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A REMARK ON TRANS-SASAKIAN 3-MANIFOLDS

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Abstract. Let M be a trans-Sasakian 3-manifold of type (α, β). In this paper, we give a negative answer to the question proposed by S. Deshmukh [Mediterr. J. Math. **13** (2016), no. 5, 2951–2958], namely we prove that the differential equation $\nabla\beta = \xi(\beta)\xi$ on M does not necessarily imply that M is homothetic to either a Sasakian or cosymplectic manifold even when M is compact. Many examples are constructed to illustrate this result.

1. Introduction

A trans-Sasakian manifold M is an almost contact metric manifold such that the product $M \times \mathbb{R}$ belongs to the class W_4 of Hermitian manifolds (see [23]). Hermitian manifolds of class W_4 are closely related to locally conformally Kähler manifolds (see [13]). In [18], Marrero gave the local structures of trans-Sasakian manifolds, namely a connected trans-Sasakian manifold of dimension greater than three must be of class C_5 or C_6. A trans-Sasakian manifold is said to be proper if it is one of the above two cases. However, there exist many trans-Sasakian 3-manifolds which are not proper (see [3, 20]), namely neither α nor β is zero. Therefore, to find on what condition a trans-Sasakian 3-manifold is proper is an interesting problem. Recently, S. Desmukh et al. in [8, 9, 10, 11, 12] obtained various conditions under which a compact trans-Sasakian 3-manifold is homothetic to either a Sasakian or a cosymplectic 3-manifold. Trans-Sasakian 3-manifolds under some curvature restrictions were also studied by U. C. De et al. in [5, 6, 7] and Wang [24].

In [8], without the compactness assumption, Deshmukh proved

Theorem 1.1 ([8])**.** *If a connected trans-Sasakian 3-manifold M of type (α, β) satisfies $\nabla\alpha = \xi(\alpha)\xi$ with $\alpha \neq 0$, then M is homothetic to a Sasakian manifold.*

In the proof of the above theorem, Deshmukh found that $\nabla\alpha = \xi(\alpha)\xi$ and $\alpha \neq 0$ imply $\nabla\beta = \xi(\beta)\xi$. In view of similar but different properties of α and β, the author in [8] proposed the following interesting question:

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Question 1.1. Does the differential equation $\nabla\beta = \xi(\beta)\xi$ give a result similar to Theorem 1.1 or not?

The above question is equivalent to asking if differential equation $\nabla\beta = \xi(\beta)\xi$ on a trans-Sasakian 3-manifold implies that the manifold is homothetic to either a Sasakian or cosymplectic 3-manifold. In this paper, we aim to investigate this question and give a negative answer. Namely, we prove

Theorem 1.2. *If a connected compact trans-Sasakian* 3*-manifold of type* (α, β) *satisfies* $\nabla\beta = \xi(\beta)\xi$*, then we have either* $\alpha = 0$ *or* $\beta = 0$*. However, the vanishing of* α *or* β *does not necessarily imply that the other one is a constant.*

After giving proof of the above Theorem 1.2 in Section 3, we construct many concrete examples to illustrate our main results.

2. Trans-Sasakian manifolds

According to D. E. Blair [2], an almost contact metric structure defined on a smooth differentiable manifold M of dimension $2n + 1$ is a (ϕ, ξ, η, g)-structure satisfying

$$\begin{gathered} \phi^2 = -\mathrm{id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \\ \phi^* g = g - \eta \otimes \eta, \end{gathered} \tag{2.1}$$

where ϕ is a $(1, 1)$-type tensor field, ξ is a tangent vector field called the characteristic or the Reeb vector field, and η is a 1-form called the almost contact form. A Riemannian manifold M furnished with an almost contact metric structure is said to be an *almost contact metric manifold*, denoted by (M, ϕ, ξ, η, g).

Let M be an almost contact metric manifold of dimension $2n+1$. On the product $M \times \mathbb{R}$ there exists an almost complex structure J defined by

$$J\left(X, f\frac{\mathrm{d}}{\mathrm{d}t}\right) = \left(\phi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}\right),$$

where X denotes a vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is a $\mathcal{C}^\infty$-function on $M^{2n+1} \times \mathbb{R}$. An almost contact metric manifold is said to be normal if the above almost complex structure J is integrable and this is equivalent to $[\phi, \phi] = -2\mathrm{d}\eta \otimes \xi$, where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ.

An almost contact metric manifold is said to be a trans-Sasakian manifold (see [18]) if it is normal and $\mathrm{d}\eta = \alpha\Phi$, $\mathrm{d}\Phi = 2\beta\eta\wedge\Phi$, where $\alpha = \frac{1}{2n}\operatorname{tr}(\phi\nabla\xi)$, $\beta = \frac{1}{2n}\operatorname{div}\xi$ and $\Phi(\cdot, \cdot) = g(\cdot, \phi\cdot)$. It is known (see [3]) that an almost contact metric manifold M is trans-Sasakian if and only if there exist two smooth functions α and β satisfying

$$(\nabla_X\phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.2}$$

for any vector fields X and Y.

A trans-Sasakian manifold is denoted by $(M, \phi, \xi, \eta, \alpha, \beta)$ and is called a trans-Sasakian manifold of type (α, β). From the definition of tran-Sasakian manifolds, putting $Y = \xi$ in (2.2) and using (2.1) we have

$$\nabla_X\xi = -\alpha\phi X + \beta(X - \eta(X)\xi) \tag{2.3}$$

for any vector field X.

By Propositions 1 and 2 and Corollary 1 of [19], we observe that a normal almost contact metric 3-manifold is always trans-Sasakian. Therefore, by the definition of trans-Sasakian manifolds, we state that an almost contact metric 3-manifold is trans-Sasakian if and only if it is normal.

3. Some remarks on trans-Sasakian 3-manifolds

Before giving the proof of our main results, firstly we must clarify some important concepts that are easily confused.

As seen in the Introduction, a trans-Sasakian 3-manifold of type (α, β) is an α-Sasakian manifold (see [14]) if $\alpha \in \mathbb{R}^*$ and $\beta = 0$; or a β-Kenmotsu manifold if $\beta \in \mathbb{R}^*$ and $\alpha = 0$ (see [14]); or a cosymplectic manifold if $\alpha = \beta = 0$ (see [2]). An α-Sasakian manifold becomes a Sasakian manifold (see [2]) if $\alpha = 1$. Similarly, a β-Kenmotsu manifold becomes a Kenmotsu manifold (see [15]) if $\beta = 1$.

A trans-Sasakian manifold of type (α, β) is of C_6-class if $\beta = 0$ (see [4]). As seen in [18, Lemma 3.1], α, on a trans-Sasakian manifold of C_6-class of dimension greater than three, is a constant. Then the trans-Sasakian manifolds of C_6-class of dimension greater than 3 are just α-Sasakian manifolds (see [14]). However, α on a trans-Sasakian 3-manifold of C_6-class is not necessarily a constant even when the manifold is compact. The proof for this is given after proving Theorem 3.1.

A trans-Sasakian manifold of type (α, β) is of C_5-class if $\alpha = 0$ (see [4]). On such manifolds of dimension greater than three there holds naturally $\mathrm{d}\beta \wedge \eta = 0$ (see [22]), or equivalently, $\nabla\beta = \xi(\beta)\xi$. However, the above equation does not necessarily hold for dimension three. The set of all β-Kenmotsu manifolds is a proper subset of that of all trans-Sasakian manifolds of C_5-class. For trans-Sasakian manifolds of C_5-class with non-constant function β we refer the reader to [1, 3, 5, 22]. Note that a trans-Sasakian manifold of C_5-class is also called a f-cosymplectic manifold (see [1]) or an f-Kenmotsu manifold (see [16, 22, 25]). In this paper, in order to answer Question 1.1 we also construct some compact trans-Sasakian 3-manifolds of C_5-class with non-constant function β.

The following lemma was proved in [7] (see also [12]).

Lemma 3.1 ([7, Theorem 3.2])**.** *On a trans-Sasakian 3-manifold of type (α, β) we have*

$$\xi(\alpha) + 2\alpha\beta = 0.$$

In this paper, we denote by ∇f the gradient of a smooth function f on M. Moreover, putting $n = 1$ in [7, Proposition 3.4] we obtain the following lemma.

Lemma 3.2 ([7, Proposition 3.4])**.** *On a trans-Sasakian 3-manifold of type (α, β) we have*

$$Q\xi = \phi(\nabla\alpha) - \nabla\beta + (2(\alpha^2 - \beta^2) - \xi(\beta))\xi,$$

where Q denotes the Ricci operator associated with the Ricci tensor S which is defined by $S(\cdot, \cdot) = \mathrm{trace}\{X \to R(X, \cdot)\cdot\}$.

Now we are ready to prove the following

Theorem 3.1. *If a connected trans-Sasakian* 3*-manifold of type* (α, β) *satisfies* $\nabla\beta = \xi(\beta)\xi$, *then one of the following cases occurs:*

(1) $\alpha \neq 0$, $\beta \in \mathbb{R}^*$, *and* $\xi(\alpha) \neq 0$.
(2) $\alpha \neq 0$, $\beta = 0$, *and* $\xi(\alpha) = 0$.
(3) $\alpha = 0$, $\beta \neq 0$, $\xi(\beta) \neq 0$, *and* $\nabla\xi(\beta) = \xi(\xi(\beta))\xi$.
(4) $\alpha = 0$, $\beta \in \mathbb{R}$.

Proof. If on a connected trans-Sasakian 3-manifold of type (α, β) there holds $\nabla\beta = \xi(\beta)\xi$, taking the covariant derivative of this equation we have

$$\nabla_X \nabla\beta = X(\xi(\beta))\xi + \xi(\beta)(-\alpha\phi X + \beta X - \beta\eta(X)\xi) \tag{3.1}$$

for any vector field X, where we have used (2.3). Note that the Hessian H_β is a symmetric bilinear form defined by

$$H_\beta(X, Y) = g(\nabla_X \nabla\beta, Y)$$

for any vector fields X, Y. Thus, the inner product of (3.1) with Y gives

$$H_\beta(X, Y) = X(\xi(\beta))\eta(Y) - \alpha\xi(\beta)g(X, \phi Y) + \beta\xi(\beta)g(X, Y) - \beta\xi(\beta)\eta(X)\eta(Y).$$

Interchanging X and Y in the above equation gives

$$H_\beta(Y, X) = Y(\xi(\beta))\eta(X) - \alpha\xi(\beta)g(Y, \phi X) + \beta\xi(\beta)g(X, Y) - \beta\xi(\beta)\eta(X)\eta(Y).$$

In view of the symmetry of H_β, subtracting the above equation from the previous one gives

$$X(\xi(\beta))\eta(Y) - Y(\xi(\beta))\eta(X) - 2\alpha\xi(\beta)g(X, \phi Y) = 0 \tag{3.2}$$

for any vector fields X, Y. Putting $Y = \xi$ in (3.2) gives $X(\xi(\beta)) - \xi(\xi(\beta))\eta(X) = 0$ for any vector field X. This is equivalent to $\nabla\xi(\beta) = \xi(\xi(\beta))\xi$.

Let X in (3.2) be an arbitrary unit vector field orthogonal to the Reeb vector field ξ. Putting $Y = \phi X$ in (3.2) we obtain $\alpha\xi(\beta) = 0$ and this implies that one of the following three cases occurs: $\alpha = 0$ and $\xi(\beta) \neq 0$, or $\alpha \neq 0$ and $\xi(\beta) = 0$, or $\alpha = \xi(\beta) = 0$. Notice that when $\xi(\beta) = 0$, by the assumption of the theorem we see that β is a constant. The proof follows from Lemma 3.1. □

From equation (2.3) we obtain $\operatorname{div}\xi = 2\beta$. Therefore, on any compact trans-Sasakian 3-manifold, β cannot be a non-zero constant. The following corollary follows directly from Theorem 3.1.

Corollary 3.1. *On a non-cosymplectic compact trans-Sasakian* 3*-manifold of type* (α, β) *satisfying* $\nabla\beta = \xi(\beta)\xi$, *we have either* $\alpha \neq 0$, $\beta = 0$ *and* $\xi(\alpha) = 0$, *or* $\alpha = 0$, $\beta \neq 0$, $\xi(\beta) \neq 0$ *and* $\nabla\xi(\beta) = \xi(\xi(\beta))\xi$.

Corollary 3.1 proves the first conclusion of Theorem 1.2. Next, we need only to show a proof of the last conclusion of Theorem 1.2, namely we show that there exist many trans-Sasakian 3-manifolds which are of C_5-class with β a non-constant function or of C_6-class with α a non-constant function. All the constructions in these examples depend on the following lemma.

Lemma 3.3 ([18]). *Let (M, ϕ, ξ, η, g) be a trans-Sasakian 3-manifold of type (α, β) and f be a positive function on M. Then, (M, ϕ, ξ, η, g') is also a trans-Sasakian 3-manifold of type $(\frac{\alpha}{f}, \beta + \frac{1}{2f}\xi(f))$, where the Riemannian metric g' is defined by $g' = fg + (1-f)\eta \otimes \eta$.*

Example 3.1. Let (x, y, z) be the canonical Cartesian coordinates in $\mathbb{R}^3$. On $\mathbb{R}^3$ there exists a standard Sasakian structure (see Blair [2, p. 60]) defined as

$$\xi = 2\frac{\partial}{\partial z}, \quad \eta = \frac{1}{2}(dz - ydx),$$
$$\phi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & y & 0 \end{pmatrix} \quad \text{and} \quad g = \frac{1}{4}\begin{pmatrix} 1+y^2 & 0 & -y \\ 0 & 1 & 0 \\ -y & 0 & 1 \end{pmatrix}.$$

The orthonormal ϕ-basis is given by $\{\xi, e_1 := 2\frac{\partial}{\partial y}, e_2 := \phi e_1 = 2(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z})\}$. Let f be a positive function on $\mathbb{R}^3$. From Lemma 3.3, $(\mathbb{R}^3, \phi, \xi, \eta, g')$ is a trans-Sasakian 3-manifold of type $(\frac{1}{f}, \frac{1}{2f}\xi(f))$, where $g' = fg + (1-f)\eta \otimes \eta$.

Obviously, let $f = f(x, y)$ be a non-constant positive function on $\mathbb{R}^3$; then the non-compact trans-Sasakian manifold $(\mathbb{R}^3, \phi, \xi, \eta, g')$ of type $(\frac{1}{f}, 0)$ is of C_6-class but $\frac{1}{f}$ is not a constant and satisfies $\xi\left(\frac{1}{f}\right) = 0$.

Example 3.2. Let S^3 be the standard unit sphere which is defined by

$$S^3 = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 = 1\}.$$

On S^3 we consider the following three vector fields:

$$e_1 = -z\frac{\partial}{\partial x} + w\frac{\partial}{\partial y} + x\frac{\partial}{\partial z} - y\frac{\partial}{\partial w},$$
$$e_2 = -w\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} + x\frac{\partial}{\partial w},$$
$$e_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - w\frac{\partial}{\partial z} + z\frac{\partial}{\partial w}.$$

On S^3 there exists a standard Sasakian structure $(S^3, \phi, \xi, \eta, g)$ (see [2] and also [17, p. 158, Theorem 3.2]) as follows:

$$\xi = e_1, \quad \eta = g(e_1, \cdot),$$
$$\phi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with respect to the ϕ-basis $\{e_1, e_2, e_3\}$. Let f be a positive function on S^3. From Lemma 3.3, $(S^3, \phi, \xi, \eta, g')$ is a trans-Sasakian 3-manifold of type $\left(\frac{1}{f}, \frac{1}{2f}\xi(f)\right)$, where $g' = fg + (1-f)\eta \otimes \eta$.

Let f be a non-constant positive function on S^3 satisfying the partial differential equation $-y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} - w\frac{\partial f}{\partial z} + z\frac{\partial f}{\partial w} = 0$. For example, let be $f = \ln(x^2 + y^2)$ or $f = \ln(z^2 + w^2)$. Then, the compact trans-Sasakian 3-manifold $(S^3, \phi, \xi, \eta, g')$ of type $(\frac{1}{f}, 0)$ is of C_6-class but $\frac{1}{f}$ is not a constant and satisfies $\xi\left(\frac{1}{f}\right) = 0$.

From Examples 3.1 and 3.2 we have

Proposition 3.1. *A trans-Sasakian* 3*-manifold M of type* $(\alpha, 0)$ *is not necessarily α-Sasakian even when the manifold is compact.*

Example 3.3. Let $\mathbb{H}^3(-1)$ be the usual hyperbolic 3-space which is defined by $\mathbb{H}^3(-1) = \{(x, y, z) \in \mathbb{R}^3 : x > 0\}$ with Riemannian metric $g = \frac{1}{x^2}(dx \otimes dx + dy \otimes dy + dz \otimes dz)$. On $\mathbb{H}^3(-1)$ there exists a Kenmotsu structure given as (see [4]):

$$\xi = -x\frac{\partial}{\partial x}, \quad \eta = g(\xi, \cdot),$$
$$\phi\xi = 0, \quad \phi\frac{\partial}{\partial y} = \frac{\partial}{\partial z}, \quad \phi\frac{\partial}{\partial z} = -\frac{\partial}{\partial y}.$$

Let $f = f(x) \neq x^c$, c a constant, be an arbitrary non-constant positive function defined on $\mathbb{H}^3$. Therefore, from Lemma 3.3, the non-compact trans-Sasakian 3-manifold $(\mathbb{H}^3, \phi, \xi, \eta, g')$ of type $(0, 1 - \frac{x}{2f}\frac{\partial f}{\partial x})$ is of C_5-class satisfying $\nabla\beta = \xi(\beta)\xi$ but neither β-Kenmotsu nor cosymplectic.

Example 3.4. Let S be the usual unit circle $S = \{e^{it}\}$ and T^2 be the torus with the standard Kähler structure (J, g_{T^2}). Then, from [19], the product $(S \times T^2, g)$ with the Riemannian metric $g = dt \otimes dt + g_{T^2}$ is a cosymplectic 3-manifold with $\xi = \frac{\partial}{\partial t}$ and $\eta = dt$. Let $f = f(t) \neq e^{ct}$, c a constant, be an arbitrary non-constant positive function defined on $S \times T^2$. Therefore, from Lemma 3.3, the compact trans-Sasakian 3-manifold $(S \times T^2, \phi, \xi, \eta, g')$ of type $(0, \frac{1}{2f}\frac{\partial f}{\partial t})$ is of C_5-class satisfying $\nabla\beta = \xi(\beta)\xi$ but neither β-Kenmotsu nor cosymplectic.

From Examples 3.1 and 3.4 we have

Proposition 3.2. *A trans-Sasakian* 3*-manifold M of type* $(0, \beta)$ *satisfying* $\nabla\beta = \xi(\beta)\xi$ *is not necessarily β-Kenmotsu or cosymplectic even when M is compact.*

Remark 3.1. The proof of Theorem 1.2 follows from Corollary 3.1 and Propositions 3.1 and 3.2.

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