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SEMI-SLANT LIGHTLIKE SUBMANIFOLDS OF INDEFINITE KAEHLER MANIFOLDS

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ABSTRACT. We introduce the notion of semi-slant lightlike submanifolds of indefinite Kaehler manifolds giving a characterization theorem with some non-trivial examples of such submanifolds. Integrability conditions of distributions D_1, D_2 and $\operatorname{Rad} TM$ on semi-slant lightlike submanifolds of indefinite Kaehler manifolds have been obtained. We also obtain necessary and sufficient conditions for foliations determined by the above distributions to be totally geodesic.

1. INTRODUCTION

The theory of lightlike submanifolds of a semi-Riemannian manifold was introduced by Duggal and Bejancu ([6]). A submanifold M of a semi-Riemannian manifold $\overline{M}$ is said to a be lightlike submanifold if the induced metric g on M is degenerate, i.e. there exists a non-zero $X \in \Gamma(TM)$ such that $g(X,Y) = 0$, $\forall Y \in \Gamma(TM)$. Lightlike geometry has its applications in general relativity, particularly in black hole theory, which gave impetus to study lightlike submanifolds of semi-Riemannian manifolds equipped with certain structures. The geometry of slant and screen-slant lightlike submanifolds of indefinite Hermitian manifolds was studied by Sahin in [12, 13]. In [3], B.Y. Chen defined slant immersions in complex geometry as a natural generalization of both holomorphic immersions and totally real immersions. The geometry of semi-slant submanifolds of Kaehler manifolds was studied by N. Papaghuic in [11].

The theory of slant, Cauchy–Riemann lightlike submanifolds of indefinite Kaehler manifolds has been studied in [6, 7]. In this paper we introduce the notion of semi-slant lightlike submanifolds of indefinite Kaehler manifolds. This new class of lightlike submanifolds of an indefinite Kaehler manifold includes slant, Cauchy–Riemann lightlike submanifolds as its sub-cases. The paper is arranged as follows. Section 2 contains some basic results. In section 3, we study semi-slant lightlike

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submanifolds of an indefinite Kaehler manifold, giving some examples. Section 4 is devoted to the study of foliations determined by distributions on semi-slant lightlike submanifolds of indefinite Kaehler manifolds.

2. Preliminaries

A submanifold (M^m, g) immersed in a semi-Riemannian manifold $(\overline{M}^{m+n}, \overline{g})$ is called a lightlike submanifold ([6]) if the metric g induced from $\overline{g}$ is degenerate and the radical distribution $\operatorname{Rad} TM$ is of rank r, where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $\operatorname{Rad} TM$ in TM, that is

$$TM = \operatorname{Rad} TM \oplus_{\text{orth}} S(TM). \tag{2.1}$$

Now consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $\operatorname{Rad} TM$ in $TM^{\perp}$. Since for any local basis $\{\xi_i\}$ of $\operatorname{Rad} TM$ there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $\operatorname{ltr}(TM)$ locally spanned by $\{N_i\}$. Let

$$\operatorname{tr}(TM) = \operatorname{ltr}(TM) \oplus_{\text{orth}} S(TM^{\perp}). \tag{2.2}$$

Then $\operatorname{tr}(TM)$ is a complementary (but not orthogonal) vector bundle to TM in $T\overline{M}|_M$, i.e.

$$T\overline{M}|_M = TM \oplus \operatorname{tr}(TM), \tag{2.3}$$

and therefore

$$T\overline{M}|_M = S(TM) \oplus_{\text{orth}} [\operatorname{Rad} TM \oplus \operatorname{ltr}(TM)] \oplus_{\text{orth}} S(TM^{\perp}). \tag{2.4}$$

Following, we give the four possible cases for a lightlike submanifold:

- Case 1: r-lightlike if $r < \min(m, n)$,
- Case 2: co-isotropic if $r = n < m$, $S\left(TM^{\perp}\right) = \{0\}$,
- Case 3: isotropic if $r = m < n$, $S\left(TM\right) = \{0\}$,
- Case 4: totally lightlike if $r = m = n$, $S(TM) = S(TM^{\perp}) = \{0\}$.

The Gauss and Weingarten formulae are given as

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \tag{2.6}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(\operatorname{tr}(TM))$, where $\{\nabla_X Y, A_V X\}$ belong to $\Gamma(TM)$ and $\{h(X, Y), \nabla^t_X V\}$ belong to $\Gamma(\operatorname{tr}(TM))$. ∇ and ∇^t are linear connections on M and on the vector bundle $\operatorname{tr}(TM)$ respectively. The second fundamental form h is a symmetric $F(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(\operatorname{tr}(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$. From (2.5) and (2.6), for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(\mathrm{ltr}(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{2.7}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \tag{2.8}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \tag{2.9}$$

where $h^l(X,Y) = L(h(X,Y))$, $h^s(X,Y) = S(h(X,Y))$, $D^l(X,W) = L(\nabla^t_X W)$, $D^s(X,N) = S(\nabla^t_X N)$, L and S are the projection morphisms of $\mathrm{tr}(TM)$ on $\mathrm{ltr}(TM)$ and $S(TM^{\perp})$ respectively. On the other hand, ∇^l and ∇^s are linear connections on $\mathrm{ltr}(TM)$ and $S(TM^{\perp})$ called the lightlike connection and screen transversal connection on M respectively.

Now by using (2.5), (2.7)-(2.9) and metric connection $\overline{\nabla}$, we obtain

$$\overline{g}(h^s(X,Y),W) + \overline{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.10}$$

$$\overline{g}(D^s(X,N),W) = \overline{g}(N, A_W X). \tag{2.11}$$

Denote the projection of TM on $S(TM)$ by $\overline{P}$. Then from the decomposition of the tangent bundle of a lightlike submanifold, for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}\,TM)$, we have

$$\nabla_X \overline{P}Y = \nabla^*_X \overline{P}Y + h^*(X, \overline{P}Y), \tag{2.12}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{2.13}$$

where $\left\{\nabla^*_X \overline{P}Y, A^*_\xi X\right\}$ and $\left\{h^*(X, \overline{P}Y), \nabla^{*t}_X \xi\right\}$ belong to $\Gamma(S(TM))$ and $\Gamma(\mathrm{Rad}\,TM)$ respectively. It follows that ∇^* and ∇^{*t} are linear connections on $S(TM)$ and $\mathrm{Rad}\,TM$ respectively. On the other hand, h^* and A^* are $\Gamma(\mathrm{Rad}\,TM)$-valued and $\Gamma(S(TM))$-valued $F(M)$-bilinear forms on $\Gamma(TM)\times\Gamma(S(TM))$ and $\Gamma(\mathrm{Rad}(TM))\times\Gamma(TM)$, called the second fundamental forms of distributions $S(TM)$ and $\mathrm{Rad}(TM)$ respectively. By using the above equations, we obtain

$$\overline{g}(h^l(X, \overline{P}Y), \xi) = g(A^*_\xi X, \overline{P}Y), \tag{2.14}$$

$$\overline{g}(h^*(X, \overline{P}Y), N) = g(A_N X, \overline{P}Y), \tag{2.15}$$

$$\overline{g}(h^l(X,\xi),\xi) = 0, \quad A^*_\xi \xi = 0. \tag{2.16}$$

It is important to note that in general ∇ is not a metric connection. Since $\overline{\nabla}$ is a metric connection, by using (2.7) we get

$$(\nabla_X g)(Y,Z) = \overline{g}(h^l(X,Y),Z) + \overline{g}(h^l(X,Z),Y). \tag{2.17}$$

An indefinite almost Hermitian manifold $(\overline{M}, \overline{g}, \overline{J})$ is a $2m$-dimensional semi-Riemannian manifold $\overline{M}$ with semi-Riemannian metric $\overline{g}$ of constant index q, $0 < q < 2m$, and a $(1,1)$ tensor field $\overline{J}$ on $\overline{M}$ such that the following conditions are satisfied:

$$\overline{J}^2 X = -X, \tag{2.18}$$

$$\overline{g}(\overline{J}X, \overline{J}Y) = \overline{g}(X,Y), \tag{2.19}$$

for all $X, Y \in \Gamma(T\overline{M})$.

An indefinite almost Hermitian manifold $(\overline{M}, \overline{g}, \overline{J})$ is called an indefinite Kaehler manifold if $\overline{J}$ is parallel with respect to $\overline{\nabla}$, i.e.,

$$(\overline{\nabla}_X \overline{J})Y = 0, \tag{2.20}$$

for all $X, Y \in \Gamma(T\overline{M})$, where $\overline{\nabla}$ is the Levi–Civita connection with respect to $\overline{g}$.

3. Semi-slant lightlike submanifolds

In this section, we introduce the notion of semi-slant lightlike submanifolds of indefinite Kaehler manifolds. At first, we state the following lemma, which helps us to define semi-slant lightlike submanifolds of indefinite Kaehler manifolds.

Lemma 3.1. *Let M be a q-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ of index $2q$. Suppose there exists a screen distribution $S(TM)$ such that $\overline{J}\,\mathrm{Rad}\,TM \subset S(TM)$ and $\overline{J}\,\mathrm{ltr}(TM) \subset S(TM)$. Then $\overline{J}\,\mathrm{Rad}\,TM \cap \overline{J}\,\mathrm{ltr}(TM) = \{0\}$ and any complementary distribution to $\overline{J}\,\mathrm{Rad}\,TM \oplus \overline{J}\,\mathrm{ltr}(TM)$ in $S(TM)$ is Riemannian.*

Proof. Let M be an m-dimensional q-lightlike submanifold of an $(m+n)$-dimensional indefinite Kaehler manifold $\overline{M}$ of index $2q$. Suppose $\overline{J}\,\mathrm{Rad}\,TM \cap \overline{J}\,\mathrm{ltr}(TM) \neq \{0\}$. Then there is an $N \in \Gamma(\mathrm{ltr}(TM))$ such that $\overline{J}N \in \Gamma(\overline{J}\,\mathrm{Rad}\,TM)$. Thus $\overline{g}(\overline{J}N, \overline{J}\xi) = 0$, for all $\xi \in \Gamma(\mathrm{Rad}\,TM)$, which implies $\overline{g}(N, \xi) = 0$, for all $\xi \in \Gamma(\mathrm{Rad}\,TM)$. This is impossible. Hence $\overline{J}\,\mathrm{Rad}\,TM \cap \overline{J}\,\mathrm{ltr}(TM) = \{0\}$. We denote the complementary distribution to $\overline{J}\,\mathrm{Rad}\,TM \oplus \overline{J}\,\mathrm{ltr}(TM)$ in $S(TM)$ by D'. Then we have a local quasi orthonormal frame of fields on $\overline{M}$ along M

$$\left\{\xi_i, N_i, \overline{J}\xi_i, \overline{J}N_i, X_\alpha, W_a\right\},$$

$i \in \{1, \dots, q\}$, $\alpha \in \{3q+1, \dots, m\}$, $a \in \{q+1, \dots, n\}$, where $\{\xi_i\}$ and $\{N_i\}$ are lightlike bases of $\mathrm{Rad}\,TM$ and $\mathrm{ltr}(TM)$, respectively, and $\{X_\alpha\}$ and $\{W_a\}$ are orthonormal bases of D' and $S(TM^\perp)$, respectively.

Now, from the basis $\left\{\xi_1, \dots, \xi_q, N_1, \dots, N_q, \overline{J}\xi_1, \dots, \overline{J}\xi_q, \overline{J}N_1, \dots, \overline{J}N_q\right\}$ of $\mathrm{Rad}\,TM \oplus \mathrm{ltr}(TM) \oplus \overline{J}\,\mathrm{Rad}\,TM \oplus \overline{J}\,\mathrm{ltr}(TM)$, we can construct an orthonormal basis $\{U_1, \dots, U_{2q}, V_1, \dots, V_{2q}\}$ as follows:

$$\begin{array}{ll} U_1 = \frac{1}{\sqrt{2}}(\xi_1 + N_1) & U_2 = \frac{1}{\sqrt{2}}(\xi_1 - N_1) \\ U_3 = \frac{1}{\sqrt{2}}(\xi_2 + N_2) & U_4 = \frac{1}{\sqrt{2}}(\xi_2 - N_2) \\ \cdots & \cdots \end{array}$$

$$\begin{array}{ll}
\dots & \dots \\
U_{2q-1} = \dfrac{1}{\sqrt{2}}(\xi_q + N_q) & U_{2q} = \dfrac{1}{\sqrt{2}}(\xi_q - N_q) \\
V_1 = \dfrac{1}{\sqrt{2}}(\overline{J}\xi_1 + \overline{J}N_1) & V_2 = \dfrac{1}{\sqrt{2}}(\overline{J}\xi_1 - \overline{J}N_1) \\
V_3 = \dfrac{1}{\sqrt{2}}(\overline{J}\xi_2 + \overline{J}N_2) & V_4 = \dfrac{1}{\sqrt{2}}(\overline{J}\xi_2 - \overline{J}N_2) \\
\dots & \dots \\
\dots & \dots \\
V_{2q-1} = \dfrac{1}{\sqrt{2}}(\overline{J}\xi_q + \overline{J}N_q) & V_{2q} = \dfrac{1}{\sqrt{2}}(\overline{J}\xi_q - \overline{J}N_q).
\end{array}$$

Hence, span $\{\xi_i, N_i, \overline{J}\xi_i, \overline{J}N_i\}$ is a non-degenerate space of constant index $2q$. Thus we conclude that $\operatorname{Rad} TM \oplus \operatorname{ltr}(TM) \oplus \overline{J} \operatorname{Rad} TM \oplus \overline{J} \operatorname{ltr}(TM)$ is non-degenerate and of constant index $2q$ on $\overline{M}$. Since $\operatorname{index}(T\overline{M}) = \operatorname{index}(\operatorname{Rad} TM \oplus \operatorname{ltr}(TM) \oplus \overline{J} \operatorname{Rad} TM \oplus \overline{J} \operatorname{ltr}(TM)) + \operatorname{index}(D' \oplus_{\mathrm{orth}} S(TM^\perp))$, we have $2q = 2q + \operatorname{index}(D' \oplus_{\mathrm{orth}} S(TM^\perp))$. Thus, $D' \oplus_{\mathrm{orth}} S(TM^\perp)$ is Riemannian, i.e., $\operatorname{index}(D' \oplus_{\mathrm{orth}} S(TM^\perp)) = 0$. Hence D' is Riemannian. □

Definition 3.1. Let M be a q-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ of index $2q$ such that $2q < \dim(M)$. Then we say that M is a semi-slant lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

(i) $\overline{J} \operatorname{Rad} TM$ is a distribution on M such that $\operatorname{Rad} TM \cap \overline{J} \operatorname{Rad} TM = \{0\}$;
(ii) there exist non-degenerate orthogonal distributions D_1 and D_2 on M such that $S(TM) = (\overline{J} \operatorname{Rad} TM \oplus \overline{J} \operatorname{ltr}(TM)) \oplus_{\mathrm{orth}} D_1 \oplus_{\mathrm{orth}} D_2$;
(iii) the distribution D_1 is an invariant distribution, i.e. $\overline{J}D_1 = D_1$;
(iv) the distribution D_2 is slant with angle $\theta(\neq 0)$, i.e. for each $x \in M$ and each non-zero vector $X \in (D_2)_x$, the angle θ between $\overline{J}X$ and the vector subspace $(D_2)_x$ is a non-zero constant, which is independent of the choice of $x \in M$ and $X \in (D_2)_x$.

This constant angle θ is called the slant angle of distribution D_2. A semi-slant lightlike submanifold is said to be proper if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $\theta \neq \frac{\pi}{2}$.

From the above definition, we have the following decomposition:

$$TM = \operatorname{Rad} TM \oplus_{\mathrm{orth}} (\overline{J} \operatorname{Rad} TM \oplus \overline{J} \operatorname{ltr}(TM)) \oplus_{\mathrm{orth}} D_1 \oplus_{\mathrm{orth}} D_2. \tag{3.1}$$

In particular, we have

(i) if $D_1 = 0$, then M is a slant lightlike submanifold;
(ii) if $D_1 \neq 0$ and $\theta = \pi/2$, then M is a CR-lightlike submanifold.

Thus the above new class of lightlike submanifolds of an indefinite Kaehler manifold includes slant, Cauchy–Riemann lightlike submanifolds as its sub-cases which have been studied in [6, 7].

Let $(\mathbb{R}^{2m}_{2q}, \overline{g}, \overline{J})$ denote the manifold $\mathbb{R}^{2m}_{2q}$ with its usual Kaehler structure given by

$$\overline{g} = \frac{1}{4}(-\sum_{i=1}^{q} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i),$$

$$\overline{J}(\sum_{i=1}^{m}(X_i \partial x_i + Y_i \partial y_i)) = \sum_{i=1}^{m}(Y_i \partial x_i - X_i \partial y_i),$$

where (x^i, y^i) are the Cartesian coordinates on $\mathbb{R}^{2m}_{2q}$. Now, we construct some examples of semi-slant lightlike submanifolds of an indefinite Kaehler manifold.

Example 1. Let $(\mathbb{R}^{12}_2, \overline{g}, \overline{J})$ be an indefinite Kaehler manifold, where $\overline{g}$ is of signature $(-,+,+,+,+,+,-,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6\}$.

Suppose M is a submanifold of $\mathbb{R}^{12}_2$ given by $-x^1 = y^2 = u_1$, $x^2 = u_2$, $y^1 = u_3$, $x^3 = u_4 \cos\beta$, $y^3 = -u_5 \cos\beta$, $x^4 = u_5 \sin\beta$, $y^4 = u_4 \sin\beta$, $x^5 = u_6 \sin u_7$, $y^5 = u_6 \cos u_7$, $x^6 = \sin u_6$, $y^6 = \cos u_6$.

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$, where

$$\begin{aligned}
Z_1 &= 2(-\partial x_1 + \partial y_2), \quad Z_2 = 2\partial x_2, \quad Z_3 = 2\partial y_1,\\
Z_4 &= 2(\cos\beta \partial x_3 + \sin\beta \partial y_4),\\
Z_5 &= 2(\sin\beta \partial x_4 - \cos\beta \partial y_3),\\
Z_6 &= 2(\sin u_7 \partial x_5 + \cos u_7 \partial y_5 + \cos u_6 \partial x_6 - \sin u_6 \partial y_6),\\
Z_7 &= 2(u_6 \cos u_7 \partial x_5 - u_6 \sin u_7 \partial y_5).
\end{aligned}$$

Hence $\operatorname{Rad} TM = \operatorname{span}\{Z_1\}$ and $S(TM) = \operatorname{span}\{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$.

Now $\operatorname{ltr}(TM)$ is spanned by $N = \partial x_1 + \partial y_2$ and $S(TM^\perp)$ is spanned by

$$\begin{aligned}
W_1 &= 2(\sin\beta \partial x_3 - \cos\beta \partial y_4),\\
W_2 &= 2(\cos\beta \partial x_4 + \sin\beta \partial y_3),\\
W_3 &= 2(\sin u_7 \partial x_5 + \cos u_7 \partial y_5 - \cos u_6 \partial x_6 + \sin u_6 \partial y_6),\\
W_4 &= 2(u_6 \sin u_6 \partial x_6 + u_6 \cos u_6 \partial y_6).
\end{aligned}$$

It follows that $\overline{J}Z_1 = Z_2 + Z_3$ and $\overline{J}N = 1/2(Z_2 - Z_3)$, which implies that $\overline{J}\operatorname{Rad} TM$ and $\overline{J}\operatorname{ltr}(TM)$ are distributions on M. On the other hand, we can see that $D_1 = \operatorname{span}\{Z_4, Z_5\}$ such that $\overline{J}Z_4 = Z_5$, $\overline{J}Z_5 = -Z_4$, which implies that D_1 is invariant with respect to $\overline{J}$ and $D_2 = \operatorname{span}\{Z_6, Z_7\}$ is a slant distribution with slant angle $\frac{\pi}{4}$. Hence M is a semi-slant 2-lightlike submanifold of $\mathbb{R}^{12}_2$.

Example 2. Let $(\mathbb{R}^{12}_2, \overline{g}, \overline{J})$ be an indefinite Kaehler manifold, where $\overline{g}$ is of signature $(-,+,+,+,+,+,-,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6\}$.

Suppose M is a submanifold of $\mathbb{R}^{12}_2$ given by $x^1 = y^2 = u_1$, $x^2 = u_2$, $y^1 = u_3$, $x^3 = y^4 = u_4$, $x^4 = -y^3 = u_5$, $x^5 = u_6$, $y^5 = u_7$, $x^6 = k \cos u_7$, $y^6 = k \sin u_7$, where k is any constant.

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$, where

$$\begin{aligned}
&Z_1 = 2(\partial x_1 + \partial y_2), \quad Z_2 = 2\partial x_2, \quad Z_3 = 2\partial y_1,\\
&Z_4 = 2(\partial x_3 + \partial y_4), \quad Z_5 = 2(\partial x_4 - \partial y_3),\\
&Z_6 = 2(\partial x_5),\\
&Z_7 = 2(\partial y_5 - k\sin u_7 \partial x_6 + k\cos u_7 \partial y_6).
\end{aligned}$$

Hence $\operatorname{Rad} TM = \operatorname{span}\{Z_1\}$ and $S(TM) = \operatorname{span}\{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$.

Now $\operatorname{ltr}(TM)$ is spanned by $N = -\partial x_1 + \partial y_2$ and $S(TM^\perp)$ is spanned by

$$\begin{aligned}
&W_1 = 2(\partial x_3 - \partial y_4), \quad W_2 = 2(\partial x_4 + \partial y_3),\\
&W_3 = 2(k\cos u_7 \partial x_6 + k\sin u_7 \partial y_6),\\
&W_4 = 2(k^2 \partial y_5 + k\sin u_7 \partial x_6 - k\cos u_7 \partial y_6).
\end{aligned}$$

It follows that $\overline{J}Z_1 = Z_2 - Z_3$ and $\overline{J}N = 1/2(Z_2 + Z_3)$, which implies that $\overline{J}\operatorname{Rad} TM$ and $\overline{J}\operatorname{ltr}(TM)$ are distributions on M. On the other hand, we can see that $D_1 = \operatorname{span}\{Z_4, Z_5\}$ such that $\overline{J}Z_4 = Z_5$, $\overline{J}Z_5 = -Z_4$, which implies that D_1 is invariant with respect to $\overline{J}$ and $D_2 = \operatorname{span}\{Z_6, Z_7\}$ is a slant distribution with slant angle $\theta = \arccos(1/\sqrt{1+k^2})$. Hence M is a semi-slant 2-lightlike submanifold of $\mathbb{R}^{12}_2$.

Now, for any vector field X tangent to M, we put $\overline{J}X = PX + FX$, where PX and FX are tangential and transversal parts of $\overline{J}X$ respectively. We denote the projections on $\operatorname{Rad} TM$, $\overline{J}\operatorname{Rad} TM$, $\overline{J}\operatorname{ltr}(TM)$, D_1 and D_2 in TM by P_1, P_2, P_3, P_4, and P_5 respectively. Similarly, we denote the projections of $\operatorname{tr}(TM)$ on $\operatorname{ltr}(TM)$ and $S(TM^\perp)$ by Q_1 and Q_2 respectively. Thus, for any $X \in \Gamma(TM)$, we get

$$X = P_1X + P_2X + P_3X + P_4X + P_5X. \tag{3.2}$$

Now applying $\overline{J}$ to (3.2), we have

$$\overline{J}X = \overline{J}P_1X + \overline{J}P_2X + \overline{J}P_3X + \overline{J}P_4X + \overline{J}P_5X, \tag{3.3}$$

which gives

$$\overline{J}X = \overline{J}P_1X + \overline{J}P_2X + \overline{J}P_3X + \overline{J}P_4X + fP_5X + FP_5X, \tag{3.4}$$

where fP_5X (resp. FP_5X) denotes the tangential (resp. transversal) component of $\overline{J}P_5X$. Thus we get $\overline{J}P_1X \in \Gamma(\overline{J}\operatorname{Rad} TM)$, $\overline{J}P_2X \in \Gamma(\operatorname{Rad} TM)$, $\overline{J}P_3X \in \Gamma(\operatorname{ltr}(TM))$, $\overline{J}P_4X \in \Gamma(D_1)$, $fP_5X \in \Gamma(D_2)$ and $FP_5X \in \Gamma(S(TM^\perp))$. Also, for any $W \in \Gamma(\operatorname{tr}(TM))$, we have

$$W = Q_1W + Q_2W. \tag{3.5}$$

Applying $\overline{J}$ to (3.5), we obtain

$$\overline{J}W = \overline{J}Q_1W + \overline{J}Q_2W, \tag{3.6}$$

which gives

$$\overline{J}W = \overline{J}Q_1W + BQ_2W + CQ_2W, \tag{3.7}$$

where BQ_2W (resp. CQ_2W) denotes the tangential (resp. transversal) component of $\overline{J}Q_2W$. Thus we get $\overline{J}Q_1W \in \Gamma(\overline{J}\operatorname{ltr}(TM))$, $BQ_2W \in \Gamma(D_2)$ and $CQ_2W \in \Gamma(S(TM^\perp))$.

Now, by using (2.20), (3.4), (3.7) and (2.7)-(2.9) and identifying the components on $\operatorname{Rad} TM$, $\overline{J}\operatorname{Rad} TM$, $\overline{J}\operatorname{ltr}(TM)$, D_1, D_2, $\operatorname{ltr}(TM)$ and $S(TM^{\perp})$, we obtain

$$\begin{aligned} P_1(\nabla_X \overline{J}P_1Y) + P_1(\nabla_X \overline{J}P_2Y) + P_1(\nabla_X \overline{J}P_4Y) + P_1(\nabla_X fP_5Y) \\ = P_1(A_{FP_5Y}X) + P_1(A_{\overline{J}P_3Y}X) + \overline{J}P_2\nabla_X Y, \end{aligned} \tag{3.8}$$

$$\begin{aligned} P_2(\nabla_X \overline{J}P_1Y) + P_2(\nabla_X \overline{J}P_2Y) + P_2(\nabla_X \overline{J}P_4Y) + P_2(\nabla_X fP_5Y) \\ = P_2(A_{FP_5Y}X) + P_2(A_{\overline{J}P_3Y}X) + \overline{J}P_1\nabla_X Y, \end{aligned} \tag{3.9}$$

$$\begin{aligned} P_3(\nabla_X \overline{J}P_1Y) + P_3(\nabla_X \overline{J}P_2Y) + P_3(\nabla_X \overline{J}P_4Y) + P_3(\nabla_X fP_5Y) \\ = P_3(A_{FP_5Y}X) + P_3(A_{\overline{J}P_3Y}X) + \overline{J}h^l(X,Y), \end{aligned} \tag{3.10}$$

$$\begin{aligned} P_4(\nabla_X \overline{J}P_1Y) + P_4(\nabla_X \overline{J}P_2Y) + P_4(\nabla_X \overline{J}P_4Y) + P_4(\nabla_X fP_5Y) \\ = P_4(A_{FP_5Y}X) + P_4(A_{\overline{J}P_3Y}X) + \overline{J}P_4\nabla_X Y, \end{aligned} \tag{3.11}$$

$$\begin{aligned} P_5(\nabla_X \overline{J}P_1Y) + P_5(\nabla_X \overline{J}P_2Y) + P_5(\nabla_X \overline{J}P_4Y) + P_5(\nabla_X fP_5Y) \\ = P_5(A_{FP_5Y}X) + P_5(A_{\overline{J}P_3Y}X) + fP_5\nabla_X Y + Bh^s(X,Y), \end{aligned} \tag{3.12}$$

$$\begin{aligned} h^l(X,\overline{J}P_1Y) + h^l(X,\overline{J}P_2Y) + h^l(X,\overline{J}P_4Y) + h^l(X,fP_5Y) \\ = \overline{J}P_3\nabla_X Y - \nabla^l_X \overline{J}P_3Y - D^l(X,FP_5Y), \end{aligned} \tag{3.13}$$

$$\begin{aligned} h^s(X,\overline{J}P_1Y) + h^s(X,\overline{J}P_2Y) + h^s(X,\overline{J}P_4Y) + h^s(X,fP_5Y) \\ = Ch^s(X,Y) - \nabla^s_X FP_5Y - D^s(X,\overline{J}P_3Y) + FP_5\nabla_X Y. \end{aligned} \tag{3.14}$$

Theorem 3.2. *Let M be a q-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ of index $2q$. Then M is a semi-slant lightlike submanifold if and only if*

(i) *$\overline{J}\operatorname{Rad} TM$ is a distribution on M such that $\operatorname{Rad} TM \cap \overline{J}\operatorname{Rad} TM = \{0\}$;*

(ii) *the screen distribution $S(TM)$ can be split as a direct sum*

$$S(TM) = (\overline{J}\operatorname{Rad} TM \oplus \overline{J}\operatorname{ltr}(TM)) \oplus_{\mathrm{orth}} D_1 \oplus_{\mathrm{orth}} D_2$$

such that D_1 is an invariant distribution on M, i.e. $\overline{J}D_1 = D_1$;

(iii) *there exists a constant $\lambda \in [0,1)$ such that $P^2X = -\lambda X$, for all $X \in \Gamma(D_2)$. In that case, $\lambda = \cos^2\theta$, where θ is the slant angle of D_2.*

Proof. Let M be a semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then the distribution D_1 is invariant with respect to $\overline{J}$ and $\overline{J}\operatorname{Rad} TM$ is a distribution on M such that $\operatorname{Rad} TM \cap \overline{J}\operatorname{Rad} TM = \{0\}$.

For any $X \in \Gamma(D_2)$ we have $|PX| = |\overline{J}X|\cos\theta$, i.e.

$$\cos\theta = \frac{|PX|}{|\overline{J}X|}. \tag{3.15}$$

In view of (3.15), we get $\cos^2\theta = \frac{|PX|^2}{|\overline{J}X|^2} = \frac{g(PX,PX)}{g(\overline{J}X,\overline{J}X)} = \frac{g(X,P^2X)}{g(X,\overline{J}^2X)}$, which gives

$$g(X, P^2X) = \cos^2\theta\, g(X, \overline{J}^2 X). \tag{3.16}$$

Since M is a semi-slant lightlike submanifold, $\cos^2\theta = \lambda(\text{constant}) \in [0,1)$ and therefore from (3.16) we get $g(X, P^2X) = \lambda g(X, \overline{J}^2X) = g(X, \lambda\overline{J}^2X)$, for all $X \in \Gamma(D_2)$, which implies

$$g(X, (P^2 - \lambda\overline{J}^2)X) = 0. \tag{3.17}$$

Since $(P^2 - \lambda\overline{J}^2)X \in \Gamma(D_2)$ and the induced metric $g = g|_{D_2\times D_2}$ is non-degenerate (positive definite), from (3.17) we have $(P^2 - \lambda\overline{J}^2)X = 0$, which implies

$$P^2X = \lambda\overline{J}^2X = -\lambda X, \quad \forall X \in \Gamma(D_2). \tag{3.18}$$

This proves (iii).

Conversely suppose that conditions (i), (ii) and (iii) are satisfied. From (iii) we have $P^2X = \lambda\overline{J}^2X$, for all $X \in \Gamma(D_2)$, where $\lambda(\text{constant}) \in [0,1)$.

Now $\cos\theta = \frac{g(\overline{J}X,PX)}{|\overline{J}X||PX|} = -\frac{g(X,\overline{J}PX)}{|\overline{J}X||PX|} = -\frac{g(X,P^2X)}{|\overline{J}X||PX|} = -\lambda\frac{g(X,\overline{J}^2X)}{|\overline{J}X||PX|} = \lambda\frac{g(\overline{J}X,\overline{J}X)}{|\overline{J}X||PX|}$.

From the above equation, we obtain

$$\cos\theta = \lambda\frac{|\overline{J}X|}{|PX|}. \tag{3.19}$$

Therefore (3.15) and (3.19) give $\cos^2\theta = \lambda(\text{constant})$.

Hence M is a semi-slant lightlike submanifold. □

Theorem 3.3. *Let M be a q-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ of index $2q$. Then M is a semi-slant lightlike submanifold if and only if*

(i) *$\overline{J}\,\mathrm{Rad}\,TM$ is a distribution on M such that $\mathrm{Rad}\,TM \cap \overline{J}\,\mathrm{Rad}\,TM = \{0\}$;*

(ii) *the screen distribution $S(TM)$ can be split as a direct sum*

$$S(TM) = (\overline{J}\,\mathrm{Rad}\,TM \oplus \overline{J}\,\mathrm{ltr}(TM)) \oplus_{\mathrm{orth}} D_1 \oplus_{\mathrm{orth}} D_2$$

such that D_1 is an invariant distribution on M, i.e. $\overline{J}D_1 = D_1$;

(iii) *there exists a constant $\mu \in (0,1]$ such that $BFX = -\mu X$, for all $X \in \Gamma(D_2)$. In that case, $\mu = \sin^2\theta$, where θ is the slant angle of D_2.*

Proof. Let M be a semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then the distribution D_1 is invariant with respect to $\overline{J}$ and $\overline{J}\,\mathrm{Rad}\,TM$ is a distribution on M such that $\mathrm{Rad}\,TM \cap \overline{J}\,\mathrm{Rad}\,TM = \{0\}$.

Now, for any vector field $X \in \Gamma(D_2)$, we have

$$\overline{J}X = PX + FX, \tag{3.20}$$

where PX and FX are the tangential and transversal parts of $\overline{J}X$ respectively. Applying $\overline{J}$ to (3.20) and taking the tangential component, we get

$$-X = P^2X + BFX, \quad \forall X \in \Gamma(D_2). \tag{3.21}$$

Since M is a semi-slant lightlike submanifold, $P^2X = -\lambda X$, $\forall X \in \Gamma(D_2)$, where $\lambda(\text{constant}) \in [0,1)$ and therefore from (3.21) we get

$$BFX = -\mu X, \quad \forall X \in \Gamma(D_2), \tag{3.22}$$

where $1-\lambda = \mu(\text{constant}) \in (0,1]$.

This proves (iii).

Conversely suppose that conditions (i), (ii) and (iii) are satisfied. From (3.21) we get

$$-X = P^2X - \mu X, \quad \forall X \in \Gamma(D_2), \tag{3.23}$$

which implies

$$P^2X = -\lambda X, \quad \forall X \in \Gamma(D_2), \tag{3.24}$$

where $1-\mu = \lambda(\text{constant}) \in [0,1)$.

Now the proof follows from Theorem 3.2. □

Corollary 3.1. *Let M be a semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ with slant angle θ; then, for any $X, Y \in \Gamma(D_2)$, we have*

$$g(PX, PY) = \cos^2\theta\, g(X,Y), \tag{3.25}$$

$$g(FX, FY) = \sin^2\theta\, g(X,Y). \tag{3.26}$$

The proof of the above corollary follows by using similar steps as in the proof of Corollary 3.1 of [13].

Theorem 3.4. *Let M be a semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $\operatorname{Rad} TM$ is integrable if and only if*

(i) $P_1(\nabla_X \overline{J}Y) = P_1(\nabla_Y \overline{J}X)$ *and* $P_4(\nabla_X \overline{J}Y) = P_4(\nabla_Y \overline{J}X)$;
(ii) $P_5(\nabla_X \overline{J}Y) = P_5(\nabla_Y \overline{J}X)$ *and* $h^l(Y, \overline{J}X) = h^l(X, \overline{J}Y)$;
(iii) $h^s(Y, \overline{J}X) = h^s(X, \overline{J}Y)$, *for all* $X, Y \in \Gamma(\operatorname{Rad} TM)$.

Proof. Let M be a semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. From (3.8), for any $X, Y \in \Gamma(\operatorname{Rad} TM)$, we have

$$P_1(\nabla_X \overline{J}Y) = \overline{J}P_2\nabla_X Y. \tag{3.27}$$

By interchanging X and Y in (3.27) we get

$$P_1(\nabla_Y \overline{J}X) = \overline{J}P_2\nabla_Y X. \tag{3.28}$$

From (3.27) and (3.28), we obtain

$$P_1(\nabla_X \overline{J}Y) - P_1(\nabla_Y \overline{J}X) = \overline{J}P_2[X,Y]. \tag{3.29}$$

From (3.11), for any $X, Y \in \Gamma(\operatorname{Rad} TM)$, we have

$$P_4(\nabla_X \overline{J}Y) = \overline{J}P_4\nabla_X Y. \tag{3.30}$$

By interchanging X and Y in (3.30) we get

$$P_4(\nabla_Y \overline{J}X) = \overline{J}P_4\nabla_Y X. \tag{3.31}$$

In view of (3.30) and (3.31) we get

$$P_4(\nabla_X \overline{J}Y) - P_4(\nabla_Y \overline{J}X) = \overline{J}P_4[X,Y]. \tag{3.32}$$

From (3.12), for any $X, Y \in \Gamma(\mathrm{Rad}\, TM)$, we have

$$P_5(\nabla_X \overline{J}Y) = fP_5\nabla_X Y + Bh^s(X, Y). \tag{3.33}$$

By interchanging X and Y in (3.33) we get

$$P_5(\nabla_Y \overline{J}X) = fP_5\nabla_Y X + Bh^s(Y, X). \tag{3.34}$$

From (3.33) and (3.34), we obtain

$$P_5(\nabla_X \overline{J}Y) - P_5(\nabla_Y \overline{J}X) = fP_5[X, Y]. \tag{3.35}$$

From (3.13), for any $X, Y \in \Gamma(\mathrm{Rad}\, TM)$, we have

$$h^l(X, \overline{J}Y) = \overline{J}P_3\nabla_X Y. \tag{3.36}$$

Interchanging X and Y in (3.36) we get

$$h^l(Y, \overline{J}X) = \overline{J}P_3\nabla_Y X. \tag{3.37}$$

From (3.36) and (3.37) we get

$$h^l(X, \overline{J}Y) - h^l(Y, \overline{J}X) = \overline{J}P_3[X, Y]. \tag{3.38}$$

From (3.14), for any $X, Y \in \Gamma(\mathrm{Rad}\, TM)$, we have

$$h^s(X, \overline{J}Y) = Ch^s(X, Y) + FP_5\nabla_X Y. \tag{3.39}$$

Interchanging X and Y in (3.39) we get

$$h^s(Y, \overline{J}X) = Ch^s(Y, X) + FP_5\nabla_Y X. \tag{3.40}$$

From (3.39) and (3.40), we obtain

$$h^s(X, \overline{J}Y) - h^s(Y, \overline{J}X) = FP_5[X, Y]. \tag{3.41}$$

Now the proof follows from (3.29), (3.32), (3.35), (3.38) and (3.41). □

Theorem 3.5. *Let M be a semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D_1 is integrable if and only if*

(i) $P_1(\nabla_X \overline{J}Y) = P_1(\nabla_Y \overline{J}X)$ *and* $P_2(\nabla_X \overline{J}Y) = P_2(\nabla_Y \overline{J}X)$*;*
(ii) $P_5(\nabla_X \overline{J}Y) = P_5(\nabla_Y \overline{J}X)$ *and* $h^l(Y, \overline{J}X) = h^l(X, \overline{J}Y)$*;*
(iii) $h^s(Y, \overline{J}X) = h^s(X, \overline{J}Y)$*, for all* $X, Y \in \Gamma(D_1)$*.*

Proof. Let M be a semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. From (3.8), for any $X, Y \in \Gamma(D_1)$, we have

$$P_1(\nabla_X \overline{J}Y) = \overline{J}P_2\nabla_X Y. \tag{3.42}$$

By interchanging X and Y in (3.42) we get

$$P_1(\nabla_Y \overline{J}X) = \overline{J}P_2\nabla_Y X. \tag{3.43}$$

From (3.42) and (3.43) we obtain

$$P_1(\nabla_X \overline{J}Y) - P_1(\nabla_Y \overline{J}X) = \overline{J}P_2[X, Y]. \tag{3.44}$$

From (3.9), for any $X, Y \in \Gamma(D_1)$, we have

$$P_2(\nabla_X \overline{J}Y) = \overline{J}P_1\nabla_X Y. \tag{3.45}$$

By interchanging X and Y in (3.45) we get

$$P_2(\nabla_Y \overline{J}X) = \overline{J}P_1\nabla_Y X. \tag{3.46}$$

From (3.45) and (3.46) we obtain

$$P_2(\nabla_X \overline{J}Y) - P_2(\nabla_Y \overline{J}X) = \overline{J}P_1[X, Y]. \tag{3.47}$$

From (3.12), for any $X, Y \in \Gamma(D_1)$, we have

$$P_5(\nabla_X \overline{J}Y) = fP_5\nabla_X Y + Bh^s(X, Y). \tag{3.48}$$

By interchanging X and Y in (3.48) we get

$$P_5(\nabla_Y \overline{J}X) = fP_5\nabla_Y X + Bh^s(Y, X). \tag{3.49}$$

In view of (3.48) and (3.49) we obtain

$$P_5(\nabla_X \overline{J}Y) - P_5(\nabla_Y \overline{J}X) = fP_5[X, Y]. \tag{3.50}$$

From (3.13), for any $X, Y \in \Gamma(D_1)$, we have

$$h^l(X, \overline{J}Y) = \overline{J}P_3\nabla_X Y. \tag{3.51}$$

Interchanging X and Y in (3.51) we get

$$h^l(Y, \overline{J}X) = \overline{J}P_3\nabla_Y X. \tag{3.52}$$

From (3.51) and (3.52) we obtain

$$h^l(X, \overline{J}Y) - h^l(Y, \overline{J}X) = \overline{J}P_3[X, Y]. \tag{3.53}$$

From (3.14), for any $X, Y \in \Gamma(D_1)$, we have

$$h^s(X, \overline{J}Y) = Ch^s(X, Y) + FP_5\nabla_X Y. \tag{3.54}$$

Interchanging X and Y in (3.54) we get

$$h^s(Y, \overline{J}X) = Ch^s(Y, X) + FP_5\nabla_Y X. \tag{3.55}$$

Also from (3.54) and (3.55) we obtain

$$h^s(X, \overline{J}Y) - h^s(Y, \overline{J}X) = FP_5[X, Y]. \tag{3.56}$$

Now the proof follows from (3.44), (3.47), (3.50), (3.53) and (3.56). □

Theorem 3.6. *Let M be a semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D_2 is integrable if and only if*

(i) $P_1(\nabla_X fY - \nabla_Y fX) = P_1(A_{FY}X - A_{FX}Y)$;
(ii) $P_2(\nabla_X fY - \nabla_Y fX) = P_2(A_{FY}X - A_{FX}Y)$;
(iii) $P_4(\nabla_X fY - \nabla_Y fX) = P_4(A_{FY}X - A_{FX}Y)$;
(iv) $h^l(X, fY) - h^l(Y, fX) = D^l(Y, FX) - D^l(X, FY)$,

for all $X, Y \in \Gamma(D_2)$.

Proof. Let m be a semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{m}$. From (3.8), for any $x, y \in \gamma(d_2)$ we have

$$P_1(\nabla_X fY) - P_1(A_{FY}X) = \overline{J}P_2\nabla_X Y. \tag{3.57}$$

Interchanging X and Y in (3.57) we get

$$P_1(\nabla_Y fX) - P_1(A_{FX}Y) = \overline{J}P_2\nabla_Y X. \tag{3.58}$$

From (3.57) and (3.58) we obtain

$$P_1(\nabla_X fY - \nabla_Y fX) - P_1(A_{FY}X - A_{FX}Y) = \overline{J}P_2[X, Y]. \tag{3.59}$$

From (3.9), for any $X, Y \in \Gamma(D_2)$ we have

$$P_2(\nabla_X fY) - P_2(A_{FY}X) = \overline{J}P_1\nabla_X Y. \tag{3.60}$$

Interchanging X and Y in (3.60) we get

$$P_2(\nabla_Y fX) - P_2(A_{FX}Y) = \overline{J}P_1\nabla_Y X. \tag{3.61}$$

In view of (3.60) and (3.61) we obtain

$$P_2(\nabla_X fY - \nabla_Y fX) - P_2(A_{FY}X - A_{FX}Y) = \overline{J}P_1[X, Y]. \tag{3.62}$$

From (3.11), for any $X, Y \in \Gamma(D_2)$ we have

$$P_4(\nabla_X fY) - P_4(A_{FY}X) = \overline{J}P_4\nabla_X Y. \tag{3.63}$$

Interchanging X and Y in (3.63) we get

$$P_4(\nabla_Y fX) - P_4(A_{FX}Y) = \overline{J}P_4\nabla_Y X. \tag{3.64}$$

From (3.63) and (3.64) we obtain

$$P_4(\nabla_X fY - \nabla_Y fX) - P_4(A_{FY}X - A_{FX}Y) = \overline{J}P_4[X, Y]. \tag{3.65}$$

From (3.13), for any $X, Y \in \Gamma(D_2)$ we have

$$h^l(X, fY) + D^l(X, FY) = \overline{J}P_3\nabla_X Y. \tag{3.66}$$

Interchanging X and Y in (3.66) we get

$$h^l(Y, fX) + D^l(Y, FX) = \overline{J}P_3\nabla_Y X. \tag{3.67}$$

Also from (3.66) and (3.67) we obtain

$$h^l(X, fY) - h^l(Y, fX) + D^l(X, FY) - D^l(Y, FX) = \overline{J}P_3[X, Y]. \tag{3.68}$$

The proof follows from (3.59), (3.62), (3.65) and (3.68). □

4. Foliations determined by distributions

In this section, we obtain necessary and sufficient conditions for foliations determined by distributions on a semi-slant lightlike submanifold of an indefinite Kaehler manifold to be totally geodesic.

Definition 4.1. A semi-slant lightlike submanifold M of an indefinite Kaehler manifold $\overline{M}$ is said to be mixed geodesic if its second fundamental form h satisfies $h(X,Y)=0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$. Thus M is a mixed geodesic semi-slant lightlike submanifold if $h^l(X,Y)=0$ and $h^s(X,Y)=0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$.

Theorem 4.1. *Let M be a semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $\operatorname{Rad} TM$ defines a totally geodesic foliation if and only if*

$$\overline{g}(\nabla_X \overline{J}P_2Z + \nabla_X \overline{J}P_4Z + \nabla_X fP_5Z, \overline{J}Y) = \overline{g}(A_{\overline{J}P_3Z}X + A_{FP_5Z}X, \overline{J}Y),$$

for all $X \in \Gamma(\operatorname{Rad} TM)$ and $Z \in \Gamma(S(TM))$.

Proof. Let M be a semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. To prove that $\operatorname{Rad} TM$ defines a totally geodesic foliation, it is sufficient to show that $\nabla_X Y \in \operatorname{Rad} TM$, for all $X, Y \in \Gamma(\operatorname{Rad} TM)$. Since $\overline{\nabla}$ is a metric connection, using (2.7) and (2.19), for any $X, Y \in \Gamma(\operatorname{Rad} TM)$ and $Z \in \Gamma(S(TM))$, we get

$$\overline{g}(\nabla_X Y, Z) = \overline{g}((\overline{\nabla}_X \overline{J})Z - \overline{\nabla}_X \overline{J}Z, \overline{J}Y). \tag{4.1}$$

Now from (2.20), (3.4) and (4.1) we get

$$\overline{g}(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X(\overline{J}P_2Z + \overline{J}P_3Z + \overline{J}P_4Z + fP_5Z + FP_5Z), \overline{J}Y). \tag{4.2}$$

In view of (2.7)-(2.9) and (4.2), for any $X, Y \in \Gamma(\operatorname{Rad} TM)$ and $Z \in \Gamma(S(TM))$ we obtain

$$\begin{aligned}\overline{g}(\nabla_X Y, Z) =& \overline{g}(A_{\overline{J}P_3Z}X + A_{FP_5Z}X - \nabla_X \overline{J}P_2Z \\ &- \nabla_X \overline{J}P_4Z - \nabla_X fP_5Z, \overline{J}Y),\end{aligned} \tag{4.3}$$

which completes the proof. □

Theorem 4.2. *Let M be a semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D_1 defines a totally geodesic foliation if and only if*

(i) $\overline{g}(A_{FZ}X, \overline{J}Y) = \overline{g}(\nabla_X fZ, \overline{J}Y)$;

(ii) $A_{\overline{J}W}X$ *and* $\nabla_X \overline{J}N$ *have no component in* D_1,

for all $X, Y \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$, $W \in \Gamma(\overline{J}\operatorname{ltr}(TM))$ and $N \in \Gamma(\operatorname{ltr}(TM))$.

Proof. Let M be a semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. The distribution D_1 defines a totally geodesic foliation if and only if $\nabla_X Y \in D_1$, for all $X, Y \in \Gamma(D_1)$. Since $\overline{\nabla}$ is a metric connection, from (2.7), (2.19) and (2.20), for any $X, Y \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$ we get

$$\overline{g}(\nabla_X Y, Z) = \overline{g}(\overline{\nabla}_X \overline{J}Y, \overline{J}Z) = -\overline{g}(\overline{\nabla}_X \overline{J}Z, \overline{J}Y). \tag{4.4}$$

In view of (2.7), (2.9) and (4.4) we obtain

$$\overline{g}(\nabla_X Y, Z) = \overline{g}(A_{FZ}X - \nabla_X fZ, \overline{J}Y). \tag{4.5}$$

Now, from (2.7), (2.19) and (2.20), for any $X, Y \in \Gamma(D_1)$ and $N \in \Gamma(\mathrm{ltr}(TM))$ we have

$$\overline{g}(\nabla_X Y, N) = \overline{g}(\overline{\nabla}_X \overline{J}Y, \overline{J}N) = -\overline{g}(\overline{J}Y, \overline{\nabla}_X \overline{J}N). \tag{4.6}$$

From (2.7) and (4.6) we get

$$\overline{g}(\nabla_X Y, N) = -\overline{g}(\overline{J}Y, \nabla_X \overline{J}N). \tag{4.7}$$

Also, from (2.7), (2.19) and (2.20), for any $X, Y \in \Gamma(D_1)$ and $W \in \Gamma(\overline{J}\,\mathrm{ltr}(TM))$ we have

$$\overline{g}(\nabla_X Y, W) = \overline{g}(\overline{\nabla}_X \overline{J}Y, \overline{J}W) = -\overline{g}(\overline{J}Y, \overline{\nabla}_X \overline{J}W). \tag{4.8}$$

In view of (2.8) and (4.8) we obtain

$$\overline{g}(\nabla_X Y, W) = \overline{g}(\overline{J}Y, A_{\overline{J}W}X), \tag{4.9}$$

which completes the proof. □

Theorem 4.3. *Let M be a semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D_2 defines a totally geodesic foliation if and only if*

(i) $\overline{g}(\nabla_X \overline{J}Z, fY) = -\overline{g}(h^s(X, \overline{J}Z), FY)$,
(ii) $\overline{g}(fY, \nabla_X \overline{J}N) = -\overline{g}(FY, h^s(X, \overline{J}N))$,
(iii) $\overline{g}(fY, A_{\overline{J}W}X) = \overline{g}(FY, D^s(X, \overline{J}W))$,

for all $X, Y \in \Gamma(D_2)$, $Z \in \Gamma(D_1)$, $W \in \Gamma(\overline{J}\,\mathrm{ltr}(TM))$ and $N \in \Gamma(\mathrm{ltr}(TM))$.

Proof. Let M be a semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. The distribution D_2 defines a totally geodesic foliation if and only if $\nabla_X Y \in D_2$, for all $X, Y \in \Gamma(D_2)$. Since $\overline{\nabla}$ is a metric connection, from (2.7), (2.19) and (2.20), for any $X, Y \in \Gamma(D_2)$ and $Z \in \Gamma(D_1)$ we obtain

$$\overline{g}(\nabla_X Y, Z) = \overline{g}(\overline{\nabla}_X \overline{J}Y, \overline{J}Z) = -\overline{g}(\overline{\nabla}_X \overline{J}Z, \overline{J}Y). \tag{4.10}$$

From (2.7) and (4.10) we get

$$\overline{g}(\nabla_X Y, Z) = -\overline{g}(\nabla_X \overline{J}Z, fY) - \overline{g}(h^s(X, \overline{J}Z), FY). \tag{4.11}$$

Now, from (2.7), (2.19) and (2.20), for any $X, Y \in \Gamma(D_2)$ and $N \in \Gamma(\mathrm{ltr}(TM))$ we have

$$\overline{g}(\nabla_X Y, N) = \overline{g}(\overline{\nabla}_X \overline{J}Y, \overline{J}N) = -\overline{g}(\overline{J}Y, \overline{\nabla}_X \overline{J}N). \tag{4.12}$$

From (2.7) and (4.12) we get

$$\overline{g}(\nabla_X Y, N) = -\overline{g}(fY, \nabla_X \overline{J}N) - \overline{g}(FY, h^s(X, \overline{J}N)). \tag{4.13}$$

Also, from (2.7), (2.19) and (2.20), for any $X, Y \in \Gamma(D_2)$ and $W \in \Gamma(\overline{J}\,\mathrm{ltr}(TM))$ we have

$$\overline{g}(\nabla_X Y, W) = \overline{g}(\overline{\nabla}_X \overline{J}Y, \overline{J}W) = -\overline{g}(\overline{J}Y, \overline{\nabla}_X \overline{J}W). \tag{4.14}$$

In view of (2.8) and (4.14) we obtain

$$\overline{g}(\nabla_X Y, W) = \overline{g}(fY, A_{\overline{J}W} X) - \overline{g}(FY, D^s(X, \overline{J}W)), \tag{4.15}$$

which completes the proof. □

Theorem 4.4. *Let M be a mixed geodesic semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D_2 defines a totally geodesic foliation if and only if*

(i) *$\nabla_X \overline{J}Z$ has no component in D_2,*
(ii) *$\overline{g}(fY, \nabla_X \overline{J}N) = -\overline{g}(FY, h^s(X, \overline{J}N))$,*
(iii) *$\overline{g}(fY, A_{\overline{J}W} X) = \overline{g}(FY, D^s(X, \overline{J}W))$,*

for all $X, Y \in \Gamma(D_2)$, $Z \in \Gamma(D_1)$, $W \in \Gamma(\overline{J}\,\mathrm{ltr}(TM))$ and $N \in \Gamma(\mathrm{ltr}(TM))$.

Proof. Since M is a mixed geodesic semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$, we have $h^s(Z, X) = 0$, for all $Z \in \Gamma(D_1)$ and $X \in \Gamma(D_2)$. Now, the proof follows from Theorem 4.3. □

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