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A GRAPH THEORETICAL MODEL FOR THE TOTAL BALANCEDNESS OF COMBINATORIAL GAMES

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ABSTRACT. In this paper we present a model for the study of the total balancedness of packing and covering games, concerning some aspects of graph theory. We give an alternative proof of van Velzen's characterization of totally balanced covering games. We introduce new types of graph perfection, which allows us to give another approach to the open problem of characterizing totally balanced packing games.

1. INTRODUCTION

A cooperative game (N, v) consists of a set of *players* N and a *characteristic function* $v : 2^N \to \mathbb{R}_+$. Given (N, v) and $T \subseteq N$, (T, v_T) is the *subgame* induced by T if $v_T(S) = v(S)$, for each subset S of T. A vector $x \in \mathbb{R}^N_+$ is called an *imputation* (or *preimputation*) if $\sum_{i\in N} x_i = v(N)$.

Depending on the application areas, we have *revenue* games and *cost* games. For a revenue game, an imputation x is in the *core* if and only if $x(S) \geq v(S)$, for every $S \subseteq N$. Similarly, for a cost game an imputation x is in the core if and only if $x(S) \leq v(S)$, for every $S \subseteq N$.

Following [8], a game is *balanced* if its core is non-empty. A balanced game is *totally balanced* if all its subgames are balanced.

Partly following the terminology in [5] and [9], we consider two particular classes of combinatorial optimization games: packing and covering games.

Given a $0, 1$ matrix $A = (a_{ij})$ of order $n \times m$ without zero columns, $N = \{1, 2, \dots, n\}$ and $M = \{1, 2, \dots, m\}$, (N, v^p_A) is a *(simple) packing game* and (M, v^c_A) is a *(simple) covering game* associated with A if

- for $S \subseteq N$, $v^p_A(S) = \max\{\mathbf{1}^t x : Ax \leq e_S,\ x \in \{0, 1\}^m\}$;
- for $T \subseteq M$, $v^c_A(T) = \min\{\mathbf{1}^t y : y^t A \geq e^t_T,\ y \in \{0, 1\}^n\}$;

where $\mathbf{1}$ is the vector of all ones and $e_R \in \{0, 1\}^k$, the characteristic vector of $R \subseteq \{1, \dots, k\}$.

The subgames preserve the combinatorial structure of the original games. More precisely, if for each $S \subseteq N$ we consider

$$T(S) = \{j \in M : a_{ij} = 0, \forall i \in N \setminus S\},$$

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it is not difficult to see that the subgame of (N, v_A^p) induced by $S \subseteq N$ is the packing game $(S, v^p_{A_{S,T(S)}})$, where $A_{S,T(S)}$ is the submatrix of A indexed by rows in S and columns in $T(S)$. Similarly, the subgame of (M, v_A^c) induced by $T \subseteq M$ is the covering game $(T, v^c_{A_{N,T}})$.

Given a $0,1$ matrix A, unique packing and covering games are defined, therefore we say that A is a *balanced packing* (BP) matrix when it defines a balanced packing game, and that A is a *totally balanced packing* (TBP) matrix, in the case the packing game defined by A is totally balanced. Similarly, we say that A is a *balanced covering* (BC) matrix or a *totally balanced covering* (TBC) matrix when it defines a balanced or a totally balanced covering game.

In [5] it is shown that A is a BP matrix if and only if the linear problem

$$\max\left\{\mathbf{1}^t x : Ax \leq \mathbf{1},\ x \geq \mathbf{0}\right\}$$

has an integer optimal solution, and that A is a BC matrix if and only if the linear problem

$$\min\left\{\mathbf{1}^t y : y^t A \geq \mathbf{1}^t,\ y \geq \mathbf{0}\right\}$$

has an integer optimal solution. Moreover, in [6] it is proved that every TBC matrix is also a TBP matrix. Although the converse is not true, every TBP matrix is a BC matrix [6].

In 2005 van Velzen [9] showed that the set of TBC matrices coincides with the set of perfect matrices, i.e., those matrices A for which the polytope $\{x : Ax \leq \mathbf{1},\ x \geq \mathbf{0}\}$ has only integer extreme points. His proof is based on a result by Lovász on totally dual integral systems. Nevertheless, finding a characterization of TBP matrices is still an open problem.

Inspired by the fact that perfect matrices have been characterized by Chvátal [3] in terms of graph perfection (Theorem 2 below), the main purpose of this paper is to model totally balanced packing games in terms of graphs.

2. Preliminary results

In what follows, $A = (a_{ij})$ denotes a $0,1$ matrix of order $n \times m$ without zero columns, $N = \{1, 2, \ldots, n\}$ and $M = \{1, 2, \ldots, m\}$. A vector all of whose components are equal to zero is denoted by $\mathbf{0}$.

Given a matrix A and A' a matrix obtained from A by row permutations, A' is a TBP (TBC) matrix if and only if A is a TBP (TBC) matrix. Hence, throughout this work, equality between $0,1$ matrices is up to permutation of rows.

It is not hard to see that if

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \tag{1}$$

then A is a TBP (TBC) matrix if and only if A_1 and A_2 are TBP (TBC) matrices. Therefore, we also consider *connnected* matrices, i.e., matrices whose rows and columns cannot be reordered in such a way that they take the form in (1).

For any $i \in N$, R^i is the i-th row of A, that is, the characteristic vector of a subset of M. For the sake of simplicity, a row of A will be considered indistinctly, as a vector or as a subset of M.

If R^1 and R^2 are two rows of A, R^1 is *dominated* by R^2 if $R^1 \leq R^2$. A row of A which is not dominated by any other row is called *maximal*. The matrix of the maximal rows of A is denoted by $[A]$.

By using simple linear programming arguments, it is not difficult to see that A is a TBC matrix if and only if $[A]$ is a TBC matrix. Therefore, in what follows, we assume that a covering game defining matrix has no dominated rows.

On the other hand, if A is a TBP matrix, then $[A]$ is a TBP matrix. Nevertheless, the converse is not true. For instance, if

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then $[A]$ is a TBP matrix and $A_{S,T(S)}$ is not balanced for $S = \{1, 2, 3\}$.

The *stability* and *covering* numbers of a matrix A are defined as

$$\alpha(A) = \max \left\{ \mathbf{1}^t x : Ax \leq \mathbf{1},\ x \in \{0,1\}^m \right\}$$

and

$$\theta(A) = \min \left\{ \mathbf{1}^t y : y^t A \geq \mathbf{1}^t,\ y \in \{0,1\}^n \right\}.$$

According to these definitions and the characterization of balanced games given in [5], A is a BP matrix if and only if $\alpha(A) = \max \{\mathbf{1}^t x : Ax \leq \mathbf{1},\ x \geq \mathbf{0}\}$, and similarly, A is a BC matrix if and only if $\theta(A) = \min \{\mathbf{1}^t y : y^t A \geq \mathbf{1}^t,\ y \geq \mathbf{0}\}$.

Given a graph G, a *node induced* subgraph of G is any graph obtained by the deletion of a subset of the nodes of G. If M is the node set of G and $R \subseteq M$, we denote by $G \setminus R$ the node induced subgraph obtained by the deletion of the nodes in R.

For every matrix A, $G(A)$ denotes its *associated graph* having node set M and where two nodes are adjacent if there is a row in A with two 1's in their corresponding positions. Given a graph G with node set M, there is not a unique matrix A with $G(A) = G$. In particular, $G(A) = G([A])$.

A *clique* in a graph G is a set of pairwise adjacent nodes, and it is *maximal* if it is not contained in any other clique in G. Given a matrix A, each row of A is the characteristic vector of a clique in $G(A)$, but not necessarily maximal.

There are two particular matrices associated with a given graph G: the *clique-node matrix*, $Q(G)$, whose rows are the characteristic vectors of the maximal cliques in G, and the *edge-node matrix*, $E(G)$, whose rows are the characteristic vectors of the cliques in G of size two.

A matrix A is a *clique-node* matrix if $A = Q(G(A))$. Clique-node matrices are characterized by the following result (see [4], p. 24):

Theorem 1. *Let $A = (a_{ij})$ be a $0,1$ matrix with no zero columns and no dominated rows. Then A is a clique-node matrix if and only if for every submatrix of A of the* *form*

$$\begin{pmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & 1 & \dots & 1 \end{pmatrix}, \qquad (2)$$

with columns $j_1, \dots, j_p$ $(p \geq 3)$, A contains a row i such that $a_{ij_k} = 1$, for $k = 1, \dots, p$.

The previous result was first noted by Conforti (1996), showing that clique-node matrices can be recognized in polynomial time.

A subset of nodes of G is a *stable set* if no pair of them are adjacent. The size of a stable set in G of maximum cardinality is denoted by $\alpha(G)$.

A set of cliques $\mathcal{C}$ is a *clique cover* of G if every node of G belongs to some clique in $\mathcal{C}$. The size of a minimum clique cover of G is denoted by $\theta(G)$.

Given G and a matrix A such that $G(A) = G$, it holds that $\alpha(G) = \alpha(A)$. On the other hand, $\theta(A)$ represents the size of a minimum cover of nodes of G by the rows of A, and we denote it by $\theta_A(G)$ to stress its relation with the graph G. When A is a clique-node matrix, $\theta_A(G) = \theta(G)$. In general, these parameters satisfy

$$\alpha(A) = \alpha(G(A)) \leq \theta(G(A)) \leq \theta_A(G(A)).$$

A graph G is *perfect* when the stability and covering numbers coincide for G and for every node induced subgraph of G. The Strong Perfect Graph Theorem [1] states that perfect graphs are those graphs without odd *holes* (chordless cycles with length at least 5) or their complements as node induced subgraphs.

Chvátal's characterization of perfect matrices is the following:

Theorem 2 ([3])**.** *A $0,1$ matrix A with no zero columns and no dominated rows is perfect if and only if A is a clique-node matrix and $G(A)$ is perfect.*

From the Strong Perfect Graph Theorem, the problem of deciding if a given graph is perfect has polynomial complexity [2]. Therefore, from van Velzen's result (presented in Section 1), TBC matrices can be recognized in polynomial time.

In the next section we model, from a graph theoretical point of view, the total balancedness of both combinatorial optimization games already described.

3. Graphs associated with TBC and TBP matrices

Let us first present a straightforward result which characterizes TBC and TBP matrices.

Proposition 1. *A is a TBC (TBP) matrix if and only if every covering (packing) subgame defining matrix is both, a BP and BC matrix.*

Proof. The *if* part follows from the definition of total balancedness of a matrix.

Let $T \subseteq M$. If A is a TBC matrix, $A_{N,T}$ is TBC and then, it is also a TBP matrix. Hence, $A_{N,T}$ is a BC and BP matrix.

In a similar way, given $S \subseteq N$, if A is a TBP matrix, $A_{S,T(S)}$ is a TBP matrix. Hence, $A_{S,T(S)}$ is also a BC matrix, concluding that $A_{S,T(S)}$ is a BP and BC matrix. □

By linear programming duality, A is a BP and BC matrix if and only if $\alpha(A) = \theta(A)$. We say that such a matrix is *good*. Then, A is good if and only if $[A]$ is good. Then, Proposition 1 can be restated saying that a matrix is a TBC (TBP) matrix if and only if every covering (packing) subgame defining matrix is good.

As a consequence of this simple result, we have the following necessary condition for TBC matrices:

Lemma 2. *If A is a TBC matrix, then A is a clique-node matrix.*

Proof. If A is not a clique-node matrix, it has a submatrix as the one in (2) and A does not contain a row i such that $a_{ij_k} = 1$, for $k = 1, \dots, p$. Let $T = \{j_1, \dots, j_p\}$ with $p \geq 3$. Clearly, $\alpha(A_{N,T}) = 1$ and, since A does not have a row with all ones in the columns in T, $\theta(A_{N,T}) = 2$. Then, $A_{N,T}$ is not good and, by Proposition 1, A is not a TBC matrix. □

Let us remark that a similar result for TBP matrices does not hold. For instance, the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

is TBP but not a clique-node matrix. The square submatrix determined by the first three rows and columns is not a packing subgame defining matrix.

In order to analyze if a given matrix is TBC or TBP, we will associate graphs to every covering (packing) subgame defining matrix.

Given a matrix A, it is not difficult to see that every graph associated with a covering subgame defining submatrix is a node induced subgraph of $G(A)$. In other words, if $\overline{T}$ denotes the complement of T with respect to M, $G(A_{N,T}) = G(A) \setminus \overline{T}$, for every $T \subseteq M$. This is not true for the packing case, as the following example shows.

Example 3. *Let*

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

and consider the graph $G(A)$. In this case there is not a subset $S \subseteq N$ of rows for which $G(A_{S,T(S)}) = G(A) \setminus \{6\}$.

In this context, we are able to present an alternative proof of van Velzen's characterization of TBC matrices, which consists in proving:

Theorem 3. *A clique-node matrix A is TBC if and only if $G(A)$ is perfect.*

Proof. If A is a clique-node matrix, $[A_{N,T}]$ is the clique-node matrix of $G(A) \setminus \overline{T}$, where $\overline{T} = M \setminus T$. Then, $\theta(A_{N,T}) = \theta(G(A) \setminus \overline{T})$.

By definition, $G(A)$ is perfect if and only if $\alpha(G(A) \setminus \overline{T}) = \theta(G(A) \setminus \overline{T})$ for every $T \subseteq M$. Since $\alpha(A_{N,T}) = \alpha(G(A) \setminus \overline{T})$, $G(A)$ is perfect if and only if for any $T \subseteq M$, $[A_{N,T}]$ is good. Proposition 1 for TBC matrices completes the proof. $\square$

Besides, there is another simple way of proving the fact that if A is a TBC matrix then it is a TBP matrix (cf. [5]): Let A be a TBC matrix and $S \subseteq N$. If $A' = A_{S,T(S)}$, A' is the clique-node matrix of $G(A')$, a node induced subgraph of $G(A)$. Since $G(A)$ is perfect, $\alpha(A') = \alpha(G(A')) = \theta(G(A')) = \theta_{A'}(G(A'))$, i.e., A' is good. Then, A is a TBP matrix.

As already observed, not every node induced subgraph of $G(A)$ corresponds to a packing subgame defining matrix. In order to analyze if a given matrix A is TBP, we have to consider those subgraphs of $G(A)$ of the form $G(A_{S,T(S)})$, for $S \subseteq N$. In the remainder of this work, we write A_S instead of $A_{S,T(S)}$.

For a graph G and a matrix A such that $G = G(A)$, we say that G' is an *A-subgraph* of G, denoted by $G' \subseteq_A G$, if there exists $S \subseteq N$ such that

$$G' = G(A_S).$$

Proposition 1 for TBP matrices can be restated as: A is a TBP matrix if and only if $\alpha(G') = \theta_{A_S}(G')$ for every $G' \subseteq_A G$. This property will be referred to as the *A-perfection* of the graph G. Formally,

Definition 4. *Given a graph G and a matrix A with $G = G(A)$, G is A-*perfect *if $\alpha(G') = \theta_{A_S}(G')$ for every $G' \subseteq_A G$.*

According to this definition, a graph can be A-perfect or not depending on which matrix A we choose to describe it.

In particular, if $A = Q(G)$, A-subgraphs and A-perfect graphs are called Q-subgraphs and Q-perfect graphs, respectively. Similarly, when $A = E(G)$, we call them E-subgraphs and E-perfect graphs, respectively.

From the result in Theorem 2 and the fact that TBC implies TBP, we have that being perfect is a weaker property for a graph than being Q-perfect.

Given a graph G, A and A' such that $G(A) = G(A') = G$, we would like to relate the A-perfection and A'-perfection. Clearly, if $\theta_{A'}(G) \leq \theta_A(G)$ and every A'-subgraph of G is also an A-subgraph, then the A-perfection implies the A'-perfection of G. Let us then introduce a partial order on the set of $0,1$ matrices A, for which $G(A) = G$.

Definition 5. *Given two matrices A and A' without dominating rows such that $G(A) = G(A')$, we say that $A \prec A'$ if every row of A' is the union of some rows of A and each row of A is contained in some row of A'.*

It is clear that if $A \prec A'$ and $G = G(A)$, then every A'-subgraph H of G is also an A-subgraph of G and $\alpha(H) \leq \theta_{A'}(H) \leq \theta_A(H)$. Hence, if G is A-perfect and $A \prec A'$, then G is A'-perfect. This proves the following:

Lemma 6. *Let G be a graph and A and A' such that $G(A) = G(A') = G$ and $A \prec A'$. Then, if G is an A-perfect graph, it is also an A'-perfect graph.*

Moreover, for a graph G and a matrix A such that $G(A) = G$, since $\theta_{E(G)}(G) \geq \theta_A(G) \geq \theta_{Q(G)}(G)$, we can say that matrices $E(G)$ and $Q(G)$ are extreme cases in the following sense:

Lemma 7. *Let G and A be a graph and a matrix such that $G(A) = G$. If $E(G)$ is good then A is good, and if A is good then $Q(G)$ is good.*

However, if we consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix},$$

$G(A)$ is A-perfect but not Q-perfect: the odd hole of $G(A)$ is a Q-subgraph but not an A-subgraph of $G(A)$.

On the other hand, if

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$G(A)$ is E-perfect but not A-perfect.

Such extreme cases for the different types of perfection of a graph G would correspond to a minimum and a maximum element of the partial order already defined. Actually, $E(G)$ becomes the minimum element. Nevertheless, since the clique-node matrix of G is maximal but not maximum, the maximum does not exist. To see this, it is enough to consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix},$$

which is noncomparable with $Q(G(A))$ according to $\prec$.

The previous reasoning proves the following

Theorem 4. *Let A be a $0,1$ matrix with at least two ones per row. If $G(A)$ is E-perfect, then A is a TBP matrix. If A has exactly two ones per row, A is a TBP matrix if and only if $G(A)$ is E-perfect.*

We believe that the previous study shows the intrinsic hardness of the problem. Having a characterization of E-perfection of graphs can be a good first step for obtaining a complete characterization of TBP matrices. In fact, if E-perfect graphs were proved to be *easily* recognized, we would have an *easy* sufficient condition for a matrix to be TBP, if it has at least two ones per row.

If, on the contrary, recognizing E-perfect graphs were proved to be a *hard* problem, the general question of recognizing TBP matrices would be proved to be also a *hard* problem.

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