REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 56, No. 2, 2015, Pages 85–94
Published online: September 17, 2015

DETECTION OF THE TORSION CLASSES IN THE BRIESKORN MODULES OF HOMOGENEOUS POLYNOMIALS

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ABSTRACT. Let $f \in \mathbb{C}[X_1, \ldots, X_n]$ be a homogeneous polynomial and $B(f)$ be the corresponding Brieskorn module, which is the quotient of the polynomial ring by some specific $\mathbb{C}$-vector space and it has a $\mathbb{C}[t]$-module structure. The main results detect torsion classes in the Brieskorn module using explicit computations with differential forms. We compute the torsion of the Brieskorn module $B(f)$ for two variables in case of non-isolated singularities and show that torsion order is at most 1. In addition, we find some interesting families in which $B(f)$ is torsion free even in case of non-isolated singularities. We exhibit several examples to compute the monomial basis for $B(f)$ and the construction of torsion elements for $n > 2$.

1. Introduction

The Brieskorn module $B(f)$ associated to a homogeneous polynomial f is a free $\mathbb{C}[t]$-module of rank $\mu(f)$, the Milnor number of f, in the case of isolated singularities (Theorem 3.1). We derive an important consequence, saying that $B(f)_{\text{tors}} = 0$ if and only if the homogeneous polynomial f has an isolated singularity at the origin (Corollary 3.4). Here $C(f) \subset B(f)$ is a naturally defined $\mathbb{C}[t]$-submodule, such that $B(f)/C(f) = M(f)$. Then we discuss the case of two variables but allow arbitrary singularities for the homogeneous polynomial f. In this case $C(f)$ is free of rank $b_1(F)$, the first Betti number of the generic fiber F of f (Theorem 4.1). By the result mentioned above, this implies that all torsion elements in $B(f)$ have torsion order one, i.e. $t \cdot B(f)_{\text{tors}} = 0$ (Corollary 4.3). This result is exemplified on the family of polynomials $f = x^p y^q$ with $(p, q) = 1$ which is completely discussed in [11].

In the present article, we find one family $f = x^p y^q (x + y)^r$, with $1 \leq p \leq q \leq r$ and $\gcd(p, q, r) = 1$, in which the Brieskorn module is torsion free even in case of non-isolated singularities for $n = 2$. This leads us to the general statement that the torsion order of the Brieskorn module is at most 1. Also for $n = 3$, we compute explicitly monomial basis for the Brieskorn module in Examples 5.1 and 5.2.

2010 *Mathematics Subject Classification.* Primary 32S40; Secondary 14J70, 14F45, 32S25.
Key words and phrases. torsion order, Brieskorn module, Milnor algebra, homogeneous polynomials, eigenvalues of the monodromy.

In the general case of n variables, there is a very subtle result due to Dimca and Saito which relates the eigenvalues of the monodromy of f to a specific (possibly torsion) element in $B(f)$, see Corollary 5.3. Then we exhibit several examples of torsion elements constructed using this general approach. We find one more very interesting Example 5.4 which provides torsion elements of order 2. The last Example 5.5 shows that even for rather complicated examples (here the zero set of f is a surface S with non-isolated singularities) one may still have 1 as the t-torsion order. These torsion classes of $B(f)$, in case of non-isolated singularities are usually very interesting for the topology, e.g. some of them are related to the monodromy of the corresponding global Milnor fibration of the polynomial, see for instance Corollary 1.10 in [8].

2. Milnor algebra and Brieskorn module

Let $f \in R = \mathbb{C}[x_1, \dots, x_n]$ be a homogeneous polynomial of degree $d > 1$. Then one can identify the following complex to the Koszul complex of the partial derivatives $f_j = \frac{\partial f}{\partial x_j}$; $j = 1, \dots, n$ in R

$$0 \longrightarrow \Omega^0 \xrightarrow{df\wedge} \Omega^1 \xrightarrow{df\wedge} \cdots \xrightarrow{df\wedge} \Omega^{n-1} \xrightarrow{df\wedge} \Omega^n \longrightarrow 0,$$

where Ω^j denotes the regular differential forms of degree j on $\mathbb{C}^n$.

Let J_f be a Jacobian ideal, spanned by the partial derivatives f_j, $j = 1, \dots, n$, in R, and $M(f) = R/J_f$ be the Milnor algebra corresponding to f. We can have the following isomorphism of graded vector spaces:

$$M(f)(-n) = \frac{\Omega^n}{df \wedge \Omega^{n-1}}.$$

Here, for any graded $\mathbb{C}[t]$-module M, the shifted module $M(m)$ is defined by setting $M(m)_s = M_{m+s}$ for all $s \in \mathbb{Z}$.

Now we define the (algebraic) Brieskorn module as the quotient

$$B(f) = \frac{\Omega^n}{df \wedge d(\Omega^{n-2})}$$

in analogy with the (analytic) local situation considered in [5], as well as the submodule

$$C(f) = \frac{df \wedge \Omega^{n-1}}{df \wedge d(\Omega^{n-2})}.$$

These modules are modules over the ring $\mathbb{C}[t]$ and the action of t is defined naturally by multiplying the polynomial f. Sometimes $B(f)$ is denoted by $G_f^{(0)}$ and $C(f)$ by $G_f^{(-1)}$, see [9, 6]. Using Euler's formula, we can see that

$$f.B(f) \subset C(f).$$

We have the following basic relation between the Milnor algebra and the Brieskorn module (see [8, prop. 1.6]).

Theorem 2.1.
$$df \wedge \Omega^{n-1} = df \wedge d(\Omega^{n-2}) + f.\Omega^n.$$
In particular
$$B(f)/f.B(f) \simeq M(f).$$

Remark 2.2. The $C[f]$-module $P(f)$, which is defined as the quotient of all $(n-1)$-forms by the closed $(n-1)$-forms,
$$P(f) = \frac{\Omega^{n-1}}{(df \wedge \Omega^{n-2} + d\Omega^{n-2})} \supseteq H_f^{n-1}.$$
The latter is an extension of the relative cohomology group H_f^{n-1}. The quotient $P(f)/H_f^{n-1}$ is naturally isomorphic to $\mathbb{C}$-space $M(f)$. In several sources, $P(f)$ is referred to as the Petrov module. Using analytic tools or theory of perverse sheaves and D-modules, they prove that under certain genericity-type assumptions on f, the highest relative cohomology module H_f^{n-1} and $P(f)$ are finitely generated over ring $\mathbb{C}[f]$. The exterior differential naturally projects as a bijective map $d : P(f) \to B(f)$ which obviously is not a $C[f]$-module homomorphism. For details one can see [12].

3. The case of an isolated singularity

Here we assume that $f \in \mathbb{C}[x_1, \ldots, x_n]$ is the homogeneous polynomial and $M(f)$ is the corresponding Milnor algebra which is a graded $\mathbb{C}$-algebra with dimension $\mu(f)$, the Milnor number when f has an isolated singularity at the origin. The Brieskorn module $B(f)$ is completely determined in this setting as follows.

Theorem 3.1. *The $\mathbb{C}[t]$-module $B(f)$ is free of rank $\mu(f)$.*

Proof. Indeed, a homogeneous polynomial f having an isolated singularity at the origin induces a tame mapping $f : \mathbb{C}^n \to \mathbb{C}$ to which the results in [6], [7] and [10] apply. □

Corollary 3.2. *As a graded module over the graded ring $\mathbb{C}[t]$, one has an isomorphism*
$$B(f) = M(f) \otimes_{\mathbb{C}} \mathbb{C}[t].$$
In particular, one has, at the level of the associated Poincaré series, the following equality:
$$P_{B(f)}(t) = P_{M(f)}(t).\frac{1}{1-t} = \frac{(1-t^{d-1})^n}{(1-t)^{n+1}}.$$

Example 3.3. For $n = 3$, let $f(x, y, z) = x^3+y^3+z^3$ be a homogeneous polynomial of degree 3;then a $\mathbb{C}$-basis of $M(f)$ is given as $1, x, y, z, xy, yz, xz, xyz$. The same 8 monomials form a basis of $B(f)$ as a free $\mathbb{C}[t]$-module.

In this case $P_{B(f)}(t) = \frac{(1-t^2)^3}{(1-t)^4} = \frac{(1+t)^3}{1-t} = (1 + 3t + 3t^2 + t^3) + t(1 + 3t + 3t^2 + t^3) + t^2(1 + 3t + 3t^2 + t^3) + \cdots = 1 + 4t + 7t^2 + 8t^3 + 8t^4 + 8t^5 + \ldots$. In general, if $f = x_0^d + x_1^d + \cdots + x_n^d$ then a $\mathbb{C}$-basis of $M(f)$ is given by
$$\langle x_0^{a_0}, \ldots, x_n^{a_n}; 0 \leq a_j \leq d-2\rangle$$

which is also $\mathbb{C}[t]$ basis of $B(f)$.

Corollary 3.4. *If* 0 *is an isolated singularity of the homogeneous polynomial* f *then the* $\mathbb{C}[t]$*-module* $B(f)$ *is torsion free.*

Proof. If 0 is an isolated singularity of the homogeneous polynomial f then clearly the $\mathbb{C}[t]$-module $B(f)$ is torsion free. To prove the converse we prove its contrapositive statement. Suppose that 0 is not an isolated singularity, then the Milnor algebra $M(f)$ is an infinite dimensional $\mathbb{C}$-vector space. The isomorphism in Theorem 2.1 implies that in this case $B(f)$ is not finitely generated over $\mathbb{C}[t]$. It follows that the canonical projection

$$B(f) \to \overline{B}(f)$$

is not an isomorphism, hence $B(f)_{\text{tors}} \neq 0$. □

4. The case of non-isolated singularity $(n = 2)$

In this section we suppose that $f \in \mathbb{C}[x, y]$ is a homogeneous polynomial of degree d which is not the power g^r(which we will always assume) of some other polynomial $g \in \mathbb{C}[x, y]$ for some $r > 1$. This condition is equivalent to asking the generic fiber of f to be connected, and such polynomials are sometimes called "primitive". For more on this, see [4], final Remark, part (I).

In this case of non-isolated singularities the Milnor algebras $M(f)$ and the Brieskorn modules $B(f)$ are not finitely generated. So we find an invariant other than rank, called torsion of $B(f)$, which is not finitely generated in general. We completely classify the Brieskorn module in the case of two variables with non-isolated singularities and show that torsion order is at most 1.

The following result follows from Proposition 7, part (ii) in [9].

Theorem 4.1. *The submodule* $C(f)$ *of the Brieskorn module* $B(f)$ *is torsion free.*

Definition 4.2. For $b \in B(f)$, we say that b is t-torsion of order $k \geq 1$ if $t^k \cdot b = 0$ and $t^{k-1} \cdot b \neq 0$.

Corollary 4.3. *The* $\mathbb{C}[t]$*-torsion elements in* $B(f)$ *are only those elements whose* t*-torsion order is* 1.

Proof. Let $b \in B(f)_{\text{tors}}$, the submodule of $\mathbb{C}[t]$-torsion elements. Since $f : \mathbb{C}^2 \to \mathbb{C}$ induces a locally trivial fibration over $\mathbb{C}^*$, it follows that b is t-torsion, say of order k. If $k > 1$, then

$$0 = t^k \cdot b = t^{k-1} \cdot (tb).$$

By Theorem 2.1 we know that $t \cdot b \in C(f)$. Applying Theorem 3.1 we get that $t \cdot b = 0$ in $C(f)$, i.e. $t \cdot b = 0$ in $B(f)$.

This is possible only if $k = 1$, since by Theorem 3.1 the submodule $C(f)$ is torsion free. □

The above corollary can be restated by saying that the following is a exact sequence

$$0 \to B(f)_{\text{tors}} \to B(f) \xrightarrow{t} C(f) \to 0.$$

If we define $\overline{B}(f) = B(f)/B(f)_{\text{tors}}$, we get an isomorphism of graded $\mathbb{C}[t]$-module

$$\overline{B}(f)(-d) \simeq C(f).$$

Here, for any graded $\mathbb{C}[t]$-module M, the shifted module $M(m)$ is defined by setting $M(m)_s = M_{m+s}$ for all $s \in \mathbb{Z}$.

The following are some examples for infinite family given by the polynomial f and the torsion order of the elements of $B(f)$ is computed.

Example 4.4. Let $f = x^p y^q$ with $(p, q) = 1$. The detailed computations for this example could be found in [11]. Which clearly shows that the torsion order in this case is exactly one.

The following is the case when $B(f)$ is torsion free in case of non-isolated singularities.

Example 4.5. Let $f = x^p y^q (x + y)^r$ with $1 \le p \le q \le r$ and $\gcd(p, q, r) = 1$. Then $df \wedge \Omega^1 = J_f \cdot \Omega^2$, where ideal

$$J_f = \langle px^{p-1}y^q(x+y)^r + rx^py^q(x+y)^{r-1}, qx^py^{q-1}(x+y)^r + rx^py^q(x+y)^{r-1}\rangle.$$

It follows that $\frac{B(f)}{C(f)} \simeq \frac{S}{J_f}$ is an infinite dimensional $\mathbb{C}$-vector space spanned by all monomials say $x^a y^b$ belongs to $S \backslash LT\{J_f\}$, where $LT\{J_f\} = \{LT(g); g \in J_f\}$ which we can find after finding the Gröbner basis of J_f. Using Buchberger's criterion and then Macaulay's basis theorem implies all these information (see for instance [1, p. 329]).

In our case, we first compute the Gröbner basis of J_f and then we take leading term of that ideal and get this monomial ideal, i.e.

$$LT(J_f) = \langle x^{p+r}y^{q-1}, x^{p+r-1}y^q, x^{p+r-2}y^{q+2}\rangle.$$

It follows that

$$\frac{B(f)}{C(f)} \simeq \frac{S}{J_f} = \begin{cases} x^a y^b, & a + b \le p + q + r - 2; \\ x^a y^b, & a + b = p + q + r - 1;\ (a, b) \notin \{(p + r, q - 1), (p + r - 1, q)\}; \\ x^a y^b, & a + b \ge p + q + r;\ a \le p + r - 3 \text{ or } b \le q - 2. \end{cases}$$

For such a monomial $x^a y^b$ consider the 2-form $w = x^a y^b dx \wedge dy$. Then $[w] \ne 0$ in $B(f)$ since $[w] \ne 0$ in $B(f)/C(f)$.

Now compute $df \wedge d\Omega^0$:

$$\begin{aligned} df \wedge d(P(x,y)) &= (px^{p-1}y^q(x+y)^r + rx^py^q(x+y)^{r-1}dx + qx^py^{q-1}(x+y)^r \\ &\quad + rx^py^q(x+y)^{r-1}dy) \wedge (\tfrac{\partial P}{\partial x}dx + \tfrac{\partial P}{\partial y}dy) \\ &= ((px^{p-1}y^q(x+y)^r + rx^py^q(x+y)^{r-1})\tfrac{\partial P}{\partial y} \\ &\quad - (qx^py^{q-1}(x+y)^r + rx^py^q(x+y)^{r-1})\tfrac{\partial P}{\partial x})dx \wedge dy, \end{aligned}$$

where $P(x, y) \in \mathbb{C}[x, y]$.

To find torsion part in $B(f)$. We know that for $n = 2$, the submodule $C(f)$ is torsion free using Theorem 3.1 and the only $\mathbb{C}[t]$-torsion elements in $B(f)$ are

t-torsion of order 1 using Corollary 4.3. Hence

$$B(f)_{\text{tors}} = \{w \in B(f) \text{ such that } t \cdot [w] = [f \cdot w] = 0 \text{ in } B(f)\}.$$

It follows that $w \in B(f)_{\text{tors}}$ if $f \cdot w \in df \wedge d\Omega^0$, which is equivalent to saying that the corresponding system of equations getting from $f \cdot w = df \wedge d(P(x,y))$ has a solution. So we have checked that for any triplet $(p,q,r) = 1$, we don't have a solution for any $w \in \frac{B(f)}{C(f)} \subseteq B(f)$ of the system corresponding to $f \cdot w = x^{a+p}y^{b+q}(x+y)^r = df \wedge d(P(x,y))$, where $P(x,y)$ is polynomial with suitable degree, which we can obtain after expansion and comparing coefficients of both sides of the preceding equations and see that the ranks of the corresponding homogeneous and non-homogeneous systems of equations do not agree. Hence our $B(f)$ is torsion free.

5. Eigenvalues of the monodromy and torsion in the general case

Now we compute the Brieskorn lattice explicitly for some examples in the case of more than two variables.

Example 5.1. Let $f(x,y,z) = xyz$. Then $df \wedge \Omega^2 = J_f \cdot \Omega^3$, where $J_f = \langle yz.xz, xy\rangle$. It follows that $\frac{B(f)}{C(f)} \simeq \frac{S}{J_f}$ is an infinite dimensional $\mathbb{C}$-vector space with a monomial basis given by $\langle 1, x^k, y^k, z^k\rangle_{k\geq 1}$.

For such a monomial $x^a y^b z^c$ consider the 3-form $w = x^a y^b z^c dx \wedge dy \wedge dz$. Then $[w] \neq 0$ in $B(f)$ since $[w] \neq 0$ in $\frac{B(f)}{C(f)}$.

For sake of simplicity we take $w_1 = x^a y^b z^c dx$, $w_2 = x^\alpha y^\beta z^\gamma dy$ and $w_3 = x^\phi y^\varphi z^\psi dz$, where $w_1, w_2, w_3 \in \Omega^1$. Now we compute $df \wedge dw_i$ in each case and then put them together. But here in the case of monomial, due to symmetry it is enough to work with one of the above forms, say $w_1 = x^a y^b z^c dx$; the remaining ones will give us the same result. So $df \wedge dw_1 = (cx^{a+1}y^b z^c - bx^{a+1}y^b z^c)dx \wedge dy \wedge dz = (c-b)x^{a+1}y^b z^c dx \wedge dy \wedge dz$.

The coefficient is 0 iff there is a $k \in \mathbb{N}$ such that $b = c = k \ for \ k \geq 1$.

Since

$$C(f) = \frac{J_f \cdot \Omega^3}{df \wedge d\Omega^1} = \frac{\langle yz, xz, xy\rangle\Omega^3}{df \wedge d\Omega^1}$$

we want to find a system of generators of the $\mathbb{C}$-vector space $C(f)$. This system is given by the classes of the elements $w \in \langle yz, xz, xy\rangle\Omega^3$, which do not belong to $df \wedge d\Omega^1$. To compute them it is enough to look at monomials, i.e.

$$\{x^\alpha y^\beta z^\gamma dx \wedge dy \wedge dz, \text{ with } \alpha \geq 1,\ \beta \geq 1,\ \gamma \geq 1 \Leftrightarrow x^\alpha y^\beta z^\gamma \in \langle yz, xz, xy\rangle\}.$$

We know that

$$df \wedge d\Omega^1 = (c-b)x^{a+1}y^b z^c dx \wedge dy \wedge dz.$$

So if $a+1 = \alpha$, $b = \beta$, $c = \gamma$, and $c - b \neq 0$, i.e. $a = \alpha - 1$, $b = \beta$, $c = \gamma$, with $a \geq 0$, $b, c \geq 1$, then

$$df \wedge \frac{d\Omega^1}{c-b} = x^\alpha y^\beta z^\gamma dx \wedge dy \wedge dz.$$

Hence, the only elements $x^\alpha y^\beta z^\gamma dx\wedge dy\wedge dz$ which do not appear in the denominator of $C(f)$ are $x^{a+1}y^bz^cdx\wedge dy\wedge dz$, where $c=b\Leftrightarrow c=k$, $b=k$; $k\geq 1$, $a=j$, $j\geq 0$. It follows that $C(f)$, as a $\mathbb{C}$-vector space, is spanned by $x^ky^kz^k$ for $k\geq 1$. Hence the monomial basis for $B(f)=\langle 1,x^k,y^k,z^k,(xyz)^k\rangle_{k\geq 1}$. It is clear that

$$B(f)_{\text{tors}}=\langle (x^k,y^k,z^k)_{k\geq 1}\rangle.$$

Moreover,

$$\overline{B}(f)=\frac{B(f)}{B(f)_{\text{tors}}}\simeq C(f)$$

is free $\mathbb{C}[t]$-module of rank $1=b_2(F)$, where $F: xyz-1=0$. Indeed $F\simeq\mathbb{C}^*\times\mathbb{C}^*$, which is homotopically equivalent to $S^1\times S^1$.

Example 5.2. Let $f=xyz+x^3+y^3$ be nodal cubic. Then $df\wedge\Omega^2=J_f\cdot\Omega^3$ where ideal

$$J_f=\langle yz+3x^2,xz+3y^2,xy\rangle.$$

It follows that $\frac{B(f)}{C(f)}\simeq\frac{S}{J_f}$ is an infinite dimensional $\mathbb{C}$-vector space spanned by all monomials say $x^ay^bz^c$ belongs to $S\setminus LT\{J_f\}$, where $LT\{J_f\}=\{LT(g); g\in J_f\}$, which we can find after finding the Gröbner basis of J_f. Using Buchberger's criterion and then Macaulay's basis Theorem implies all this information (see for instance [1, p. 329]).

In our case, we first compute Gröbner basis of J_f, and the leading term ideal is given by

$$LT(J_f)=\langle xy,y^2,x^2,yz^2,xz^2\rangle.$$

It follows that

$$\frac{S}{J_f}=\{x^ay^bz^c;\ a=1,\ b=0,\ 0\leq c\leq 1 \text{ or } a=0,\ b=1,\ 0\leq c\leq 1 \text{ or } a=0=b\}.$$

For the sake of simplicity, we take $w_1=x^ay^bz^cdx$, $w_2=x^\alpha y^\beta z^\gamma dy$ and $w_3=x^\phi y^\varphi z^\psi dz$, where $w_1,w_2,w_3\in\Omega^1$; then compute $df\wedge d\Omega^1$ in each case and put them together. After the computations in each case of the above forms we get the following three expressions:

$$\begin{aligned}
df\wedge dw_1&=((b-c)x^{a+1}y^bz^c-3cx^ay^{b+2}z^{c-1})dx\wedge dy\wedge dz,\\
df\wedge dw_2&=((\gamma-\alpha)x^\alpha y^{\beta+1}z^\gamma-3\gamma x^{\alpha+2}y^\beta z^{\gamma-1})dx\wedge dy\wedge dz,\\
df\wedge dw_3&=((\varphi-\phi)x^\phi y^\varphi z^{\psi+1}+3\varphi x^{\phi+2}y^{\varphi-1}z^\psi-3\phi x^{\phi-1}y^{\varphi+2}z^\psi)dx\wedge dy\wedge dz.
\end{aligned}$$

Since $B(f)$ is a graded module, we can consider the graded pieces one at a time. Elements in $B(f)$ are equivalence classes, so keeping in mind the three expressions above we just take one representative of each class and get $B^0(f)=\mathbb{C}\langle 1\rangle$. For degrees greater than one we have that if degree $=d=3s, s\geq 1$, then $B^d(f)=\langle xyz^{d-2},z^d\rangle$; otherwise $B^d(f)=\langle xz^{d-1},yz^{d-1},z^d\rangle$, so the monomial basis for $B(f)$ is

$$B(f)=\langle xz^{3k},xz^{3k+1},yz^{3k},yz^{3k+1},xyz^{3k+1},z^k\rangle_{k\geq 0}.$$

It is clear that $B(f)_{\text{tors}} = \langle z^k;\ k \geq 1 (k \text{ is not a multiple of } 3)\rangle$. Moreover,

$$\overline{B}(f) = \frac{B(f)}{B(f)_{\text{tors}}} = \begin{cases} 1 \\ xz^k, & k \geq 0 \text{ but } k \neq 3s-1,\ s \geq 1 \\ yz^k, & k \geq 0 \text{ but } k \neq 3s-1,\ s \geq 1 \\ xyz^{3k+1}, & k \geq 0 \\ z^{3k}, & k \geq 1 \end{cases}$$

is free $\mathbb{C}[t]$-module of rank $5 = b_2(F)$, where $F : xyz + x^3 + y^3 - 1 = 0$.

$\chi(F) = 1 - b_1 + b_2$ where $b_i's$ are Betti numbers and $C : \{xyz + x^3 + y^3 = 0\}$. The map $F : \{xyz + x^3 + y^3 = 1\} \longrightarrow U = \mathbb{P}^2 \setminus C$ is $\mathbb{Z}_3$-covering, which implies that $\pi_1(F) \rightarrowtail \pi_1(U) = \mathbb{Z}_3$ of index 3, where $U = \mathbb{P}^2 \setminus C$ and then $b_1(F) = 0$.

Since $\chi(F) = 3 \cdot \chi(U)$ where $\chi(U) = \chi(\mathbb{P}^2) \setminus \chi(C) = 3 - 1 = 2$ ($\chi(C) = 1$ since C is homotopically equivalent to $s^1 \vee s^2$). So $\chi(F) = 3 \cdot 2 = 6$ and $\chi(F) = 1 - b_1 + b_2$ imply $b_2(F) = 5$. For more details see for instance [3, Ch. 4].

According to [8], just before Remark 1.9, one has for $q \leq n$ an inclusion

$$t^{n-q-1} : \overline{\mathcal{H}}_{f,(q+1)d-j} \to \overline{\mathcal{H}}_{f,nd-j} = H^{n-1}(F, \mathbb{C})_\lambda,$$

where $\lambda = \exp(\frac{2\pi j \sqrt{-1}}{d})$, with $j = 0, 1, \ldots, d-1$.

Solve the equation $n = (q+1)d - j$, i.e. $n = qd + (d-j)$ is the Euclidean division if $j \neq 0$.

We get the following (see [8, Corollary 1.10]):

Corollary 5.3. *Assume that* $\exp(\frac{2\pi n \sqrt{-1}}{d})$ *is not an eigenvalue of the monodromy of f. Then there is a $k \geq 0$ such that*

$$t^k \cdot [w] = [f^k \cdot dx_1 \wedge \cdots \wedge dx_n] = 0$$

in $B(f)$, which is equivalent to the following: There is a positive integer k such that $f^k w \in df \wedge d\Omega^{n-2}$.

Using the above corollary, we compute the torsion order of $B(f)$ in the general case.

Example 5.4. Let $f = x^2y^2z + x^5 + y^5$, $n = 3,\ d = 5$.

For $w = dx \wedge dy \wedge dz$, to find the value of k such that $t^k \cdot [w] = 0$ in $B(f)$ is equivalent to saying that $f^k \cdot w \in df \wedge d\Omega^1$. We have to check for which value of k we have solutions of the equation

$$f^k \cdot w = \Big[(2xy^2z + 5x^4)(\tfrac{\partial R}{\partial y} - \tfrac{\partial Q}{\partial z}) + (2x^2yz + 5y^4)(\tfrac{\partial P}{\partial z} - \tfrac{\partial R}{\partial x}) + x^2y^2(\tfrac{\partial Q}{\partial x} - \tfrac{\partial P}{\partial y})\Big]\, dx \wedge dy \wedge dz,$$

where $P, Q, R \in \mathbb{C}[x, y, z]$ are homogeneous polynomials of degree $5k - 3$.

For $k = 1$, we get a system of non-homogeneous linear equations in which we have 15 unknowns and 13 equations. Then using the software MATLAB we compute the rank of the corresponding homogeneous system (containing 13 rows and 15 columns) and get 8. On the other hand, the rank of the non-homogeneous system (containing 13 rows and 16 columns) is 9, which shows that this system has no solution.

For $k = 2$, we get another system of non-homogeneous linear equations containing 102 unknowns and 58 equations. Then using MATLAB we compute the rank of the corresponding homogeneous system (containing 58 rows and 102 columns) and get 56. This time the corresponding non-homogeneous system (containing 58 rows and 103 columns) has also rank 56, which shows that this system of equations has a solution. An explicit solution for $k = 2$ is $P = -6x^5yz$, $Q = 6x^4y^3 - x^3y^2z^2 - 4xy^5z$ and $R = 1/5(x^6y - xy^6)$. Hence $[w]$ has t-torsion order 2 in $B(f)$.

The last example shows that even for rather complicated examples (here the zero set of f is a surface S with non-isolated singularities) one may still have 1 as the t-torsion order of $[\omega_n]$.

Example 5.5. Let $f = x^2y^2 + y^2z^2 + x^2z^2 - 2xyz(x + y + z)$, $n = 3$, $d = 4$.

For $w = dx \wedge dy \wedge dz$, to find the value of k such that $t^k \cdot [w] = 0$ in $B(f)$ is equivalent to saying that $f^k \cdot w \in df \wedge d\Omega^1$. We have to check for which value of k we have solutions of the equation

$$\begin{aligned} f^k \cdot w = \Big[&(2xy^2 + 2xz^2 - 4xyz - 2y^2z - 2yz^2)(\tfrac{\partial R}{\partial y} - \tfrac{\partial Q}{\partial z}) \\ &+ (2x^2y + 2yz^2 - 2x^2z - 4xyz - 2xz^2)(\tfrac{\partial P}{\partial z} - \tfrac{\partial R}{\partial x}) \\ &+ (2y^2z + 2x^2z - 2x^2y - 2xy^2 - 4xyz)(\tfrac{\partial Q}{\partial x} - \tfrac{\partial P}{\partial y})\Big]\, dx \wedge dy \wedge dz, \end{aligned}$$

where $P, Q, R \in \mathbb{C}[x, y, z]$ are homogeneous polynomials of degree $3k - 1$.

For $k = 1$, we get a system of non-homogeneous linear equations in which we have 15 unknowns and 12 equations. Then using MATLAB we compute the rank of the corresponding homogeneous system (containing 12 rows and 15 columns) and get 9. On the other hand, the rank of the non-homogeneous system (containing 12 rows and 16 columns) is also 9, which shows that this system has a solution. Hence $[w]$ has t-torsion order 1 in $B(f)$.

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Received: March 26, 2014
Accepted: February 9, 2015