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REAL HYPERSURFACES IN COMPLEX GRASSMANNIANS OF RANK TWO WITH SEMI-PARALLEL STRUCTURE JACOBI OPERATOR

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Abstract. We prove that there does not exist any real hypersurface in complex Grassmannians of rank two with semi-parallel structure Jacobi operator. With this result, the non-existence of real hypersurfaces in complex Grassmannians of rank two with recurrent structure Jacobi operator is proved.

1. Introduction

Let $\hat{M}^m(c)$ be the compact complex Grassmannian $SU_{m+2}/S(U_2U_m)$ of rank two (resp. noncompact complex Grassmannian $SU_{2,m}/S(U_2U_m)$ of rank two) for $c > 0$ (resp. $c < 0$), where $c = \max \|K\|/8$ is a scaling factor for the Riemannian metric g and K is the sectional curvature for $\hat{M}^m(c)$. It is an irreducible Riemannian symmetric space equipped with a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J.

Let M be a connected, oriented real hypersurface isometrically immersed in $\hat{M}^m(c)$, $m \geq 2$, and N be a unit normal vector field on M. Denote by the same g the Riemannian metric on M. The Reeb vector field ξ is defined by $\xi = -JN$, and we define $\xi_a = -J_aN$, $a \in \{1, 2, 3\}$, where $\{J_1, J_2, J_3\}$ is a canonical local basis of $\mathfrak{J}$. Denote by $D^\perp$ (resp. $\mathfrak{D}^\perp$) the distribution on M spanned by ξ (resp. $\{\xi_1, \xi_2, \xi_3\}$). A real hypersurface M in a Kähler manifold is said to be Hopf if the Reeb vector field is principal, that is, $A\xi = \alpha\xi$.

The study of real hypersurfaces in $\hat{M}^m(c)$ was initiated by Berndt and Suh in [1, 2]. They considered the invariance of $\mathfrak{D}^\perp$ under the shape operator A of Hopf hypersurfaces M in $\hat{M}^m(c)$ and proved a classification of such Hopf hypersurfaces in $\hat{M}^m(c)$.

The structures J and $\mathfrak{J}$ of the ambient space impose several restrictions on the geometry of its real hypersurfaces; for example, there does not exist any semi-parallel real hypersurface in $SU_{m+2}/S(U_2U_m)$ [13], while the non-existence problem of Hopf hypersurfaces in $\hat{M}^m(c)$ with parallel Ricci tensor was studied in [15, 16].

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Besides the shape operator and the Ricci tensor, there are particularly two operators on a real hypersurface M which draw much attention, namely the normal Jacobi operator R_N and the structure Jacobi operator R_ξ. Denote by $\hat{R}$ and R the curvature tensor on $\hat{M}^m(c)$ and that induced on M, respectively. We define $R_N(X) = \hat{R}(X,N)N$ and $R_\xi(X) = R(X,\xi)\xi$ for any vector field X tangent to M.

A $(1,s)$-tensor field P is said to be *semi-parallel* if $R \cdot P = 0$, where the curvature tensor R acts on P as a derivation. More precisely,

$$\begin{aligned}(R(X,Y)\cdot P)&(X_1,\dots,X_s)\\ &= R(X,Y)P(X_1,\dots,X_s) - \sum_{j=1}^{s} P(X_1,\dots,R(X,Y)X_j,\dots,X_s).\end{aligned}$$

The tensor field P is said to be *recurrent* if there exists a 1-form ω on M such that

$$(\nabla_X P)(X_1,\dots,X_s) = \omega(X)P(X_1,\dots,X_s).$$

Clearly, a vanishing ω leads to parallelism of P.

Recently, we proved the non-existence of real hypersurfaces in $SU_{m+2}/S(U_2U_m)$, $m \geq 3$, with pseudo-parallel normal Jacobi operator [5]. On the other hand, related to the structure Jacobi operator R_ξ, Jeong et al. proved that there does not exist any Hopf hypersurface in $SU_{m+2}/S(U_2U_m)$, $m \geq 3$, with parallel structure Jacobi operator [10]. Also, the non-existence of Hopf hypersurfaces with $D^\perp$-parallel structure Jacobi operator is obtained under certain conditions [9]. Jeong et al. considered Reeb-parallel structure Jacobi operator and proved the following:

Theorem 1.1 ([8]). *Let M be a Hopf hypersurface in $SU_{m+2}/S(U_2U_m)$, $m \geq 3$, with Reeb parallel structure Jacobi operator. If the principal curvature of the Reeb vector field ξ on M is non-vanishing and constant along the direction of the Reeb vector field ξ, then M is an open part of a tube around a totally geodesic $SU_{m+1}/S(U_2U_{m-1})$ in $SU_{m+2}/S(U_2U_m)$ with radius $r \in (0, \frac{\pi}{4\sqrt{2}}) \cup (\frac{\pi}{4\sqrt{2}}, \frac{\pi}{\sqrt{8}})$.*

We say that a real hypersurface M has commuting structure Jacobi operator if it commutes with any other Jacobi operator defined on M, that is, $R_\xi \cdot R_X = R_X \cdot R_\xi$ for any X tangent to M. Machado et al. proved the non-existence of Hopf real hypersurfaces in $SU_{m+2}/S(U_2U_m)$, $m \geq 3$, with commuting structure Jacobi operator under certain conditions [14]. They also classified real hypersurfaces in $SU_{m+2}/S(U_2U_m)$, $m \geq 3$, with $\hat{R}_N \cdot R_\xi = R_\xi \cdot \hat{R}_N$. In [11], Jeong et al. proved the following:

Theorem 1.2 ([11]). *There does not exist any Hopf hypersurface in $SU_{m+2}/S(U_2U_m)$, $m \geq 3$, with recurrent structure Jacobi operator if the distribution $\mathfrak{D}$ or $\mathfrak{D}^\perp$-component of the Reeb vector field is invariant under the shape operator.*

On the other hand, under certain restrictions, Hwang et al. obtained the following non-existence result.

Theorem 1.3 ([6]). *There does not exist any Hopf hypersurface in $SU_{m+2}/S(U_2U_m)$, $m \geq 3$, with semi-parallel structure Jacobi operator if the smooth function $\alpha = g(A\xi,\xi)$ is constant along the direction of ξ.*

Motivated from the above studies, a natural question arises:

Problem 1.1. *Does there exist a real hypersurface in $\hat{M}^m(c)$ with parallel, recurrent or semi-parallel structure Jacobi operator?*

In the present paper we first prove the following:

Theorem 1.4. *There does not exist any connected real hypersurface in $\hat{M}^m(c)$, $m \geq 3$, with semi-parallel structure Jacobi operator.*

The non-existence of real hypersurfaces with semi-parallel structure Jacobi operator in a non flat complex space form has been proved in [3, 7]. We also remark that Theorem 1.4 holds for non-Hopf real hypersurfaces as well and no further conditions are imposed. By a result in [4], we learn that if a tensor field is recurrent, it is always semi-parallel. Hence, as a corollary we obtain the following:

Corollary 1.1. *There does not exist any real hypersurface in $\hat{M}^m(c)$, $m \geq 3$, with parallel or recurrent structure Jacobi operator.*

2. Preliminaries

In this section, we recall some fundamental identities for real hypersurfaces in complex Grassmannians of rank two, which have been proven in [1, 2, 12, 13].

Let M be a connected, oriented real hypersurface isometrically immersed in $\hat{M}^m(c)$, $m \geq 3$. The almost contact metric 3-structure $(\phi_a, \xi_a, \eta_a, g)$ on M is given by

$$J_aX = \phi_aX + \eta_a(X)N, \quad J_aN = -\xi_a, \quad \eta_a(X) = g(X, \xi_a),$$

for any $X \in TM$, where $\{J_1, J_2, J_3\}$ is a canonical local basis of $\mathfrak{J}$ on $\hat{M}^m(c)$. It follows that

$$\begin{aligned} \phi_a\phi_{a+1} - \xi_a \otimes \eta_{a+1} &= \phi_{a+2} \\ \phi_a\xi_{a+1} = \xi_{a+2} &= -\phi_{a+1}\xi_a \end{aligned}$$

for $a \in \{1, 2, 3\}$. The indices in the preceding equations are taken modulo three.

The Kähler structure J induces on M an almost contact metric structure (ϕ, ξ, η, g), namely,

$$JX = \phi X + \eta(X)N, \quad JN = -\xi, \quad \eta(X) = g(X, \xi).$$

Let $\mathfrak{D}^\perp = \mathfrak{J}TM^\perp$, and $\mathfrak{D}$ its orthogonal complement in TM. We define a local $(1,1)$-tensor field θ_a on M by

$$\theta_a := \phi_a\phi - \xi_a \otimes \eta.$$

Denote by ∇ the Levi-Civita connection on M. Then there exist local 1-forms q_a, $a \in \{1, 2, 3\}$, such that

$$\left.\begin{aligned} \nabla_X\xi &= \phi AX \\ \nabla_X\xi_a &= \phi_aAX + q_{a+2}(X)\xi_{a+1} - q_{a+1}(X)\xi_{a+2} \\ \nabla_X\phi\xi_a &= \theta_aAX + \eta_a(\xi)AX + q_{a+2}(X)\phi\xi_{a+1} - q_{a+1}(X)\phi\xi_{a+2}. \end{aligned}\right\} \tag{2.1}$$

The following identities are known.

Lemma 2.1 ([12]).

(a) θ_a *is symmetric,*
(b) $\phi\xi_a = \phi_a\xi$,
(c) $\theta_a\xi = -\xi_a, \quad \theta_a\xi_a = -\xi, \quad \theta_a\phi\xi_a = \eta(\xi_a)\phi\xi_a$,
(d) $\theta_a\xi_{a+1} = \phi\xi_{a+2} = -\theta_{a+1}\xi_a$,
(e) $-\theta_a\phi\xi_{a+1} + \eta(\xi_{a+1})\phi\xi_a = \xi_{a+2} = \theta_{a+1}\phi\xi_a - \eta(\xi_a)\phi\xi_{a+1}$.

Lemma 2.2 ([12]). *If* $\xi \in \mathfrak{D}$ *everywhere, then* $A\phi\xi_a = 0$ *for* $a \in \{1,2,3\}$.

For each $x \in M$, we define a subspace $\mathcal{H}^\perp$ of T_xM by

$$\mathcal{H}^\perp := \operatorname{span}\{\xi, \xi_1, \xi_2, \xi_3, \phi\xi_1, \phi\xi_2, \phi\xi_3\}.$$

Let $\mathcal{H}$ be the orthogonal complement of $\mathcal{H}^\perp$ in T_xM. Then $\dim\mathcal{H} = 4m-4$ (resp. $\dim\mathcal{H} = 4m-8$) when $\xi \in \mathfrak{D}^\perp$ (resp. $\xi \notin \mathfrak{D}^\perp$). Moreover, $\theta_{a|\mathcal{H}}$ has two eigenvalues: 1 and -1. Denote by $\mathcal{H}_a(\varepsilon)$ the eigenspace corresponding to the eigenvalue ε of $\theta_{a|\mathcal{H}}$. Then $\dim\mathcal{H}_a(1) = \dim\mathcal{H}_a(-1)$ is even, and

$$\begin{aligned}\phi\mathcal{H}_a(\varepsilon) &= \phi_a\mathcal{H}_a(\varepsilon) = \theta_a\mathcal{H}_a(\varepsilon) = \mathcal{H}_a(\varepsilon)\\ \phi_b\mathcal{H}_a(\varepsilon) &= \theta_b\mathcal{H}_a(\varepsilon) = \mathcal{H}_a(-\varepsilon), \quad (a \neq b).\end{aligned}$$

We define the tensor fields θ, $\phi^\perp$, $\xi^\perp$, and $\eta^\perp$ on M as follows:

$$\theta := \sum_{a=1}^{3}\eta_a(\xi)\theta_a, \quad \phi^\perp := \sum_{a=1}^{3}\eta_a(\xi)\phi_a, \quad \xi^\perp := \sum_{a=1}^{3}\eta_a(\xi)\xi_a, \quad \eta^\perp := \sum_{a=1}^{3}\eta_a(\xi)\eta_a.$$

Then for each $x \in M$ with $\xi^\perp \neq 0$, $\theta_{|\mathcal{H}}$ has two eigenvalues $\varepsilon\|\xi^\perp\|$, $\varepsilon \in \{1,-1\}$. Let $\mathcal{H}(\varepsilon)$ be the eigenspace of $\theta_{|\mathcal{H}}$ corresponding to $\varepsilon\|\xi^\perp\|$. Then

(a) $\phi\mathcal{H}(\varepsilon) = \phi^\perp\mathcal{H}(\varepsilon) = \mathcal{H}(\varepsilon)$,
(b) $\dim\mathcal{H}(1) = \dim\mathcal{H}(-1)$ is even.

Moreover, we can take a canonical local basis of $\mathfrak{J}$ on a neighborhood $G \subset M$ of such a point x such that

$$\begin{aligned}&\xi_1 = \frac{\xi^\perp}{\|\xi^\perp\|}, \quad 0 < \eta_1(\xi) = \|\xi^\perp\| \leq 1, \quad \eta_2(\xi) = \eta_3(\xi) = 0\\ &\mathcal{H}(\varepsilon) = \mathcal{H}_1(\varepsilon), \quad \theta = \eta_1(\xi)\theta_1, \quad \phi^\perp = \eta_1(\xi)\phi_1, \quad \eta^\perp = \eta_1(\xi)\eta_1.\end{aligned}$$

In particular, if $\|\xi^\perp\| = 1$ at x, then

$$\xi_1 = \xi = \xi^\perp, \quad \xi_2 = \theta\xi_2 = \phi\xi_3, \quad \xi_3 = \theta\xi_3 = -\phi\xi_2.$$

Throughout this paper, we always consider such a local orthonormal frame $\{\xi_1, \xi_2, \xi_3\}$ on $\mathfrak{D}^\perp$ under these situations.

A straightforward calculation gives

$$(\nabla_X\theta)Y = \eta^\perp(\phi Y)AX - g(AX,Y)\phi\xi^\perp + 2\sum_{a=1}^{3}\eta_a(\phi AX)\theta_aY. \tag{2.2}$$

The equations of Gauss and Codazzi are respectively given by

$$\begin{aligned}R(X,Y)Z = {}& g(AY,Z)AX - g(AX,Z)AY + c\{g(Y,Z)X - g(X,Z)Y \\ &+ g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\} \\ &+ c\sum_{a=1}^{3}\{g(\phi_a Y,Z)\phi_a X - g(\phi_a X,Z)\phi_a Y - 2g(\phi_a X,Y)\phi_a Z \\ &\quad + g(\theta_a Y,Z)\theta_a X - g(\theta_a X,Z)\theta_a Y\},\end{aligned}$$

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X = {}& c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi\} \\ &+ c\sum_{a=1}^{3}\{\eta_a(X)\phi_a Y - \eta_a(Y)\phi_a X - 2g(\phi_a X,Y)\xi_a \\ &\quad + \eta_a(\phi X)\theta_a Y - \eta_a(\phi Y)\theta_a X\}.\end{aligned}$$

As M is a real hypersurface in $\hat{M}^m(c)$, by the Gauss equation we have

$$\begin{aligned}R_\xi X = {}& \alpha AY - \eta(AY)A\xi + c\{X - \eta(X)\xi - \theta X\} \\ &- c\sum_{a=1}^{3}\{\eta_a(X)\xi_a + 3\eta_a(\phi X)\phi\xi_a\}.\end{aligned} \tag{2.3}$$

We end this section with the following general result.

Theorem 2.1. *Let M be an almost contact metric manifold. The structure Jacobi operator R_ξ is semi-parallel if and only if $R_\xi = 0$.*

Proof. Suppose the structure Jacobi operator is semi-parallel. Then

$$R(X,Y)R_\xi Z - R_\xi R(X,Y)Z = (R(X,Y)\cdot R_\xi)Z = 0$$

for any $X, Y, Z \in TM$. In particular, for $Y = Z = \xi$, we obtain $R_\xi^2 X = 0$. Since R_ξ is self-adjoint, $R_\xi = 0$. The converse is trivial. □

3. Proof of Theorem 1.4

By virtue of Theorem 2.1, it suffices to show that the structure Jacobi operator cannot be identically zero. Suppose to the contrary that $R_\xi = 0$. Then by (2.3), we have

$$\begin{aligned}&\alpha AY - \eta(AY)A\xi + c\{Y - \eta(Y)\xi - \theta Y\} \\ &\qquad - c\sum_{a=1}^{3}\{\eta_a(Y)\xi_a + 3\eta_a(\phi Y)\phi\xi_a\} = 0.\end{aligned} \tag{3.1}$$

Claim 3.1. *$\xi \notin \mathfrak{D}$ on an open dense subset of M.*

Proof. Suppose $\xi \in \mathfrak{D}$ on an open subset G of M. For each $x \in G$, we have $\theta = 0$. It follows from Lemma 2.2 that $\phi\xi_1 = 0$ after putting $Y = \phi\xi_1$ in (3.1), a contradiction. Hence we obtain the claim. □

Consider a point $x \in M$ on which $\xi \notin \mathfrak{D}$ on a neighborhood G of x in M. We define subspaces $\mathcal{F}$, $\mathcal{F}(1)$ and $\mathcal{F}(-1)$ of T_xM by

$$\mathcal{F} = \{X \in \mathcal{H} : \eta(AX) = 0\}, \quad \mathcal{F}(1) = \mathcal{F} \cap \mathcal{H}(1), \quad \mathcal{F}(-1) = \mathcal{F} \cap \mathcal{H}(-1).$$

It is clear that

$$AY = \lambda_\varepsilon Y, \quad \alpha\lambda_\varepsilon + c(1 - \varepsilon\|\xi^\perp\|) = 0$$

for any $Y \in \mathcal{F}(\varepsilon)$ and $\varepsilon \in \{1, -1\}$. By (3.1), we have

$$\begin{aligned}
&(X\alpha)AY + \alpha(\nabla_X A)Y - \eta(AY)\big\{(\nabla_X A)\xi + A\nabla_X\xi\big\} \\
&\quad - \big\{g(\nabla_X A)Y, \xi) + g(AY, \nabla_X\xi)\big\}A\xi + c\{-g(\nabla_X\xi, Y) - \eta(Y)\nabla_X\xi - (\nabla_X\theta)Y\} \\
&\quad - c\sum_{a=1}^{3}\{g(\nabla_X\xi_a, Y)\xi_a + \eta_a(Y)\nabla_X\xi_a - 3g(\nabla_X\phi\xi_a, Y)\phi\xi_a + 3\eta_a(\phi Y)\nabla_X\phi\xi_a\} = 0
\end{aligned}$$

for any $X, Y \in T_xM$. By using (2.1) and (2.2), the preceding equation becomes

$$\begin{aligned}
&(X\alpha)AY + \alpha(\nabla_X A)Y - \eta(AY)\big\{(\nabla_X A)\xi + A\phi AX\big\} \\
&\qquad - \big\{g(\nabla_X A)Y, \xi) + g(A\phi AX, Y)\big\}A\xi \\
&\qquad + c\{-g(\phi AX, Y) - \eta(Y)\phi AX + 4g(AX, Y)\phi\xi^\perp - 4\eta^\perp(\phi Y)AX\} \\
&\qquad + c\sum_{a=1}^{3}\{-g(\phi_a AX, Y)\xi_a - \eta_a(Y)\phi_a AX \\
&\qquad\qquad + 3g(\theta_a AX, Y)\phi\xi_a - 3\eta_a(\phi Y)\theta_a AX - 2\eta_a(\phi AX)\theta_a Y\} = 0.
\end{aligned}$$

By the preceding equation and the Codazzi equation, we have

$$\begin{aligned}
&(X\alpha)AY - (Y\alpha)AX - \eta(AY)\big\{(\nabla_X A)\xi + A\phi AX\big\} + \eta(AX)\big\{(\nabla_Y A)\xi + A\phi AY\big\} \\
&\quad + c\big\{\eta(X)(\phi AY + \alpha\phi Y) - \eta(Y)(\phi AX + \alpha\phi X) - 4\eta^\perp(\phi Y)AX + 4\eta^\perp(\phi X)AY\big\} \\
&\quad + 2g(c(\phi + \phi^\perp)X - A\phi AX, Y)A\xi - cg(2\alpha\phi X + (\phi A + A\phi)X, Y)\xi \\
&\quad + c\sum_{a=1}^{3}\big\{\eta_a(X)(\phi_a AY + \alpha\phi_a Y) - \eta_a(Y)(\phi_a AX + \alpha\phi_a X) \\
&\qquad - \alpha\eta_a(\phi Y)\theta_a X + \alpha\eta_a(\phi X)\theta_a Y - 3\eta_a(\phi Y)\theta_a AX + 3\eta_a(\phi X)\theta_a AY \\
&\qquad + 2\eta_a(\phi AY)\theta_a X - 2\eta_a(\phi AX)\theta_a Y - 2\eta_a(X)\eta_a(\phi Y)A\xi + 2\eta_a(Y)\eta_a(\phi X)A\xi \\
&\qquad - g(2\alpha\phi_a X + (\phi_a A + A\phi_a)X, Y)\xi_a + 3g((\theta_a A - A\theta_a)X, Y)\phi\xi_a\big\} = 0
\end{aligned} \tag{3.2}$$

for any $X, Y \in T_xM$. By putting $X, Y \in \mathcal{F}$ in (3.2), we have

$$\begin{aligned}
&(X\alpha)AY - (Y\alpha)AX \\
&\qquad + 2g(c(\phi + \phi^\perp)X - A\phi AX, Y)A\xi - cg(2\alpha\phi X + (\phi A + A\phi)X, Y)\xi \\
&\qquad + c\sum_{a=1}^{3}\big\{ - g(2\alpha\phi_a X + (\phi_a A + A\phi_a)X, Y)\xi_a + 3g((\theta_a A - A\theta_a)X, Y)\phi\xi_a\big\} = 0.
\end{aligned}$$

Since the first two terms are in $\mathcal{H}$ and the remaining are in $\mathcal{H}^\perp$, we have

$$\begin{aligned}2g(c(\phi+\phi^\perp)X - A\phi AX, Y)A\xi - cg(2\alpha\phi X + (\phi A + A\phi)X, Y)\xi \\ + c\sum_{a=1}^{3}\{-g(2\alpha\phi_a X + (\phi_a A + A\phi_a)X, Y)\xi_a \\ + 3g((\theta_a A - A\theta_a)X, Y)\phi\xi_a\} = 0\end{aligned} \tag{3.3}$$

for any $X, Y \in \mathcal{F}$. For any $\varepsilon \in \{1, -1\}$, we can further deduce from (3.3) that

$$\begin{aligned}0 &= 2g(\phi X, Y)\{c(1-\varepsilon\|\xi^\perp\|) - \lambda_\varepsilon^2\}A\xi + g(\phi X, Y)c(2\alpha + 2\lambda_\varepsilon)\left\{-\xi + \frac{\varepsilon}{\|\xi^\perp\|}\xi^\perp\right\} \\ &= g(\phi X, Y)(\alpha+\lambda_\varepsilon)\left\{-\lambda_\varepsilon A\xi - c\left(\xi - \frac{\varepsilon}{\|\xi^\perp\|}\xi^\perp\right)\right\}\end{aligned} \tag{3.4}$$

for any $X, Y \in \mathcal{F}(\varepsilon)$; and

$$0 = \sum_{a=1}^{3}\{-(2\alpha + \lambda_\varepsilon + \lambda_{-\varepsilon})g(\phi_a X, Y)\xi_a + 3(\lambda_\varepsilon - \lambda_{-\varepsilon})g(\theta_a X, Y)\phi\xi_a\} \tag{3.5}$$

for any $X \in \mathcal{F}(\varepsilon)$ and $Y \in \mathcal{F}(-\varepsilon)$.

Claim 3.2. $\dim \mathcal{H} \geq 8$.

Proof. Suppose $\dim \mathcal{H} = 4$. Since $\dim M = 4m-1 \geq 11$, we have $\xi \notin \mathfrak{D}^\perp$ or $0 < \|\xi^\perp\| < 1$. Take a unit vector $V \in \mathcal{F}(1)$ such that $\mathcal{H}(1) = \mathbb{R}V \oplus \mathbb{R}\phi_1 V$ and $\mathcal{H}(-1) = \mathbb{R}\phi_2 V \oplus \mathbb{R}\phi_3 V$.

Substituting $X = V$ and $Y = \phi_2 V$ in (3.5), we obtain

$$0 = -(2\alpha + \lambda_1 + \lambda_{-1})\xi_2 - 3(\lambda_1 - \lambda_{-1})\phi\xi_3$$

for any $Y \in \mathcal{F}(1)$. Since $\{\xi_a, \phi\xi_a\}_{a\in\{1,2,3\}}$ is linearly independent, $-2c\alpha^{-1}\|\xi^\perp\| = \lambda_1 - \lambda_{-1} = 0$, a contradiction. Hence, the claim is obtained. □

According to Claim 3.2, there exists $X \in \mathcal{F}(\varepsilon)$ such that $X \perp \phi A\xi$. Taking such a vector X and $Y = \phi X$ in (3.4), we obtain

$$(\alpha + \lambda_\varepsilon)\left\{\lambda_\varepsilon A\xi + c\left(\xi - \frac{\varepsilon}{\|\xi^\perp\|}\xi^\perp\right)\right\} = 0 \tag{3.6}$$

for any $\varepsilon \in \{1, -1\}$.

Claim 3.3. $\|\xi^\perp\| = 1$ *on* M.

Proof. Suppose $0 < \|\xi^\perp\| < 1$ on the open subset G of M. It is clear that $\alpha + \lambda_1$ and $\alpha + \lambda_{-1}$ cannot be both zero as $\lambda_1 - \lambda_{-1} = -2c\alpha^{-1}\|\xi^\perp\| \neq 0$. Fix $\varepsilon \in \{1, -1\}$ such that $\alpha + \lambda_\varepsilon \neq 0$. It follows from (3.6) that

$$\lambda_\varepsilon A\xi + c\left(\xi - \frac{\varepsilon}{\|\xi^\perp\|}\xi^\perp\right) = 0.$$

This implies that $A\xi \perp \mathcal{H}$, so $\mathcal{F}(1) = \mathcal{H}(1)$ and $F(-1) = \mathcal{H}(-1)$. Taking $X \in \mathcal{H}(1)$ and $Y = \phi_2 X$ in (3.5), we can obtain a contradiction by using a similar method as in the proof of Claim 3.2. Hence, $\|\xi^\perp\| = 1$ at the point x. By the connectedness of M and the continuity of $\|\xi^\perp\|$, we conclude that $\|\xi^\perp\| = 1$ on M. $\square$

Since $\|\xi^\perp\| = 1$ on M or $\xi \in \mathfrak{D}^\perp$ everywhere, we have $\lambda_1 = 0$ and $\lambda_{-1} = -2c/\alpha$ ($= \lambda$, for simplicity). Moreover, we have

$$-\sum_{a=1}^{3} \eta_a(\phi Y)\phi\xi_1 = \sum_{a=1}^{3} \eta_a(Y)\xi_a - \eta(Y)\xi. \tag{3.7}$$

It follows from (3.1) and (3.7) that

$$\alpha AY - \eta(AY)A\xi + c\{Y - 4\eta(Y)\xi - \theta Y\} + 2c\sum_{a=1}^{3} \eta_a(Y)\xi_a = 0. \tag{3.8}$$

On the other hand, we have

$$\begin{aligned}\sum_{a=1}^{3}&\{g(Y, \nabla_X \phi\xi_a)\phi\xi_a - \eta_a(\phi Y)\nabla_X \phi\xi_a\}\\ &= \sum_{a=1}^{3}\{g(Y, \nabla_X \xi_a)\xi_a + \eta_a(Y)\nabla_X \xi_a\} - g(Y, \nabla_X \xi)\xi - \eta(Y)\nabla_X \xi.\end{aligned}$$

By using (2.1), we have

$$\begin{aligned}\sum_{a=1}^{3}&\{g(Y, \theta_a AX)\phi\xi_a - \eta_a(\phi Y)\theta_a AX\}\\ &= \sum_{a=1}^{3}\{g(Y, \phi_a AX)\xi_a + \eta_a(Y)\phi_a AX\} - g(Y, \phi AX)\xi - \eta(Y)\phi AX.\end{aligned} \tag{3.9}$$

Claim 3.4. $\lambda A\xi + 2c\xi \neq 0$.

Proof. Suppose $\lambda A\xi + 2c\xi = 0$. Then $A\xi = \alpha\xi$ and $\alpha\lambda + 2c = 0$. It follows from (3.8) that

$$AY = -\frac{c}{\alpha}(Y - \theta Y) + \frac{\alpha^2 + 4c}{\alpha}\eta(Y)\,\xi - \frac{2c}{\alpha}\sum_{a=1}^{3} \eta_a(Y)\,\xi_a.$$

By [12, Theorem 6.1], we obtain

$$\alpha^2 + 4c = 0 \tag{3.10}$$

and so $c < 0$. Furthermore, we have either

$$\frac{-c}{\alpha} = \frac{\sqrt{-2c}\tanh(\sqrt{-2c}r)}{2}, \quad \frac{-2c}{\alpha} = \sqrt{-2c}\coth(\sqrt{-2c}r), \quad r > 0$$

or

$$\frac{-c}{\alpha} = \frac{\sqrt{-2c}}{2}, \quad \frac{-2c}{\alpha} = \sqrt{-2c}.$$

However, both cases contradict (3.10). Accordingly, we obtain the claim. $\square$

By using Claim 3.4, (3.6) and (3.8), there exists a unit vector field U tangent to $\mathcal{H}(-1) \oplus (\mathfrak{D}^{\perp} \ominus \mathbb{R}\xi)$ and functions τ, β ($\beta \neq 0$) on M such that

$$\left.\begin{aligned} &A\xi = \alpha\xi + \beta U \\ &AU = \beta\xi + \tau U \\ &AY = \lambda Y \quad\ (Y \in \mathcal{F}(-1)) \\ &AX = 0 \qquad (X \in \mathcal{H}(1)) \\ &\alpha\tau + \alpha^2 = \beta^2 \\ &\lambda + \alpha = 0, \quad \alpha^2 = 2c. \end{aligned}\right\} \tag{3.11}$$

By using (3.9) and (3.11), (3.2) is simplified as

$$\begin{aligned} &- \eta(AY)\big\{\alpha\phi AX + (X\beta)U + \beta\nabla_X U\big\} + \eta(AX)\big\{\alpha\phi AY + (Y\beta)U + \beta\nabla_Y U\big\} \\ &\quad + c\big\{\eta(X)(4\phi AY + \alpha\phi Y) - \eta(Y)(4\phi AX + \alpha\phi X)\big\} \\ &\quad + 2g(c(\phi + \phi^{\perp})X - A\phi AX, Y)A\xi - cg(2\alpha\phi X + 4(\phi A + A\phi)X, Y)\xi \\ &\quad + c\sum_{a=1}^{3}\big\{\eta_a(X)(-2\phi_a AY + \alpha\phi_a Y) - \eta_a(Y)(-2\phi_a AX + \alpha\phi_a X) \\ &\qquad - \alpha\eta_a(\phi Y)\theta_a X + \alpha\eta_a(\phi X)\theta_a Y + 2\eta_a(\phi AY)\theta_a X - 2\eta_a(\phi AX)\theta_a Y \\ &\qquad - 2\eta_a(X)\eta_a(\phi Y)A\xi + 2\eta_a(Y)\eta_a(\phi X)A\xi \\ &\qquad + g(-2\alpha\phi_a X + (\phi_a A + A\phi_a)X, Y)\xi_a\big\} = 0. \end{aligned} \tag{3.12}$$

On the other hand, by the Codazzi equation, we obtain

$$cg((\phi + \phi^{\perp})Y, Z) - 2c\sum_{a=1}^{3}\eta_a(Y)\eta_a(\phi Z) = g((\nabla_\xi A)Y - (\nabla_Y A)\xi, Z). \tag{3.13}$$

Substituting $Z = \xi$ in (3.13) gives

$$(\xi\beta)g(U, Y) + \beta g(\nabla_\xi U, Y) + 4\beta\alpha g(\phi U, Y).$$

Letting $Y = U$ in the preceding equation gives

$$\xi\beta = 0 \tag{3.14}$$

$$\nabla_\xi U + 4\alpha\phi U = 0 \tag{3.15}$$

Next, with the help of (3.11), after putting $Y \in \mathcal{H}(1)$ and $Z = U$ in (3.13) gives

$$Y\beta = 0 \tag{3.16}$$

for any $Y \in \mathcal{H}(1)$. By putting $X = \xi$ and $Y \perp \xi$ in (3.12), we have

$$\begin{aligned} &Y\beta + (\alpha^2 + 2\beta^2)g(Y, \phi U) + 3cg(Y, \phi U - \phi^{\perp}U) + \sum_{a=1}^{3}\{-2\alpha\eta_a(\phi AY)\eta_a(U) \\ &\quad - \alpha\beta g(\phi_a Y, U)\eta_a(U) - \alpha\beta g(\theta_a Y, U)\eta_a(\phi U) - 2c\alpha\eta_a(U)\eta_a(\phi U)\} = 0 \end{aligned} \tag{3.17}$$

for any $Y \perp \xi$. In particular, for $Y \in \mathcal{H}(1)$, with the help of (3.16) we obtain

$$0 = -\sum_{a=1}^{3}\{g(\phi_a Y, U)\eta_a(U) - g(\theta_a Y, U)\eta_a(\phi U)\} = 2\sum_{a=1}^{3}\eta_a(U)g(\phi_a U, Y)$$

for any $Y \in \mathcal{H}(1)$.

Denote by U^- the $\mathcal{H}(-1)$-component of U. If U is tangent to $\mathfrak{D}^\perp$ on an open subset G of M. Then for each $x \in G$, $\mathcal{F}(-1) = \mathcal{H}(-1)$ and so $A\mathfrak{D}^\perp \subset \mathfrak{D}^\perp$. By virtue of [12, Theorem 3.6], ξ is principal on G. This contradicts Claim 3.4. Hence, we assume that $U^- \neq 0$. By putting $Y = \phi_b U^-$, $b \in \{2,3\}$ in the preceding equation, we obtain $\eta_2(U) = \eta_3(U) = 0$. Consequently, $U = U^- \in \mathcal{H}(-1)$ and $\phi U = \phi^\perp U$. These, together with (3.14) and (3.17), give

$$Y\beta = -(\alpha^2 + 2\beta^2)g(Y, \phi U)$$

for any vector field Y tangent to M. It follows that

$$(XY - \nabla_X Y)\beta = (\alpha^2 + 2\beta^2)\{4\beta g(X, \phi U)g(Y, \phi U) - g(Y, \nabla_X \phi U)\}.$$

Hence

$$g(Y, \nabla_X \phi U) - g(X, \nabla_Y U) = 0.$$

By virtue of (3.11) and (3.15), after substituting $X = \xi$ and $Y = U$ in the preceding equation we get $4\alpha + \tau = 0$. But then $\beta^2 = \alpha\tau + \alpha^2 = -3\alpha^2$, a contradiction. This completes the proof.

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