we have $1 \in \sigma(T)$ and $\rho(T) \geq 1$, so that one can require only the semi-convergence of the iterative method (2.1). By [7], the iterative scheme (2.1) is semi-convergent if and only if the following three conditions are satisfied:

(1) $\rho(T) = 1$;
(2) Elementary divisors associated with $\lambda = 1 \in \sigma(T)$ are linear, i.e.,
$\operatorname{rank}(I - T)^2 = \operatorname{rank}(I - T)$;
(3) If $\lambda \in \sigma(T)$ with $|\lambda| = 1$, then $\lambda = 1$, i.e.,

$$\vartheta(T) = \max\{|\lambda|, \lambda \in \sigma(T), \lambda \neq 1\} < 1.$$

In this situation, the associated convergence factor is $\vartheta(T)$. We call a matrix T is semi-convergent provided it satisfies the above three conditions, and iterative method (2.1) is semi-convergent if T is a semi-convergent matrix. On the semi-convergence of the iterative method for solving general singular linear systems $Ax = b$, for more details, one can see [4, 7, 12, 14, 26, 27]. When A is singular, the following two lemmas give the semi-convergence property about the iteration method (2.1).

Lemma 2.1 [4,7] *Let $A = M - N$ with M nonsingular, $T = M^{-1}N$. Then for any initial vector x_0, the iterative scheme (2.1) is semi-convergent to a solution x of linear equations $Ax = c$ if and only if the matrix T is semi-convergent.*

Lemma 2.2 [14,30] *Let $H \in \mathbb{R}^{l\times l}$ with positive integers l. Then the partitioned matrix*

$$T = \begin{pmatrix} H & 0 \\ L & I \end{pmatrix}$$

is semi-convergent if and only if either of the following conditions holds true:

(1) $L = 0$ *and H is semi-convergent;*
(2) $\rho(H) < 1$.

In fact, based on the following matrix splitting:

$$\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} = \begin{pmatrix} Q_A & 0 \\ -B^T & Q_B \end{pmatrix} - \begin{pmatrix} Q_A - A & -B \\ 0 & Q_B \end{pmatrix},$$

the above inexact Uzawa method (1.2) can be regarded as the following iterative method

$$\begin{pmatrix} Q_A & 0 \\ -B^T & Q_B \end{pmatrix}\begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} = \begin{pmatrix} Q_A - A & -B \\ 0 & Q_B \end{pmatrix}\begin{pmatrix} x_i \\ y_i \end{pmatrix} + \begin{pmatrix} f \\ -g \end{pmatrix} \tag{2.2}$$

and the corresponding iteration matrix is

$$T = \begin{pmatrix} I_m - Q_A^{-1}A & -Q_A^{-1}B \\ Q_B^{-1}B^T(I_m - Q_A^{-1}A) & I_n - Q_B^{-1}B^TQ_A^{-1}B \end{pmatrix}. \tag{2.3}$$

When $r = n$, the matrix $\mathcal{A}$ is nonsingular, and the systems (1.1) have a unique solution. The detailed convergence analysis about the inexact Uzawa method (1.2) are given in [10] and [15] by using different method. When $r < n$, the matrix $\mathcal{A}$ is singular, and the systems (1.1) have infinity many solutions. In this paper, we suppose that the singular linear systems (1.1) are consistent. The following theorem describes the semi-convergence property about the inexact Uzawa method (1.2) for solving the singular linear systems (1.1).

Theorem 2.1 *Assume that $r < n$, then the inexact Uzawa method (1.2) is semi-convergent to a solution x of the singular saddle point problems (1.1) if and only if the preconditioners satisfy the following condition:*

$$4Q_A - 2A - BQ_B^{-1}B^T \tag{2.4}$$

is positive definite.

Proof. By Lemma 2.1, we only need to describe the semi-convergence of the iteration matrix T of the inexact Uzawa method (1.2) defined in (2.3).

Let $B = U(B_r, 0)V^*$ be the singular value decomposition of B, $B_r = (\Sigma_r, 0)^T \in \mathbb{R}^{m\times r}$ with $\Sigma_r = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$, U, V are unitary matrices. Then

$$P = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$$

is an $(m+n)$-by-$(m+n)$ unitary matrix. Define $\widehat{T} = P^*TP$, P^* denotes the conjugate transpose of P, then the matrix T has the same eigenvalues with matrix $\widehat{T}$. Hence, we only need to demonstrate the semi-convergence of the matrix $\widehat{T}$.

Let

$$\widehat{A} = U^*AU,\ \widehat{Q_A} = U^*Q_AU,\ \widehat{B} = U^*BV \text{ and } \widehat{\mathrm{Q_B}} = \mathrm{V^*Q_BV}.$$

Then it holds that $\widehat{B} = (B_r, 0)$ and

$$\widehat{Q_B}^{-1} = \begin{pmatrix} V_1^*Q_B^{-1}V_1 & V_1^*Q_B^{-1}V_2 \\ V_2^*Q_B^{-1}V_1 & V_2^*Q_B^{-1}V_2 \end{pmatrix}$$

with appropriate partitioned matrix $V = (V_1, V_2)$. By simple computation, we have

$$\widehat{T} = \begin{pmatrix} \widehat{H} & 0 \\ \widehat{L} & I_{n-r} \end{pmatrix}$$

where

$$\widehat{H} = \begin{pmatrix} I_m - \widehat{Q_A}^{-1}\widehat{A} & -\widehat{Q_A}^{-1}B_r \\ V_1^*Q_B^{-1}V_1B_r^T(I_m - \widehat{Q_A}^{-1}\widehat{A}) & I_r - V_1^*Q_B^{-1}V_1B_r^T\widehat{Q_A}^{-1}B_r \end{pmatrix}$$

and

$$\widehat{L} = (V_2^*Q_B^{-1}V_1B_r^T(I_m - \widehat{Q_A}^{-1}\widehat{A}),\ -V_2^*Q_B^{-1}V_1B_r^T\widehat{Q_A}^{-1}B_r).$$

As $\widehat{L} \neq 0$, from Lemma 2.2 we know that the matrix $\widehat{T}$ is semi-convergent if and only if $\rho(\widehat{H}) < 1$.

When the inexact Uzawa method (1.2) is applied to solve the following nonsingular saddle point problem

$$\begin{pmatrix} \widehat{A} & B_r \\ B_r^T & 0 \end{pmatrix} \begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix} = \begin{pmatrix} \widehat{f} \\ \widehat{g} \end{pmatrix}, \tag{2.5}$$

with the preconditioning matrix $\widehat{Q_A}$ and $Q_{B1} = (V_1^*Q_B^{-1}V_1)^{-1}$, and vectors $\widehat{y},\ \widehat{g} \in \mathbb{R}^r$, then the iterative matrix of the inexact Uzawa method is $\widehat{H}$. From Theorem 3.1 of [15], we obtain that $\rho(\widehat{H}) < 1$ if and only if

$$4\widehat{Q_A} - 2\widehat{A} - B_rQ_{B1}^{-1}B_r^T$$

is positive definite.

Since

$$\begin{aligned} U^*(4Q_A - 2A - BQ_B^{-1}B^T)U &= 4\widehat{Q_A} - 2\widehat{A} - (U^*BV)(V^*Q_B^{-1}V)(V^*B^TU) \\ &= 4\widehat{Q_A} - 2\widehat{A} - (B_r,\ 0)\widehat{Q_B}^{-1}(B_r,\ 0)^T \\ &= 4\widehat{Q_A} - 2\widehat{A} - B_rQ_{B1}^{-1}B_r^T, \end{aligned}$$

we know that matrix $4\widehat{Q_A} - 2\widehat{A} - B_rQ_{B1}^{-1}B_r^T$ positive definite is equivalent to the matrix $4Q_A - 2A - BQ_B^{-1}B^T$ positive definite. By the above analysis, the proof is completed. $\square$

In [13], Chen and Jiang extended the inexact Uzawa method (1.2) and proposed the following general inexact Uzawa method based on the matrix splitting

$$\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} = \begin{pmatrix} Q_A & 0 \\ (-B+Q_3)^T & Q_B \end{pmatrix} - \begin{pmatrix} Q_A - A & -B \\ Q_3^T & Q_B \end{pmatrix}$$

with $Q_A = A + Q_1$ to solve the the nonsingular linear system (1.1).

Method 2.1 (General inexact Uzawa method) [13]. *Assume that the preconditioners Q_A and Q_B are symmetric and positive definite, $Q_3 \in \mathbb{R}^{m\times n}$ is such that $BQ_B^{-1}Q_3^T$ is symmetric, for initial vectors $x_0 \in \mathbb{R}^m$ and $y_0 \in \mathbb{R}^n$, the sequence $\{x_i, y_i\}$ is defined for $i = 1, 2\ldots$ by*

$$\begin{cases} x_{i+1} = x_i + Q_A^{-1}(f - (Ax_i + By_i)), \\ y_{i+1} = y_i + Q_B^{-1}((B - Q_3)^T x_{i+1} + Q_3 x_i - g). \end{cases} \tag{2.6}$$

When the general inexact Uzawa method (2.6) is applied to solve the singular linear system (1.1), the following theorem gives the semi-convergence property of the general inexact Uzawa method (2.6).

Theorem 2.2 *Assume that $r < n$, Q_A, Q_B are symmetric and positive definite, and $Q_3 \in \mathbb{R}^{m\times n}$ is such that $BQ_B^{-1}Q_3^T$ is symmetric, then the general inexact Uzawa method (2.6) is semi-convergent if and only if:*

$$4Q_A - 2A - BQ_B^{-1}B^T + 2BQ_B^{-1}Q_3^T \ \text{and} \ A - BQ_B^{-1}Q_3^T$$

are positive definite.

Proof. The proof is similar to that given of the above Theorem 2.1, therefore, it is omitted. □

Especially, when $Q_3 = tB$(t is a real relaxation parameter), the general inexact Uzawa method (2.6) becomes

$$\begin{cases} x_{i+1} = x_i + Q_A^{-1}(f - (Ax_i + By_i)), \\ y_{i+1} = y_i + Q_B^{-1}((1-t)B^T x_{i+1} + tB^T x_i - g). \end{cases} \tag{2.7}$$

And the iterative matrix of iteration scheme (2.7) is

$$\widetilde{T} = \begin{pmatrix} Q_A & 0 \\ (t-1)B^T & Q_B \end{pmatrix}^{-1} \begin{pmatrix} Q_A - A & -B \\ tB^T & Q_B \end{pmatrix}.$$

In this setting, the general inexact Uzawa method (2.7) is semi-convergent if and only if matrices

$$4Q_A - 2A + (2t-1)BQ_B^{-1}B^T \ \text{ and } \ A - tBQ_B^{-1}B^T \tag{2.8}$$

are positive definite.

When $Q_A = \frac{1}{\omega}A$, $Q_B = \frac{1}{\omega}Q$ and $t = 1 - \frac{\tau}{\omega}$, where τ and ω are real parameters, then the above general inexact Uzawa method (2.7) is equivalent to the following generalized AOR (GAOR) method (2.9) defined in [22]:

$$\begin{cases} x_{i+1} = x_i + \omega A^{-1}(f - (Ax_i + By_i)), \\ y_{i+1} = y_i + \omega Q^{-1}(\frac{\tau}{\omega} B^T x_{i+1} + (1 - \frac{\tau}{\omega}) B^T x_i - g). \end{cases} \tag{2.9}$$

If

$$D = \begin{pmatrix} A & 0 \\ 0 & Q \end{pmatrix}, \; L = \begin{pmatrix} 0 & 0 \\ -B^T & 0 \end{pmatrix}, \; U = \begin{pmatrix} 0 & -B \\ 0 & Q \end{pmatrix},$$

then the GAOR method can be defined by the following iteration

$$(D - \tau L) \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} = [(1-\omega)D + (\omega - \tau)L + \omega U] \begin{pmatrix} x_i \\ y_i \end{pmatrix} + \omega \begin{pmatrix} f \\ -g \end{pmatrix}. \tag{2.10}$$

Theorem 2 in [22] gives a necessary and sufficient condition of the GAOR method to converge for solving the nonsingular saddle point problems (1.1). When the GAOR method is applied to solve the singular saddle point problems, the following Theorem gives a necessary and sufficient condition of the GAOR method to semi-converge.

Theorem 2.3 *Assume that $r < n$, Q is a symmetric and positive definite matrix, $0 < \omega < 2$, denote the maximum eigenvalue of $Q^{-1}B^TA^{-1}B$ by $\mu_{\max}$, then the GAOR method* (2.9) *is semi-convergent if and only if:*

$$\omega - \frac{1}{\mu_{\max}} < \tau < \frac{2-\omega}{\omega \mu_{\max}} + \frac{\omega}{2}. \tag{2.11}$$

Remark 1. In essence, the general inexact Uzawa method (2.7) is the same as the GAOR method. As a matter of fact, by simple computation, note that the condition (2.8) is equivalent to the condition (2.11).

In fact, the general inexact Uzawa method (2.7) can be taken as extension of the inexact Uzawa method (1.2). However, the semi-convergence condition of the general inexact Uzawa method (2.7) is weaker than that of the inexact Uzawa method (1.2), hence, the general inexact Uzawa method (2.7) has more application value than the inexact Uzawa method (1.2). For example, let

$$A = \mathrm{diag}\{100,\ 10,\ 1,\ 1\},\ B^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$

and

$$Q_A = \mathrm{diag}\{40,\ 20,\ 2,\ 2\},\ Q_B = I_2,$$

then all the three matrices A, Q_A and Q_B are symmetric positive definite, and rank $(B) = 1$. We know that

$$4Q_A - 2A - BQ_B^{-1}B^T = \mathrm{diag}\{-45,\ 60,\ 6,\ 6\}$$

is symmetric and indefinite. Hence, the inexact Uzawa methods is not semi-convergent. The semi-convergence can also be concluded by the eigenvalues of the iteration matrix. The iteration matrix of the inexact Uzawa algorithm (1.2) is

$$T = \begin{pmatrix} 40 & 0 & 0 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -60 & 0 & 0 & 0 & -1 & -2 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1.5 & 0 & 0 & 0 & -0.025 & -0.05 \\ 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ -1.5 & 0 & 0 & 0 & 0.975 & -0.05 \\ -3 & 0 & 0 & 0 & -0.05 & -0.9 \end{pmatrix}$$

and the eigenvalues of T are -1.5765, 0.9515, 1, 0.5, 0.5 and 0.5. So the spectral radius of T satisfies $\rho(T) > 1$, and the inexact Uzawa algorithm (1.2) is not semi-convergent.

For the general inexact Uzawa method (2.7), by simple computation, we have that

$$4Q_A - 2A + (2t-1)BQ_B^{-1}B^T = \mathrm{diag}\{10t - 45,\ 60,\ 6,\ 6\}$$

and

$$A - tBQ_B^{-1}B^T = \mathrm{diag}\{100 - 5t,\ 10,\ 1,\ 1\}.$$

Hence, we conclude that the general inexact Uzawa method is semi-convergent if and only if $4.5 < t < 20$. For example, when

- $t = 10$, the eigenvalues of $\widetilde{\mathrm{T}}$ are $-0.2771,\ 0.9021,\ 1,\ 0.5,\ 0.5$ and 0.5;
- $t = 2$, the eigenvalues of $\widetilde{\mathrm{T}}$ are $-1.3211,\ 0.9461,\ 1,\ 0.5,\ 0.5$ and 0.5;
- $t = 30$, the eigenvalues of $\widetilde{\mathrm{T}}$ are $1.125,\ 2,\ 1,\ 0.5,\ 0.5$ and 0.5.

3. Conclusion

In this paper, when the inexact Uzawa method and generalized inexact Uzawa method are applied to solve singular saddle point problems, the semi-convergence analysis of those two methods are given under certain conditions. And the example shows that the semi-convergence condition of the general inexact Uzawa method (2.7) is weaker than that of the inexact Uzawa method (1.2).

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Recibido: 17 de mayo de 2010
Aceptado: 5 de enero de 2011