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FURTHER RESULTS ON SEMISIMPLE HOPF ALGEBRAS OF DIMENSION p^2q^2

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Abstract. Let p, q be distinct prime numbers, and k an algebraically closed field of characteristic 0. Under certain restrictions on p, q, we discuss the structure of semisimple Hopf algebras of dimension p^2q^2. As an application, we obtain the structure theorems for semisimple Hopf algebras of dimension $9q^2$ over k. As a byproduct, we also prove that odd-dimensional semisimple Hopf algebras of dimension less than 600 are of Frobenius type.

1. Introduction

Throughout this paper, we will work over an algebraically closed field k of characteristic 0.

The question of classifying all Hopf algebras of a fixed dimension backdates to I. Kaplansky in 1975. This question was first solved in the Ph.D. thesis of R. Williams for dimension less than 11 [24]. In the last twenty years there has been an intense activity in classification problems of finite dimensional Hopf algebras. Many results have been found, containing mainly the semisimple case and the pointed non-semisimple case.

Quite recently, an outstanding classification result was obtained for semisimple Hopf algebras over k. That is, Etingof et al. [5] completed the classification of semisimple Hopf algebras of dimension pq^2 and pqr, where p, q, r are distinct prime numbers. Up to now, besides those mentioned above, semisimple Hopf algebras of dimension p, p^2, p^3 and pq have been completely classified. See [4, 7, 12, 13, 14, 25] for details.

Recall that a semisimple Hopf algebra H is called of Frobenius type if the dimensions of the simple H-modules divide the dimension of H. Kaplansky conjectured that every finite-dimensional semisimple Hopf algebra is of Frobenius type [8, Appendix 2]. It is still an open problem. Many examples show that a positive answer to Kaplansky's conjecture would be very helpful in the classification of semisimple Hopf algebras. See [2, 7, 18] for examples.

In a previous paper [3], we studied the structure of semisimple Hopf algebras of dimension p^2q^2, where p, q are prime numbers with $p^4 < q$. As an application, we also studied the structure of semisimple Hopf algebras of dimension $4q^2$, for all

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prime numbers q. In the present paper, we shall continue our investigation and prove that some results in [3] can be extended to the case $p^2 < q$. Moreover, the structure theorems for semisimple Hopf algebras of dimension $9q^2$ will also be given in this paper, where q is a prime number.

The paper is organized as follows. In Section 2, we recall the definitions and basic properties of semisolvability, characters and Radford's biproducts, respectively. Some useful lemmas are also obtained in this section. In particular, we give a partial answer to Kaplansky's conjecture. We prove that if $\dim H$ is odd and H has a simple module of dimension 3 then 3 divides $\dim H$. This result has already appeared in [1, Corollary 8] and [10, Theorem 4.4], respectively. In the first paper, Burciu does not assume that the characteristic of the base field is zero, but adds the assumption that H has no even-dimensional simple modules. Accordingly, his proof is rather different from ours. Our proof here is also different from that in the second paper. Under the assumption that H does not have simple modules of dimension 3 or 7, we also prove that if $\dim H$ is odd and H has a simple module of dimension 5 then 5 divides $\dim H$.

We begin our main work in Section 3. Assume that $q > p^2$ and p does not divide $q-1$. We prove that if H is a semisimple Hopf algebra of dimension p^2q^2 and has a simple module of dimension p then

(1) If $\gcd(|G(H)|, |G(H^*)|) = p^2$ then $H = R\#kG$ is a Radford biproduct, where kG is a group algebra of dimension p^2 and R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension q^2.

(2) In other cases, H is semisolvable.

In Section 4, we study the more concrete example, that is, the structure of semisimple Hopf algebras of dimension $9p^2$. Let H be a semisimple Hopf algebra of dimension $9q^2$, where $q > 3$ is a prime number. We prove that

(1) If $\gcd(|G(H^*)|, |G(H)|) = 9$ then H is isomorphic to a Radford's biproduct $R\#kG$, where kG is the group algebra of group G of order 9, R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension q^2.

(2) In all other cases, H is semisolvable.

The techniques we develop in Section 2.1 are quite useful in excluding potential candidates for Hopf algebras not of Frobenius type. In Section 5, we shall illustrate this point of view by showing that odd-dimensional semisimple Hopf algebras of dimension less than 600 are of Frobenius type.

Throughout this paper, all modules and comodules are left modules and left comodules, and moreover they are finite-dimensional over k. $\otimes$, dim mean $\otimes_k$, $\dim_k$, respectively. For two positive integers m and n, $\gcd(m,n)$ denotes the greatest common divisor of m, n. Our references for the theory of Hopf algebras are [16] or [23]. The notation for Hopf algebras is standard. For example, the group of group-like elements in H is denoted by $G(H)$.

2. PRELIMINARIES

2.1. **Characters.** Throughout this subsection, H will be a semisimple Hopf algebra over k. As an algebra, H is isomorphic to a direct product of full matrix

algebras
$$H \cong k^{(n_1)} \times \prod_{i=2}^{s} M_{d_i}(k)^{(n_i)},$$
where $n_1 = |G(H^*)|$. In this case, we say H is of type $(d_1, n_1; \ldots; d_s, n_s)$ as an algebra, where $d_1 = 1$. If H^* is of type $(d_1, n_1; \ldots; d_s, n_s)$ as an algebra, we shall say that H is of type $(d_1, n_1; \ldots; d_s, n_s)$ as a coalgebra.

Obviously, H is of type $(d_1, n_1; \ldots; d_s, n_s)$ as an algebra if and only if H has n_1 non-isomorphic irreducible characters of degree d_1, n_2 non-isomorphic irreducible characters of degree d_2, etc. In this paper, we shall use the notation X_t to denote the set of all irreducible characters of H of degree t.

Let V be an H-module. The character of V is the element $\chi = \chi_V \in H^*$ defined by $\langle \chi, h \rangle = \mathrm{Tr}_V(h)$ for all $h \in H$. The degree of χ is defined to be the integer $\deg\chi = \chi(1) = \dim V$. If U is another H-module, we have
$$\chi_{U\otimes V} = \chi_U \chi_V, \quad \chi_{V^*} = S(\chi_V),$$
where S is the antipode of H^*.

The irreducible characters of H span a subalgebra $R(H)$ of H^*, which is called the character algebra of H. By [25, Lemma 2], $R(H)$ is semisimple. The antipode S induces an anti-algebra involution $* : R(H) \to R(H)$, given by $\chi \mapsto \chi^* := S(\chi)$. The character of the trivial H-module is the counit ε.

Let $\chi_U, \chi_V \in R(H)$ be the characters of the H-modules U and V, respectively. The integer $m(\chi_U, \chi_V) = \dim\mathrm{Hom}_H(U, V)$ is defined to be the multiplicity of U in V. This can be extended to a bilinear form $m : R(H) \times R(H) \to k$.

Let $\mathrm{Irr}(H)$ denote the set of irreducible characters of H. Then $\mathrm{Irr}(H)$ is a basis of $R(H)$. If $\chi \in R(H)$, we may write $\chi = \sum_{\alpha\in\mathrm{Irr}(H)} m(\alpha, \chi)\alpha$. Let $\chi, \psi, \omega \in R(H)$. Then $m(\chi, \psi\omega) = m(\psi^*, \omega\chi^*) = m(\psi, \chi\omega^*)$ and $m(\chi, \psi) = m(\chi^*, \psi^*)$. See [19, Theorem 9].

For each group-like element g in $G(H^*)$, we have $m(g, \chi\psi) = 1$, if and only if $\psi = \chi^* g$ and 0 otherwise for all $\chi, \psi \in \mathrm{Irr}(H)$. In particular, $m(g, \chi\psi) = 0$ if $\deg\chi \neq \deg\psi$. Let $\chi \in \mathrm{Irr}(H)$. Then for any group-like element g in $G(H^*)$, $m(g, \chi\chi^*) > 0$ if and only if $m(g, \chi\chi^*) = 1$ if and only if $g\chi = \chi$. The set of such group-like elements forms a subgroup of $G(H^*)$, of order at most $(\deg\chi)^2$. See [19, Theorem 10]. Denote this subgroup by $G[\chi]$. In particular, we have
$$\chi\chi^* = \sum_{g\in G[\chi]} g + \sum_{\alpha\in\mathrm{Irr}(H),\ \deg\alpha>1} m(\alpha, \chi\chi^*)\alpha. \tag{1}$$

A subalgebra A of $R(H)$ is called a standard subalgebra if A is spanned by irreducible characters of H. Let X be a subset of $\mathrm{Irr}(H)$. Then X spans a standard subalgebra of $R(H)$ if and only if the product of characters in X decomposes as a sum of characters in X. There is a bijection between $*$-invariant standard subalgebras of $R(H)$ and quotient Hopf algebras of H. See [19, Theorem 6].

In the rest of this subsection, we shall present some results on irreducible characters and algebra types.

Lemma 2.1. *Let $\chi \in \mathrm{Irr}(H)$ be an irreducible character of H. Then*

(1) The order of $G[\chi]$ *divides* $(\deg\chi)^2$.

(2) The order of $G(H^*)$ *divides* $n(\deg\chi)^2$*, where* n *is the number of non-isomorphic irreducible characters of degree* $\deg\chi$.

Proof. It follows from Nichols-Zoeller Theorem [20]. See also [18, Lemma 2.2.2]. □

Lemma 2.2. *Assume that* $\dim H$ *is odd and* H *is of type* $(1, n_1; \dots; d_s, n_s)$ *as an algebra. Then* d_i *is odd and* n_i *is even for all* $2 \le i \le s$.

Proof. It follows from [9, Theorem 5] that d_i is odd.

If there exists $i \in \{2, \dots, s\}$ such that n_i is odd, then there is at least one irreducible character of degree d_i such that it is self-dual. This contradicts [9, Theorem 4]. □

Remark 2.3. In fact, [9, Theorem 4] is also useful when we consider the possible decompositions of $\chi\chi^*$, where $\chi \in \mathrm{Irr}(H)$. Assume that $\dim H$ is odd and $\chi \in \mathrm{Irr}(H)$. We rewrite (1) as

$$\chi\chi^* = \sum_{g \in G[\chi]} g + \sum_{\alpha_1 \in X_{q_1}} m(\alpha_1, \chi\chi^*)\alpha_1 + \cdots + \sum_{\alpha_n \in X_{q_n}} m(\alpha_n, \chi\chi^*)\alpha_n.$$

Then $\sum_{\alpha_i \in X_{q_i}} m(\alpha_i, \chi\chi^*)$ is even for all $1 \le i \le n$. Indeed, If $\sum_{\alpha_i \in X_{q_i}} m(\alpha_i, \chi\chi^*)$ is odd, then there exists at least one irreducible character α_i of degree q_i such that it is self-dual, since $\chi\chi^*$ is self-dual. This contradicts [9, Theorem 4].

Lemma 2.4. *Assume that* $\dim H$ *is odd. Then*

(1) If H *has an irreducible character* χ *of degree* 3*, then* $G[\chi] \neq \{\varepsilon\}$*. In particular,* 3 *divides* $\dim H$.

(2) Assume in addition that H *does not have irreducible characters of degree* 3 *or* 7*. If* H *has an irreducible character* χ *of degree* 5*, then* $G[\chi] \neq \{\varepsilon\}$*. In particular,* 5 *divides* $\dim H$.

Proof. (1) Let χ be an irreducible character of degree 3. By Lemma 2.2, H does not have irreducible characters of even degree. Therefore, if $G[\chi]$ is trivial then $\chi\chi^* = \varepsilon + \chi_3 + \chi_5$ for some $\chi_3 \in X_3, \chi_5 \in X_5$. This contradicts Remark 2.3. Hence, $G[\chi]$ is not trivial for every $\chi \in X_3$. By Lemma 2.1 (1), the order of $G[\chi]$ is 3 or 9. Thus, 3 divides $|G(H^*)|$ since $G[\chi]$ is a subgroup of $G(H^*)$ for every $\chi \in X_3$.

(2) Let χ be an irreducible character of degree 5. By assumption and Lemma 2.2, if $G[\chi]$ is trivial then there are four possible decomposition of $\chi\chi^*$:

$\chi\chi^* = \varepsilon + \chi_{11} + \chi_{13}; \chi\chi^* = \varepsilon + \chi_9 + \chi_{15}; \chi\chi^* = \varepsilon + \chi_5 + \chi_{19}; \chi\chi^* = \varepsilon + \chi_5^1 + \chi_5^2 + \chi_5^3 + \chi_9,$

where χ_i, χ_j^k are irreducible characters of degree i, j. This contradicts Remark 2.3. Therefore, $G[\chi]$ is not trivial for every $\chi \in X_5$.

The rest of the statement can be obtained by the Nichols–Zoeller Theorem. □

Lemma 2.5. *Assume that* $\dim H$ *is odd and* H *is of type* $(1, n; 3, m; \dots)$ *as an algebra. If*

(1) H *does not have irreducible characters of degree* 9*, or*

(2) there exists a non-trivial subgroup G *of* $G(H^*)$ *such that* $G[\chi] = G$ *for all* $\chi \in X_3$,

then H has a quotient Hopf algebra of dimension $n+9m$.

Proof. Let χ, ψ be irreducible characters of degree 3. Both assumptions (1) and (2) imply that $\chi\psi$ is not irreducible. Indeed, this is immediate for assumption (1), while for assumption (2) it is a consequence of [17, Lemma 2.4.1].

If there exists $\chi_5 \in X_5$ such that $m(\chi_5, \chi\psi) > 0$ then $\chi\psi = \chi_5 + \chi_3 + g$ for some $\chi_3 \in X_3$ and $g \in G(H^*)$, by Lemma 2.2. From $m(g, \chi\psi) = m(\chi, g\psi^*) = 1$, we get $\chi = g\psi^*$. Then $\chi\psi = g\psi^*\psi = \chi_5 + \chi_3 + g$ shows that $\psi^*\psi = g^{-1}\chi_5 + g^{-1}\chi_3 + \varepsilon$. This contradicts Lemma 2.4. Similarly, we can show that there does not exist $\chi_7 \in X_7$ such that $m(\chi_7, \chi\psi) > 0$. Therefore, $\chi\psi$ is a sum of irreducible characters of degree 1 or 3. It follows that irreducible characters of degree 1 and 3 span a standard subalgebra of $R(H)$ and H has a quotient Hopf algebra of dimension $n+9m$. □

Lemma 2.6. *Assume that* $\dim H$ *is odd and H does not have simple modules of dimension 3 and 7. If H has a simple module of dimension 5, then 5 divides the order of $G(H^*)$. In particular, 5 divides* $\dim H$.

Proof. Let χ be an irreducible character of degree 5. By assumption and Lemma 2.2, if $G[\chi]$ is trivial then there are four possible decomposition of $\chi\chi^*$:

$$\chi\chi^* = \varepsilon+\chi_{11}+\chi_{13}; \chi\chi^* = \varepsilon+\chi_9+\chi_{15}; \chi\chi^* = \varepsilon+\chi_5+\chi_{19}; \chi\chi^* = \varepsilon+\chi_5^1+\chi_5^2+\chi_5^3+\chi_9,$$

where χ_i, χ_j^k are irreducible characters of degree i, j. This contradicts Remark 2.3. Therefore, $G[\chi]$ is not trivial for every $\chi \in X_5$. Hence, 5 divides the order of $G(H^*)$ by Lemma 2.1 (1). □

Proposition 2.7. *Let q be a prime number. If H is of type $(1, q^2; q, m; \dots)$ as an algebra, where q does not divide m, then H^* has a Hopf subalgebra K of dimension $\geq 2q^2$. Moreover, $kG(H^*)$ is a normal Hopf subalgebra of K.*

Proof. The group $G(H^*)$ acts by left multiplication on the set X_q. The set X_q is a union of orbits which have length $1, q$ or q^2. Since q does not divide $|X_q| = m$, there exists at least one orbit with length 1. That is, there exists an irreducible character $\chi_q \in X_q$ such that $G[\chi_q] = G(H^*)$. In addition, [17, Lemma 2.1.4] shows that $G[\chi_q^*] = G(H^*)$ in this case. This means that $g\chi_q = \chi_q = \chi_q g$ for all $g \in G(H^*)$.

Let C be the q^2-dimensional simple subcoalgebra of H^*, corresponding to χ_q. Then $gC = C = Cg$ for all $g \in G(H^*)$. By [17, Proposition 3.2.6], $kG(H^*)$ is normal in $K := k[C]$, where $k[C]$ denotes the subalgebra generated by C. It is a Hopf subalgebra of H^* containing $G(H^*)$. Counting dimensions, we find that $\dim K \geq 2q^2$. □

2.2. **Semisolvability.** Let B be a finite-dimensional Hopf algebra over k. A Hopf subalgebra $A \subseteq B$ is called normal if $h_1 A S(h_2) \subseteq A$, for all $h \in B$. If B does not contain proper normal Hopf subalgebras then it is called simple. The notion of simplicity is self-dual, that is, B is simple if and only if B^* is simple.

Let $\pi : H \to B$ be a Hopf algebra map and consider the subspaces of coinvariants

$$H^{co\pi} = \{h \in H | (id \otimes \pi)\Delta(h) = h \otimes 1\}, \text{and}$$

$$^{co\pi}H = \{h \in H | (\pi \otimes id)\Delta(h) = 1 \otimes h\}.$$

Then $H^{co\pi}$ (respectively, $^{co\pi}H$) is a left (respectively, right) coideal subalgebra of H. Moreover, we have

$$\dim H = \dim H^{co\pi} \dim \pi(H) = \dim {}^{co\pi}H \dim \pi(H).$$

The left coideal subalgebra $H^{co\pi}$ is stable under the left adjoint action of H. Moreover $H^{co\pi} = {}^{co\pi}H$ if and only if $H^{co\pi}$ is a normal Hopf subalgebra of H. If this is the case, we shall say that the map $\pi : H \to B$ is normal. See [22] for more details.

The following lemma is taken from [17, Section 1.3].

Lemma 2.8. *Let $q : H \to B$ be a Hopf epimorphism and A a Hopf subalgebra of H such that $A \subseteq H^{coq}$. Then $\dim A$ divides $\dim H^{coq}$.*

The notions of upper and lower semisolvability for finite-dimensional Hopf algebras have been introduced in [15], as generalizations of the notion of solvability for finite groups. By definition, H is called lower semisolvable if there exists a chain of Hopf subalgebras

$$H_{n+1} = k \subseteq H_n \subseteq \cdots \subseteq H_1 = H$$

such that H_{i+1} is a normal Hopf subalgebra of H_i, for all i, and all quotients $H_i/H_iH_{i+1}^+$ are trivial. That is, they are isomorphic to a group algebra or a dual group algebra. Dually, H is called upper semisolvable if there exists a chain of quotient Hopf algebras

$$H_{(0)} = H \xrightarrow{\pi_1} H_{(1)} \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_n} H(n) = k$$

such that $H_{(i-1)}^{co\pi_i} = \{h \in H_{(i-1)} | (id \otimes \pi_i)\Delta(h) = h \otimes 1\}$ is a normal Hopf subalgebra of $H_{(i-1)}$, and all $H_{(i-1)}^{co\pi_i}$ are trivial. H is called semisolvable if it is upper or lower semisolvable.

In analogy with the situations for finite groups, it is enough for many applications to know that a Hopf algebra is semisolvable.

By [15, Corollary 3.3], we have that H is upper semisolvable if and only if H^* is lower semisolvable. If this is the case, then H can be obtained from group algebras and their duals by means of (a finite number of) extensions.

Let K be a proper normal Hopf subalgebra of H. Then

$$k \to K \to H \to \overline{H} := H/HK^+ \to k$$

is an exact sequence of Hopf algebras. If K is lower semisolvable and $\overline{H}$ is trivial then H is lower semisolvable. On the other hand, is K is trivial and $\overline{H}$ is upper semisolvable then H is upper semisolvable. As an immediate consequence of this observation, we obtain the following result.

Proposition 2.9. *Let H be a semisimple Hopf algebra of dimension p^2q^2, where p, q are distinct prime numbers. If H is not simple as a Hopf algebra then it is semisolvable.*

Proof. By assumption, H has a proper normal Hopf subalgebra K. Then $\dim K$ divides $\dim H$ by Nichols-Zoeller Theorem [20]. Hence, by the classification of semisimple Hopf algebras [4, 5, 13, 25], K is upper and lower semisolvable. The proposition then follows from a direct check for every possible $\dim K$. □

2.3. **Radford's biproduct.** Let A be a semisimple Hopf algebra and let ${}^A_A\mathcal{YD}$ denote the braided category of Yetter-Drinfeld modules over A. Let R be a semisimple Yetter-Drinfeld Hopf algebra in ${}^A_A\mathcal{YD}$. Denote by $\rho: R \to A \otimes R$, $\rho(a) = a_{-1} \otimes a_0$, and $\cdot : A \otimes R \to R$, the coaction and action of A on R, respectively. We shall use the notation $\Delta(a) = a^1 \otimes a^2$ and S_R for the comultiplication and the antipode of R, respectively.

Since R is in particular a module algebra over A, we can form the smash product (see [15, Definition 4.1.3]). This is an algebra with underlying vector space $R \otimes A$, multiplication given by

$$(a \otimes g)(b \otimes h) = a(g_1 \cdot b) \otimes g_2 h, \text{ for all } g, h \in A, a, b \in R,$$

and unit $1 = 1_R \otimes 1_A$.

Since R is also a comodule coalgebra over A, we can dually form the smash coproduct. This is a coalgebra with underlying vector space $R \otimes A$, comultiplication given by

$$\Delta(a \otimes g) = a^1 \otimes (a^2)_{-1} g_1 \otimes (a^2)_0 \otimes g_2, \text{ for all } h \in A, a \in R,$$

and counit $\varepsilon_R \otimes \varepsilon_A$.

As observed by D. E. Radford (see [21, Theorem 1]), the Yetter-Drinfeld condition assures that $R \otimes A$ becomes a Hopf algebra with these structures. This Hopf algebra is called the Radford's biproduct of R and A. We denote this Hopf algebra by $R\#A$ and write $a\#g = a \otimes g$ for all $g \in A, a \in R$. Its antipode is given by

$$S(a\#g) = (1\#S(a_{-1}g))(S_R(a_0)\#1), \text{ for all } g \in A, a \in R.$$

A biproduct $R\#A$ as described above is characterized by the following property (see [21, Theorem 3]): suppose that H is a finite-dimensional Hopf algebra endowed with Hopf algebra maps $\iota: A \to H$ and $\pi: H \to A$ such that $\pi\iota: A \to A$ is an isomorphism. Then the subalgebra $R = H^{co\pi}$ has a natural structure of Yetter-Drinfeld Hopf algebra over A such that the multiplication map $R\#A \to H$ induces an isomorphism of Hopf algebras.

Lemma 2.10. [3, Theorem 2.6] *Let H be a semisimple Hopf algebra of dimension p^2q^2, where p, q are distinct prime numbers. If $\gcd(|G(H)|, |G(H^*)|) = p^2$, then $H \cong R\#kG$ is a biproduct, where kG is the group algebra of group G of order p^2, R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension q^2.*

3. SEMISIMPLE HOPF ALGEBRAS OF DIMENSION p^2q^2

Let p, q be distinct prime numbers with $p < q$. Throughout this section, H will be a semisimple Hopf algebra of dimension p^2q^2, unless otherwise stated. By Nichols-Zoeller Theorem [20], the order of $G(H^*)$ divides $\dim H$. Moreover, $|G(H^*)| \neq 1$ by [5, Proposition 9.9]. By [3, Lemma 2.2], H is of Frobenius type.

This fact is also a consequence of [5, Theorem 1.5]. Therefore, the dimension of a simple H-module can only be $1, p, p^2$ or q. Let a, b, c be the number of non-isomorphic simple H-modules of dimension p, p^2 and q, respectively. It follows that we have an equation $p^2q^2 = |G(H^*)| + ap^2 + bp^4 + cq^2$. In particular, if $|G(H^*)| = p^2q^2$ then H is a dual group algebra; if $|G(H^*)| = pq^2$ then H is upper semisolvable by the following lemma.

Lemma 3.1. *If H has a Hopf subalgebra K of dimension pq^2 then H is lower semisolvable.*

Proof. Since the index of K in H is p which is the smallest prime number dividing $\dim H$, the main result in [11] shows that K is a normal Hopf subalgebra of H. The lemma then follows from Proposition 2.9. □

Lemma 3.2. *If the order of $G(H^*)$ is q^2 then H is upper semisolvable.*

Proof. If $p = 2$ and $q = 3$ then it is the case discussed in [17, Chapter 8]. Hence, H is upper semisolvable. Throughout the remainder of the proof, we assume that $p \geq 3$.

By Lemma 2.1 (2), if $a \neq 0$ then $ap^2 \geq p^2q^2$, a contradiction. Hence, $a = 0$. Similarly, $b = 0$. If follows that H is of type $(1, q^2; q, p^2 - 1)$ as an algebra. In addition, q does not divides $p^2 - 1$, since $q > p \geq 3$. Therefore, by Proposition 2.7, H has a quotient Hopf algebra K of dimension $\geq 2q^2$ and $kG(H^*)$ is a normal Hopf subalgebra of K. Since $\dim K$ divides $\dim H$, we know $\dim K = pq^2$ or p^2q^2. If $\dim K = pq^2$ then Lemma 3.1 shows that H^* is lower semisolvable. If $\dim K = p^2q^2$ then $K = H^*$. Since $kG(H^*)$ is a group algebra and the quotient $H^*/H^*(kG(H^*))^+$ is trivial (see [13]), H^* is lower semisolvable. Hence, H is upper semisolvable. This completes the proof. □

Lemma 3.3. *If $q > p^2$ then the order of $G(H^*)$ can not be q.*

Proof. Suppose on the contrary that $|G(H^*)| = q$. Then $p^2q^2 = q + ap^2 + bp^4 + cq^2$. It is easily observed that $a = 0$. Indeed, if $a \neq 0$ and $\chi \in X_p$ then the decomposition of $\chi\chi^*$ gives rise to a contradiction, by the fact that $p^2 < q$ and $G[\chi]$ is a subgroup of $G(H^*)$ of order dividing p^2, also by Lemma 2.1. Moreover, a direct check shows that $b \neq 0$ and $c \neq 0$. Hence, H is of type $(1, q; p^2, b; q, c)$ as an algebra.

The group $G(H^*)$ acts by left multiplication on the set X_q. The set X_q is a union of orbits which have length 1 or q. If there exists an orbit with length q then $cq^2 \geq q^3 \geq p^2q^2 = \dim H$. It is impossible. Therefore, every orbit has length 1. It follows that $g\chi = \chi = \chi g$ for all $g \in G(H^*)$ and $\chi \in X_q$.

Let χ be an irreducible character of degree p^2. Since $G[\chi]$ is a subgroup of $G(H^*)$ and the order of $G[\chi]$ divides p^4, we have

$$\chi\chi^* = \varepsilon + \sum_i m(\varphi_i, \chi\chi^*)\varphi_i + \sum_j m(\psi_j, \chi\chi^*)\psi_j,$$

where $\deg\varphi_i = p^2, \deg\psi_j = q$ for all i, j. It is obvious that there must exist some ψ_j such that $m(\psi_j, \chi\chi^*) \neq 0$. We denote this ψ_j by ψ. Since $p^2 < q$, we have

$m(\psi, \chi\chi^*) < p^2$, and hence

$$0 < m(\chi, \psi\chi) < p^2.$$

Consider the decomposition of $\psi\chi$. If there exists an irreducible character ω of degree q such that $m(\omega, \psi\chi) > 0$ then we have $qp^2 = mp^2 + nq$ for some positive integers m, n. This leads to the contradiction $p^2(q-m) = nq$. Moreover, irreducible characters of degree 1 can not appear in the decomposition of $\psi\chi$, since $\deg\psi \neq \deg\chi$. It follows that $\psi\chi$ is a sum of irreducible characters of degree p^2. Write

$$\psi\chi = m(\chi, \psi\chi)\chi + \sum_{\deg\alpha_k = p^2, \alpha_k \neq \chi} m(\alpha_k, \psi\chi)\alpha_k.$$

Note that the relation above implies that $m(\chi, \psi\chi) = 1$. Indeed, the left hand side of this equality is stable under left multiplication by $G[\psi] = G(H^*)$. Then all the $G(H^*)$-conjugates of χ appear in $\psi\chi$ with the same multiplicity as χ does. Since $G[\chi] = \{\varepsilon\}$, there are q of them. Whence this multiplicity must be 1, by dimension restrictions. Thus $\psi\chi = \sum_{g\in G(H^*)} g\chi$. Since $G[\psi] = G[\psi^*]$, we also get $\psi(h\chi) = \psi\chi = \sum_{g\in G(H^*)} g\chi$, for all $h \in G(H^*)$.

Let $C \subseteq H^*$ be the sum of the q simple subcoalgebras of dimension p^4 containing the conjugates of χ, and let $K \subseteq H^*$ be the Hopf subalgebra generated as an algebra by the simple subcoalgebra of dimension q^2 containing ψ.

It follows that $KC \subseteq C$ and therefore C is a (K, H^*)-Hopf module. Then by the Nichols-Zoeller theorem, $\dim K$ divides $\dim C = qp^4$. In particular, $K \neq H^*$. Since $G(H^*) \subseteq K$, then q divides $\dim K$. Hence $\dim K = pq$ or qp^2.

If $\dim K = pq$, then K is commutative and since $p^2 < q$, it has group-likes of order p, which contradicts the assumption on the order of $G(H^*)$. Also, in view of the classification of semisimple Hopf algebras of dimension qp^2 with $p^2 < q$, the possibility $\dim K = qp^2$ implies a similar contradiction. This finishes the proof. □

Notice that if $|G(H^*)| = p$ or pq then $p^2q^2 = |G(H^*)| + ap^2 + bp^4 + cq^2$ shows that c can not be 0.

Lemma 3.4. *Assume that $q > p^2$ and $|G(H^*)| = p$. If $a \neq 0$ and $p \nmid q-1$ then H is upper semisolvable.*

Proof. It is clear that $G[\chi] = G(H^*)$ for all $\chi \in X_p$. By [17, Lemma 2.4.1], $\chi\varphi$ is not irreducible for all $\chi, \varphi \in X_p$. Hence, $G(H^*) \cup X_p$ spans a standard subalgebra of $R(H)$. It follows that H has a quotient Hopf algebra of dimension $p + ap^2$. Since $c \neq 0$, $p + ap^2 < p^2q^2$. By Nichols-Zoeller Theorem, $p + ap^2$ divides p^2q^2. If $p + ap^2 = pq^2$ then H is upper semisolvable by Lemma 3.1. If $p + ap^2 = pq$ then $a = (q-1)/p$, which contradicts the assumption. If $p + ap^2 = p^2q$ then $1 = p(q-a)$, which is impossible. □

Lemma 3.5. *Assume that $q > p^2$ and $|G(H^*)| = pq$. If $a \neq 0$ then H is upper semisolvable.*

Proof. It is clear that the order of $G[\chi]$ is p for all $\chi \in X_p$. In addition, a is not divisible by q^2, since otherwise $ap^2 \geq p^2q^2$, which is impossible. Hence, by [17, Proposition 1.2.6], $G(H^*)$ is abelian and $G[\chi]$ is the unique subgroup of $G(H^*)$ of order p. By a similar argument as in the proof of Lemma 2.5, $\chi\varphi$ is not irreducible for all $\chi, \varphi \in X_p$. Hence, $G(H^*) \cup X_p$ spans a standard subalgebra of $R(H)$. It follows that H has a quotient Hopf algebra of dimension $pq + ap^2$. Since $c \neq 0$, $pq + ap^2 < p^2q^2$. By Nichols-Zoeller Theorem, $pq + ap^2$ divides p^2q^2. If $pq + ap^2 = pq^2$ then H is upper semisolvable by Lemma 3.1. If $pq+ap^2 = p^2q$ then $q = p(q-a)$, a contradiction. □

Theorem 3.6. *Assume that $q > p^2$ and p does not divide $q - 1$. If H has a simple module of dimension p then*

(1) If $\gcd(|G(H)|, |G(H^*)|) = p^2$ then $H = R\#kG$ is a Radford biproduct, where kG is a group algebra of dimension p^2 and R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension q^2.

(2) In other cases, H is semisolvable.

Proof. It follows from Lemma 2.10 and Lemmas 3.1– 3.5. □

Notice that if $|G(H^*)| = p^2$ or p^2q then Lemma 2.1 (2) shows that $X_q = \varnothing$.

Corollary 3.7. *Assume that $q > p^2$ and p does not divide $q^2 - 1$. If H has a simple module of dimension p then H is semisolvable.*

Proof. By the discussions above, we only consider the case that $\gcd(|G(H)|, |G(H^*)|) = p^2$. Let $K \subseteq G(H)$ and $G \subseteq G(H^*)$ be subgroups of order p^2. Considering the projection $\pi : H \to (kG)^*$ obtained by transposing the inclusion $kG \subseteq H^*$, we have that $\dim H^{co\pi} = q^2$. If there exists $1 \neq g \in K$ such that $g \in H^{co\pi}$ then $k\langle g\rangle \subseteq H^{co\pi}$, since $H^{co\pi}$ is an algebra, where $\langle g\rangle$ denotes the subgroup generated by g. It contradicts Lemma 2.8, since $\dim k\langle g\rangle$ does not divide $\dim H^{co\pi} = q^2$. Therefore, $kK \cap H^{co\pi} = k1$. We then have 2 possible decompositions of H^{coq} as a coideal of H:

$$H^{coq} = k1 \oplus \sum_i V_i \oplus \sum_j W_j, \text{ or } H^{coq} = kL \oplus \sum_i V_i \oplus \sum_j W_j,$$

where V_i is an irreducible left coideal of H of dimension p, W_i is an irreducible left coideal of H of dimension p^2 and L is a subgroup of $G(H)$ of order q. Counting dimensions on both sides, we have $q^2 = 1 + mp$ or $q^2 = q + np$ for some positive integers m, n. This contradicts the assumption that p does not divide $q - 1$ and $q + 1$. This completes the proof. □

4. SEMISIMPLE HOPF ALGEBRAS OF DIMENSION $9q^2$

In this section, we shall investigate the structure of a semisimple Hopf algebra H of dimension $9q^2$, where q is a prime number. The structure of semisimple Hopf algebras of dimension 36 is presented in [17, Chapter 8], which are semisolvable up to a cocycle twist. By [15, Theorem 3.5], semisimple Hopf algebras of dimension 81 are semisolvable. Hence, we assume that $q > 3$ in the remainder of this section. Moreover, by [3, Theorem 3.7], it suffices to consider the case that $q < 81$.

Let a, b, c be the number of non-isomorphic simple H-modules of dimension $3, 9$ and q, respectively. It follows that we have an equation $9q^2 = |G(H^*)| + 9a + 81b +$

cq^2. Since dim$H$ is odd, Lemma 2.2 shows that a, b, c are even. We shall prove the following proposition whose proof involves four lemmas.

Proposition 4.1. *Assume that H is a semisimple Hopf algebra of dimension $9q^2$, where $3 < q < 81$. Then*

(1) If $\gcd(|G(H^*)|, |G(H)|) = 9$ then H is isomorphic to a Radford's biproduct $R\#kG$, where kG is the group algebra of group G of order 9, R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension q^2.

(2) In all other cases, H is semisolvable.

4.1. **The case $3 < q < 9$.**

Lemma 4.2. *If $q = 5$ then Proposition 4.1 holds true.*

Proof. By Lemma 2.1, 2.2, 2.4, 2.6 and the fact that H is of Frobenius type, if dim$H = 3^2 \times 5^2$ then H is of one of the following types as an algebra:

$$(1, 25; 5, 8), (1, 75; 5, 6), (1, 3; 3, 8; 5, 6), (1, 9; 3, 6; 9, 2), (1, 9; 3, 24), (1, 45; 3, 20).$$

If H is of type $(1, 25; 5, 8)$ as an algebra then Lemma 3.2 shows that H is upper semisolvable. If H is of type $(1, 75; 5, 6)$ as an algebra then Lemma 3.1 shows that H is upper semisolvable. If H is of type $(1, 3; 3, 8; 5, 6)$ as an algebra then Lemma 2.5 shows that H has a quotient Hopf algebra of dimension 75. Hence, Lemma 3.1 shows that H is upper semisolvable. The lemma then follows from Lemma 2.10. □

Lemma 4.3. *If $q = 7$ then Proposition 4.1 holds true.*

Proof. By Lemma 2.1, 2.2, 2.4 and the fact that H is of Frobenius type, if dim$H = 3^2 \times 7^2$ then H is of one of the following types as an algebra:

$$(1, 3; 3, 14; 5, 6; 9, 2), (1, 3; 3, 32; 5, 6), (1, 3; 3, 16; 7, 6), (1, 21; 3, 14; 7, 6),$$

$$(1, 49; 7, 8), (1, 147; 7, 6), (1, 9; 3, 12; 9, 4), (1, 9; 3, 30; 9, 2), (1, 9; 3, 48), (1, 63; 3, 42).$$

Lemma 2.5 shows that H can not be of type $(1, 3; 3, 14; 5, 6; 9, 2), (1, 3; 3, 32; 5, 6)$ as an algebra, since it contradicts Nichols-Zoeller Theorem. The lemma then follows from a similar argument as in Lemma 4.2. □

4.2. **The case $9 < q < 81$.** By the discussion in Section 3 and Lemma 2.10, it suffices to prove that H is upper semisolvable when the order of $G(H^*)$ is 3 or $3q$.

Lemma 4.4. *If the order of $G(H^*)$ is 3 then H is upper semisolvable.*

Proof. By Lemma 3.4, it is enough to consider the case that $a = 0$ or 3 divides $q - 1$.

We first consider the case $a = 0$. In this case, $9q^2 = 3 + 81b + cq^2$. Since c is even, is divisible by 3 and is not 0, we have $c = 6$. Hence, $q^2 = 1 + 27b$. A direct check, for $9 < q < 81$, shows that the equation holds true only when $q = 53$ and $b = 104$. That is, H is of type $(1, 3; 9, 104; 53, 6)$ as an algebra. We shall prove that it is impossible.

Suppose on the contrary that H is of type $(1, 3; 9, 104; 53, 6)$ as an algebra. Let χ be an irreducible character of degree 9. From the decomposition of $\chi\chi^*$, we have two equations: $81 = 3 + 9m + 53n$ and $81 = 1 + 9m + 53n$, where m, n are non-negative integers. It is easy to check that the first equation can not hold true, and the second one holds true only when $m = 3$ and $n = 1$, which contradicts Remark 2.3.

We then consider the case that 3 divides $q-1$. Let χ be an irreducible character of degree 9. From the decomposition of $\chi\chi^*$, we have two equations: $81 = 1+3m+9n+qs$ and $81 = 3+3m+9n+qs$, where m, n, s are non-negative integers which are even by Remark 2.3. A direct check, for $q = 13, 19, 31, 37, 43, 61, 67, 73, 79$, shows that the first equation can not hold true, and the second one holds true only when $s = 0$. This means that $G[\chi] = G(H^*)$ for all $\chi \in X_9$ and $\chi\chi^*$ is a sum of irreducible characters of degree $1, 3$ or 9.

Let χ, ψ be two distinct irreducible characters of degree 9. We shall prove that $\chi\psi^*$ is a sum of irreducible characters of degree $1, 3$ or 9. In fact, if there exists an irreducible character $\varphi \in X_q$ such that $m(\varphi, \chi\psi^*) > 0$ then there must exist $\varepsilon \neq g \in G(H^*)$ such that $m(g, \chi\psi^*) = 1$. From $m(g, \chi\psi^*) = m(\chi, g\psi) = 1$, we have $\chi = g\psi$. Hence, $m(\varphi, \chi\psi^*) = m(\varphi, g\psi\psi^*) > 0$. This contradicts the fact that $\psi\psi^*$ does not contain any irreducible characters of degree q.

It follows that irreducible characters of degree $1, 3$ and 9 span a standard subalgebra of $R(H)$, and hence H has a quotient Hopf algebra $\overline{H}$ of dimension $3+9a+81b$. Since $c \neq 0$, then $\dim\overline{H} < 9q^2$. Therefore $\dim\overline{H} = 3, 3q, 9q, 3q^2$ or 9. Moreover, $\dim\overline{H} \neq 9$, since otherwise $(\overline{H})^* \subseteq kG(H^*)$ by [13], but $9 = \dim\overline{H}$ does not divide $|G(H^*)| = 3$.

The possibilities $\dim\overline{H} = 3, 3q$ or $9q$ lead, respectively to the contradictions $9q^2 = 3 + cq^2$, $9q^2 = 3q + cq^2$ and $9q^2 = 9q + cq^2$. Hence they are also discarded, and therefore $\dim\overline{H} = 3q^2$. This implies that H is upper semisolvable, by Lemma 3.1. □

Lemma 4.5. *If the order of $G(H^*)$ is $3q$ then H is upper semisolvable.*

Proof. By Lemma 3.5, it is enough to consider the case that $a = 0$. In this case, $9q^2 = 3q + 81b + cq^2$, where b is even and $c = 6$. A direct check, for $9 < q < 81$, shows this equation can not hold true. □

Combining [3, Theorem 3.7] with the results obtained in this section, we obtain the structure theorem for semisimple Hopf algebras of dimension $9q^2$ over k for all prime numbers q.

Theorem 4.6. *Suppose that H is a semisimple Hopf algebra of dimension $9q^2$, where $q > 3$ is a prime number. Then*

(1) If $\gcd(|G(H^*)|, |G(H)|) = 9$ then H is isomorphic to a Radford's biproduct $R\#kG$, where kG is the group algebra of group G of order 9, R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension q^2.

(2) In all other cases, H is semisolvable.

As an immediate consequence of Theorem 4.6, we have a corollary.

Corollary 4.7. *Suppose that H is a semisimple Hopf algebra of dimension $9q^2$, where q is a prime number. If H is simple as a Hopf algebra then H is isomorphic* *to a Radford's biproduct $R\#kG$, where kG is the group algebra of group G of order* 9*, R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension q^2.*

In fact, examples of nontrivial semisimple Hopf algebras of dimension p^2q^2 which are Radford's biproducts in such a way, and are simple as Hopf algebras do exist. A construction of such examples as twisting deformations of certain groups appears in [6, Remark 4.6].

5. Semisimple Hopf algebras of Frobenius type

In this section, we shall prove the following theorem.

Theorem 5.1. *Let H be a semisimple Hopf algebra over k. If* $\dim H$ *is odd and less than* 600*, then H is of Frobenius type.*

To do this, we first restate some results from Section 2.1 in terms of algebra types, which can be easily handled by a computer.

Lemma 5.2. *Let H be a semisimple Hopf algebra over k. Then*

(1) If n_1 does not divide $\dim H$ *or $n_id_i^2 (2 \le i \le s)$, then H cannot be of type $(1, n_1; d_2, n_2; \dots; d_s, n_s)$ as an algebra.*

(2) If $\dim H$ *is odd, then H cannot be of type $(1, n_1; d_2, n_2; \dots; d_s, n_s)$ as an algebra, where there exists $i \in \{2, \dots, s\}$ such that n_i is odd.*

(3) If $\dim H$ *is odd, then H cannot be of type $(1, n_1; d_2, n_2; \dots; d_s, n_s)$ as an algebra, where there exists $i \in \{2, \dots, s\}$ such that d_i is even.*

(4) If $\dim H$ *is odd and* 3 *does not divide* $\dim H$*, then H cannot be of type $(1, m; 3, n; \dots)$ as an algebra.*

(5) If $\dim H$ *is odd, then H cannot be of type $(1, m; 3, n; \dots)$ as an algebra, where m is not divisible by* 3*.*

(6) If $\dim H$ *is odd and $3+9n$ does not divide* $\dim H$*, then H cannot be of type $(1, 3; 3, n; \dots)$ as an algebra.*

(7) If $\dim H$ *is odd, H does not have simple modules of dimension* 9 *and $m+9n$ does not divide* $\dim H$*, then H cannot be of type $(1, m; 3, n; \dots)$ as an algebra.*

(8) If $\dim H$ *is odd, H does not have simple modules of dimension $3, 7$ and* 5 *does not divide* $\dim H$*, then H cannot be of type $(1, m; 5, n; \dots)$ as an algebra.*

(9) If $\dim H$ *is odd and H does not have simple modules of dimension $3, 7$, then H cannot be of type $(1, m; 5, n; \dots)$ as an algebra, where m is not divisible by* 5*.*

(10) If H is of type $(1, 1; d_2, n_2; \dots; d_s, n_s)$ as an algebra then $\{d_i : d_i > 1\}$ has at least three elements.

(11) If t does not divide m then H can not be of type $(1, m; t, n)$ as an algebra.

Proof. Part (1) is just Lemma 2.1; parts (2),(3) are just Lemma 2.2; parts (4),(5) are just Lemma 2.4; parts (6),(7) are just Lemma 2.5; parts (8),(9) are just Lemma 2.6; part (10) follows from [26, Lemma 11].

(11) Suppose on the contrary that H is of type $(1, m; t, n)$ as an algebra. Let $\chi_i (1 \le i \le n)$ be all distinct irreducible characters of degree t, s the order of group $G[\chi_1]$, and u the number of irreducible characters of degree t in the decomposition

of $\chi_1\chi_1^*$. Then, we have $t^2 = s + ut$ from $\chi_1\chi_1^* = \sum_{g\in G[\chi_1]} g + \sum_{i=1}^n m(\chi_i, \chi_1\chi_1^*)\chi_i$. It follows that t divides s, which implies t divides m. It is a contradiction. $\square$

Proof of Theorem 5.1. Let p, q, r be distinct prime numbers. Semisimple Hopf algebras of dimension pqr are classified in [5]. These Hopf algebras are of Frobenius type. In addition, by [3, Lemma 2.2], semisimple Hopf algebras of dimension $p^m q^n$ are also of Frobenius type, where m, n are non-negative integers. Therefore, it suffices to consider the case that $\dim H = 315, 495, 525, 585$.

In the rest of the proof, the computation is partly handled by a personal computer. For example, it is easy to write a computer program by which one finds out all possible positive integers $1 = d_1, d_2, \ldots, d_s$ and $n_1, n_2, \ldots, n_s$ such that $315 = \sum_{i=1}^s n_i d_i^2$, and then one can eliminate those which can not be algebra types of H by using Lemma 5.2.

If $\dim H = 315$ then H is of one of the following types as an algebra: $(1,63;3,28)$, $(1,3;3,2;7,6)$, $(1,21;7,6)$, $(1,15;5,12)$, $(1,9;3,34)$, $(1,45;3,30)$, $(1,9;3,16;9,2)$. Clearly, H is of Frobenius type.

If $\dim H = 495$ then H is of one of the following types as an algebra: $(1,45;15,2)$, $(1,9;3,4;15,2)$, $(1,11;11,4)$, $(1,9;9,6)$, $(1,9;3,18;9,4)$, $(1,9;3,36;9,2)$, $(1,45;5,18)$, $(1,9;3,4;5,18)$, $(1,15;3,20;5,12)$, $(1,9;3,54)$, $(1,45;3,50)$, $(1,99;3,44)$. Clearly, H is of Frobenius type.

If $\dim H = 525$ then H is of one of the following types as an algebra: $(1,75;15,2)$, $(1,25;5,2;15,2)$, $(1,3;3,8;15,2)$, $(1,35;7,10)$, $(1,25;5,20)$, $(1,175;5,14)$, $(1,75;5,18)$, $(1,3;3,8;5,18)$, $(1,3;3,58)$, $(1,21;3,56)$, $(1,75;3,50)$. Clearly, H is of Frobenius type.

If $\dim H = 585$ then H is of one of the following types as an algebra: $(1,117;3,52)$, $(1,1;5,2;7,4;13,2)$, $(1,9;3,10;9,6)$, $(1,9;3,28;9,4)$, $(1,9;3,46;9,2)$, $(1,9;3,64)$, $(1,45;3,60)$. We shall prove that H can not be of type $(1,1;5,2;7,4;13,2)$ as an algebra.

Suppose on the contrary that H is of type $(1,1;5,2;7,4;13,2)$ as an algebra. Let $X_5 = \{\chi, \varphi\}$. Then $\chi\chi^* = \varepsilon + \chi + \varphi + \chi_7 + \chi_7'$ by Remark 2.3, where χ_7, χ_7' are distinct elements in X_7. From $m(\chi_7, \chi\chi^*) = m(\chi, \chi_7\chi) = 1$, we have $\chi_7\chi = \chi + \psi$, where $\deg\psi = 30$ and $m(\chi, \psi) = 0$. There are two possible decomposition of ψ: $\psi = 6\varphi$ or $\psi = \phi + \chi_{13} + 2\varphi$, where $\phi \in X_7, \chi_{13} \in X_{13}$. If the first one holds true, then $m(\varphi, \chi_7\chi) = m(\chi_7, \varphi\chi^*) = 6$, which is impossible. If the second one holds true, then $m(\chi_7, \varphi\chi^*) = 2$ implies that $\varphi\chi^* = 2\chi_7 + \omega$, where $\deg\omega = 11$. It is also impossible. This completes the proof, and hence H is of Frobenius type. $\square$

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REFERENCES

[1] S. Burciu, Representations of degree three for semisimple Hopf algebras, J. Pure Appl. Algebra, 194, 85–93 (2004). 98

[2] J. Dong, Structure of a class of semisimple Hopf algebras, Acta Mathematica Sinica, Chinese Series, 54(2):293–300 (2011). 97

[3] J. Dong, Structure of semisimple Hopf algebras of dimension p^2q^2, Comm. Algebra, 40(3): 795–806 (2012). 97, 98, 103, 106, 108, 110

[4] P. Etingof and S. Gelaki, Semisimple Hopf algebras of dimension pq are trivial, J. Algebra, 210(2), 664–669 (1998). 97, 103

[5] P. Etingof, D. Nikshych and V. Ostrik, Weakly group-theoretical and solvable fusion categories, Adv. Math., 226 (1), 176–505 (2011). 97, 103, 104, 110

[6] C. Galindo, S. Natale, Simple Hopf algebras and deformations of finite groups, Math. Res. Lett. 14 (6), 943–954 (2007). 109

[7] S. Gelaki and S. Westreich, On semisimple Hopf algebras of dimension pq, Proc. Amer. Math. Soc., 128(1), 39–47 (2000). 97

[8] I. Kaplansky, Bialgebras. University of Chicago Press, Chicago (1975). 97

[9] Y. Kashina, Y. Sommerhauser, Y. Zhu, Self-dual modules of semisimple Hopf algebras, J. Algebra, 257, 88–96 (2002). 100

[10] Y. Kashina, Y. Sommerhauser, Y. Zhu, On higher Frobenius-Schur indicators, Mem. Amer. Math. Soc., 181(2006). 98

[11] T. Kobayashi and A. Masuoka, A result extended from groups to Hopf algebras. Tsukuba J. Math. 21(1), 55-58 (1997). 104

[12] A. Masuoka, Semisimple Hopf algebras of dimension $2p$, Comm. Algebra, 23, 1931–1940 (1995). 97

[13] A. Masuoka, The p^n theorem for semisimple Hopf algebras, Proc. Amer. Math. Soc., 124, 735–737 (1996). 97, 103, 104, 108

[14] A. Masuoka, Self-dual Hopf algebras of dimension p^3 obtained by extension, J. Algebra, 178, 791-806 (1995). 97

[15] S. Montgomery and S. Whiterspoon, Irreducible representations of crossed products, J. Pure Appl. Algebra, 129, 315–326 (1998). 102, 103, 106

[16] S. Montgomery, Hopf algebras and their actions on rings. CBMS Reg. Conf. Ser. Math. 82. Amer. Math. Soc., Providence 1993. 98

[17] S. Natale, Semisolvability of semisimple Hopf algebras of low dimension, Mem. Amer. Math. Soc., 186 (2007). 101, 102, 104, 105, 106

[18] S. Natale, On semisimple Hopf algebras of dimension pq^2, J. Algebra, 221(2), 242–278 (1999). 97, 100

[19] W. D. Nichols and M. B. Richmond, The Grothendieck group of a Hopf algebra, J. Pure Appl. Algebra, 106, 297–306 (1996). 99

[20] W. D. Nichols and M. B. Zoelle, A Hopf algebra freeness theorem, Amer. J. Math., 111(2), 381–385 (1989). 100, 103

[21] D. Radford, The structure of Hopf algebras with a projection, J. Algebra, 92, 322–347 (1985). 103

[22] H.-J. Schneider, Normal basis and transitivity of crossed products for Hopf algebras. J. Algebra 152, 289–312 (1992). 102

[23] M. E. Sweedler, Hopf Algebras. Benjamin, New York (1969). 98

[24] R. Williams, Finite dimensional Hopf algebras. Ph. D. Thesis, Florida State University, 1988. 97

[25] Y. Zhu, Hopf algebras of prime dimension, Internat. Math. Res. Notices, 1, 53–59 (1994). 97, 99, 103

[26] S. Zhu, On finite dimensional semisimple Hopf algebras, Comm. Algebra, 21(11), 3871–3885 (1993). 109

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