REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 61, No. 1, 2020, Pages 169–196
Published online: June 30, 2020
https://doi.org/10.33044/revuma.v61n1a11

LOCALIZATION OPERATORS AND SCALOGRAM ASSOCIATED WITH THE GENERALIZED CONTINUOUS WAVELET TRANSFORM ON $\mathbb{R}^d$ FOR THE HECKMAN–OPDAM THEORY

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ABSTRACT. We consider the generalized wavelet transform Φ_h^W on $\mathbb{R}^d$ for the Heckman–Opdam theory. We study the localization operators associated with Φ_h^W; in particular, we prove that they are in the Schatten–von Neumann class. Next we introduce some results on the scalogram for this transform.

1. Introduction

We consider the differential-difference operators T_j, $j = 1, 2, \dots, d$, associated with a root system $\mathcal{R}$ and a multiplicity function k, introduced by Cherednik in [5], and called the Cherednik operators in the literature. These operators were helpful for the extension and simplification of the theory of Heckman–Opdam, which is a generalization of the harmonic analysis on the symmetric spaces G/K ([33, 34, 37]).

The Cherednik and Heckman–Opdam theories are based on the Opdam–Cherednik hypergeometric function $G_\lambda, \lambda \in \mathbb{C}^d$, which is the unique analytic solution of the system

$$T_j u(x) = -i\lambda_j u(x), \quad j = 1, 2, \dots, d,$$

satisfying the normalizing condition $u(0) = 1$, and the Heckman–Opdam kernel F_λ, $\lambda \in \mathbb{C}^d$, which is defined by

$$\forall\, x \in \mathbb{R}^d, \quad F_\lambda(x) = \frac{1}{|W|} \sum_{w \in W} G_\lambda(wx),$$

where W is the Weyl group associated with the root system $\mathcal{R}$ ([33, 34]).

With the kernel G_λ Opdam and Cherednik have defined in [5, 33] the Opdam–Cherednik transform $\mathcal{H}$ and have used the kernel F_λ to define the hypergeometric Fourier transform $\mathcal{H}^W$ on spaces of W-invariant functions, and have established some of their properties (see also [34]).

Very recently, many authors have been investigating the behavior of the hypergeometric Fourier transform in several problems already studied for the Fourier

2010 *Mathematics Subject Classification.* Primary 81S30, 44A05; Secondary 42B10, 42B15, 94A12.

Key words and phrases. Cherednik operators; generalized wavelet transform; localization operators; Schatten–von Neumann class; scalogram.

transform; for instance, wavelet theory [19], real Paley–Wiener theorems [23, 26], Roe's theorem [25], boundedness and compactness of two-wavelet multipliers [28], uncertainty principles [21, 24, 27], Ramanujan's master theorem [32], the heat equation [37], and so on.

In the classical setting, the notion of wavelets was first introduced by Morlet, a French petroleum engineer at ELF-Aquitaine, in connection with his study of seismic traces. The mathematical foundations were given by Grossmann and Morlet in [18]. The harmonic analyst Meyer and many other mathematicians became aware of this theory and they recognized many classical results inside it (see [6, 22, 30]). Classical wavelets have wide applications, ranging from signal analysis in geophysics and acoustics to quantum theory and pure mathematics (see [11, 16, 20] and the references therein).

Recently in [19] Hassini et al., with the aid of the harmonic analysis associated to the Heckman–Opdam theory, have defined and studied the generalized wavelet transform, and they have proved Plancherel's and inversion formulas for this transform.

One of the applications of the continuous wavelet transform is time-frequency analysis, which is a mathematical tool to define a restriction of functions to a region in the time-frequency plane, that is compatible with the uncertainty principle, and to extract time-frequency features. In this sense they have been introduced and studied by Daubechies [8, 9, 10] and Ramanathan and Topiwala [35], and they are now extensively investigated as an important mathematical tool in signal analysis and other applications [7, 12, 13, 17, 35, 41].

As the harmonic analysis associated to the Heckman–Opdam theory has known remarkable development, it is a natural question to ask whether there exists the equivalent of the theory of time-frequency analysis for the generalized wavelet transform introduced in [19].

In this paper we study only two subjects of the time-frequency analysis associated with the generalized wavelet transform. The first subject is the theory of localization operators. This theory has found many applications to time-frequency analysis, the theory of differential equations, and quantum mechanics. Many works have been written on localization operators from these points of view; we refer in particular to the papers of Balazs et al. [3, 4]. The second subject is the scalogram. We note that the scalogram has many applications; for example in [2], the authors used Morlet wavelet scalograms to detect a previously unknown coordinated contractility behavior of the atrium during ventricular fibrillation, a phenomenon which is not captured in a normal electrocardiogram. Other applications can also be found in [39], where the authors applied the scalogram to biomedical signals to detect their short-lived temporal interactions. We mention that scalograms have been studied in the context of the generalized wavelet transforms by many authors; see for example [15, 29].

The remainder of the paper is organized as follows. In Section 2 we recall the main results about the harmonic analysis associated with the Cherednik operators. Section 3 is devoted to the study of boundedness and compactness properties of the localization operators for the generalized continuous wavelet transform Φ_h^W; we show that they are in the Schatten–von Neumann class. We also give a trace formula. In the last section we study the eigenvalues and eigenfunctions of the time-frequency localization operator. Next we study the scalogram associated with the generalized continuous wavelet transform.

2. Preliminaries

This section gives an introduction to the theory of Cherednik operators, hypergeometric Fourier transform, and hypergeometric convolution. The main references are [5, 21, 31, 33, 34, 37, 40].

2.1. **Reflection groups, root systems, and multiplicity functions.** The basic ingredient in the theory of Cherednik operators are root systems and finite reflection groups, acting on $\mathbb{R}^d$ with the standard euclidean scalar product $\langle \cdot, \cdot \rangle$ for which the basis $\{e_i,\ i = 1, \dots, d\}$ is orthogonal, and $\|x\| = \sqrt{\langle x, x\rangle}$. On $\mathbb{C}^d$, $\|\cdot\|$ denotes also the standard Hermitian norm, while $\langle z, w\rangle = \sum_{j=1}^{d} z_j \overline{w}_j$.

For $\alpha \in \mathbb{R}^d\backslash\{0\}$, let $\alpha^\vee = \frac{2}{\|\alpha\|}\alpha$ be the coroot associated to α and let

$$r_\alpha(x) = x - \langle \alpha^\vee, x\rangle \alpha$$

be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α.

A finite set $\mathcal{R} \subset \mathbb{R}^d\backslash\{0\}$ is called a root system if $r_\alpha(\mathcal{R}) = \mathcal{R}$ for all $\alpha \in \mathcal{R}$.

For a given root system $\mathcal{R}$ the reflections r_α, $\alpha \in \mathcal{R}$, generate a finite group $W \subset O(d)$, called the reflection group associated with $\mathcal{R}$.

We fix a positive root system $\mathcal{R}_+ = \{\alpha \in \mathcal{R} : \langle \alpha, \beta\rangle > 0\}$ for some $\beta \in \mathbb{R}^d \backslash \bigcup_{\alpha\in\mathcal{R}} H_\alpha$. We denote by $\mathcal{R}_+^0$ the set of positive indivisible roots.

Let

$$C_+ = \left\{x \in \mathbb{R}^d : \forall\, \alpha \in \mathcal{R}_+,\ \langle \alpha, x\rangle > 0\right\}$$

be the positive chamber. We denote by $\overline{C}_+$ its closure.

A function $k : \mathcal{R} \to [0, \infty)$ is called a multiplicity function if it is invariant under the action of the associated reflection group W. For abbreviation, we introduce the index

$$\gamma = \gamma(k) = \sum_{\alpha\in\mathcal{R}_+} k(\alpha).$$

Moreover, let A_k denote the weight function

$$\forall\, x \in \mathbb{R}^d, \quad A_k(x) = \prod_{\alpha\in\mathcal{R}_+} \left|\sinh \left\langle \frac{\alpha}{2}, x\right\rangle\right|^{2k(\alpha)}.$$

We note that this function is W invariant and satisfies

$$\forall\, x \in \overline{C}_+, \quad A_k(x) \le \exp(2\langle \varrho, x\rangle),$$

where

$$\rho = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\alpha.$$

2.2. **The eigenfunctions of the Cherednik operators.** The Cherednik operators T_j, $j = 1, \dots, d$, on $\mathbb{R}^d$ associated with the finite reflection group W and the multiplicity function k are given by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \mathcal{R}_+} \frac{k(\alpha)\alpha_j}{1 - e^{-\langle \alpha, x \rangle}} \{f(x) - f(r_\alpha(x))\} - \rho_j f(x),$$

with $\alpha_j = \langle \alpha, e_j \rangle$ and $\varrho_j = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\alpha_j$.

The operators T_j can also be written in the form

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\alpha_j \coth\left\langle \frac{\alpha}{2}, x \right\rangle \{f(x) - f(r_\alpha(x))\} - \frac{1}{2} S_j f(x),$$

with

$$\forall\, x \in \mathbb{R}^d, \quad S_j f(x) = \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\alpha_j f(r_\alpha(x)).$$

In the case $k(\alpha) = 0$, for all $\alpha \in \mathcal{R}_+$, the T_j, $j = 1, 2, \dots, d$, reduce to the corresponding partial derivatives.

Example 2.1. For $d = 1$, the root systems are $\mathcal{R} = \{-\alpha, \alpha\}$, $\mathcal{R} = \{-2\alpha, 2\alpha\}$, or $\mathcal{R} = \{-2\alpha, -\alpha, \alpha, 2\alpha\}$, with α the positive root. We take the normalization $\alpha = 2$.

- For $\mathcal{R}_+ = \{\alpha\}$, we have the Cherednik operator

  $$T_1 f(x) = \frac{d}{dx} f(x) + \frac{2k(\alpha)}{1 - e^{-2x}} \{f(x) - f(-x)\} - \rho f(x),$$

  with $\rho = k(\alpha)$. This operator can also be written in the form

  $$T_1 f(x) = \frac{d}{dx} f(x) + k(\alpha)\coth(x)\{f(x) - f(-x)\} - k(\alpha) f(-x). \tag{2.1}$$

- For $\mathcal{R}_+ = \{2\alpha\}$, we have the Cherednik operator

  $$T_1 f(x) = \frac{d}{dx} f(x) + \frac{4k(2\alpha)}{1 - e^{-4x}} \{f(x) - f(-x)\} - \rho f(x).$$

  This operator can also be written in the form

  $$T_1 f(x) = \frac{d}{dx} f(x) + (k(2\alpha)\coth(x) + k(2\alpha)\tanh(x))\{f(x) - f(-x)\} - \rho f(-x), \tag{2.2}$$

  with $\rho = 2k(2\alpha)$.

- For $\mathcal{R}_+ = \{\alpha, 2\alpha\}$, we have the Cherednik operator

  $$T_1 f(x) = \frac{d}{dx} f(x) + \left(\frac{2k(\alpha)}{1 - e^{-2x}} + \frac{4k(2\alpha)}{1 - e^{-4x}} \right) \{f(x) - f(-x)\} - \rho f(x),$$

with $\rho = k(\alpha) + 2k(2\alpha)$. It can also be written as

$$T_1 f(x) = \frac{d}{dx} f(x) + ((k(\alpha)+k(2\alpha))\coth(x) + k(2\alpha)\tanh(x))\{f(x) - f(-x)\} - \rho f(-x). \tag{2.3}$$

The operators (2.1), (2.2) and (2.3) are particular cases of the differential-difference operator

$$\Lambda_{k,k'} f(x) = \frac{d}{dx} f(x) + (k\coth(x) + k'\tanh(x))\{f(x) - f(-x)\} - \rho f(-x),$$

with $k \geq k' \geq 0$ and $k \neq 0$. This operator is called the Jacobi–Cherednik operator (cf. [14]).

The Heckman–Opdam Laplacian $\triangle_k$ on $\mathbb{R}^d$ is defined by

$$\begin{aligned}\triangle_k f(x) := \sum_{j=1}^{d} T_j^2 f(x) = \triangle f(x) + \sum_{\alpha \in \mathbb{R}_+} k(\alpha) \left(\coth\left\langle \frac{\alpha}{2}, x \right\rangle\right) \langle \nabla f(x), \alpha \rangle + \|\rho\|^2 f(x) \\ - \sum_{\alpha \in R_+} k(\alpha) \frac{\|\alpha\|^2}{4(\sinh\langle \frac{\alpha}{2}, x\rangle)^2} \{f(x) - f(r_\alpha(x))\},\end{aligned}$$

where $\triangle$ and ∇ are respectively the euclidean Laplacian and the gradient operator on $\mathbb{R}^d$.

The Heckman–Opdam Laplacian on W-invariant functions is denoted by $\triangle_k^W$ and has the expression

$$\triangle_k^W f(x) = \triangle f(x) + \sum_{\alpha \in R_+} k(\alpha) \left(\coth\left\langle \frac{\alpha}{2}, x \right\rangle\right) \langle \nabla f(x), \alpha \rangle + \|\rho\|^2 f(x).$$

Example 2.2. For $d = 1$, $W = \mathbb{Z}_2$ and $k \geq k' \geq 0$, $k \neq 0$, the Heckman–Opdam Laplacian is the Jacobi operator defined for even functions f of class C^2 on $\mathbb{R}$ by

$$\triangle_{k,k'}^W f(x) = \frac{d^2}{dx^2} f(x) + (2k\coth x + 2k'\tanh x)\frac{d}{dx} f(x) + \varrho^2 f(x),$$

with $\varrho = k + k'$.

We denote by G_λ the eigenfunction of the operators T_j, $j = 1, 2, \ldots, d$. It is the unique analytic function on $\mathbb{R}^d$ that satisfies the differential-difference system

$$\begin{cases} T_j u(x) = -i\lambda_j u(x), \quad j = 1, 2, \ldots, d,\ x \in \mathbb{R}^d, \\ u(0) = 1. \end{cases}$$

G_λ is called the Opdam–Cherednik kernel.

We consider the function F_λ defined by

$$\forall\, x \in \mathbb{R}^d, \quad F_\lambda(x) = \frac{1}{|W|} \sum_{w \in W} G_\lambda(wx).$$

This function is the unique analytic W-invariant function on $\mathbb{R}^d$ that satisfies the differential equation

$$\begin{cases} p(T)u(x) = p(-i\lambda)u(x), & x \in \mathbb{R}^d,\ \lambda \in \mathbb{R}^d, \\ u(0) = 1, \end{cases}$$

for all W-invariant complex polynomials p on $\mathbb{R}^d$ and $p(T) = p(T_1, \dots, T_d)$.

In particular, for all $\lambda \in \mathbb{R}^d$ we have

$$\triangle_k^W F_\lambda(x) = -\|\lambda\|^2 F_\lambda(x).$$

The function F_λ is called the Heckman–Opdam hypergeometric function.

The functions G_λ and F_λ possess the following properties:

i) For all $x \in \mathbb{R}^d$, the functions $G_\lambda(x)$ and $F_\lambda(x)$ are entire on $\mathbb{C}^d$.

ii) The functions G_λ and F_λ satisfy the estimate

$$\forall\, x \in \mathbb{R}^d,\ \forall\, \lambda \in \mathbb{R}^d, \quad |G_\lambda(x)| \leq \sqrt{|W|},$$

and

$$\forall\, x \in \mathbb{R}^d,\ \forall\, \lambda \in \mathbb{R}^d, \quad |F_\lambda(x)| \leq 1.$$

iii) We have

$$\forall\, x \in \overline{C}_+, \quad F_0(x) \asymp e^{-\langle \rho, x\rangle} \prod_{\alpha \in R_+^0} (1 + \langle \alpha, x\rangle).$$

iv) Let p and q be polynomials of degree m and n, respectively. Then there exists a positive constant M such that for all $\lambda \in \mathbb{C}^d$ and for all $x \in \mathbb{R}^d$, we have

$$\left| p\left(\frac{\partial}{\partial \lambda}\right) q\left(\frac{\partial}{\partial x}\right) G_\lambda(x) \right| \leq M(1 + \|x\|)^m (1 + \|\lambda\|)^n F_0(x) e^{\max_{w \in W} (\mathrm{Im}\langle w\lambda, x\rangle)}.$$

v) The preceding estimate holds true for F_λ too.

Example 2.3. When $d = 1$, $W = \mathbb{Z}_2$, and $k \geq k' \geq 0$, $k \neq 0$, the Opdam–Cherednik kernel $G_\lambda(x)$ is given for all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$ by

$$G_\lambda(x) = \varphi_\lambda^{(k-\frac{1}{2}, k'-\frac{1}{2})}(x) - \frac{1}{\rho - i\lambda} \frac{d}{dx} \varphi^{(k-\frac{1}{2}, k'-\frac{1}{2})}(x),$$

where $\varphi_\lambda^{(\alpha,\beta)}(x)$ is the Jacobi function of index (α, β) defined by

$$\varphi_\lambda^{(\alpha,\beta)}(x) = {}_2F_1\left(\tfrac{1}{2}(\rho + i\lambda), \tfrac{1}{2}(\rho - i\lambda); \alpha + 1; -(\sinh x)^2\right),$$

with $\rho = \alpha + \beta + 1$ and ${}_2F_1$ is the Gauss hypergeometric function.

In this case the Heckman–Opdam kernel $F_\lambda(x)$ is given for all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$ by

$$F_\lambda(x) = \varphi_\lambda^{(k-\frac{1}{2}, k'-\frac{1}{2})}(x).$$

2.3. **The hypergeometric Fourier transform on W-invariant functions.**

Notation. We denote by

- $\mathcal{E}(\mathbb{R}^d)^W$ the space of C^∞-functions on $\mathbb{R}^d$ that are W-invariant;
- $\mathcal{D}(\mathbb{R}^d)^W$ the space of C^∞-functions on $\mathbb{R}^d$ that are W-invariant and with compact support;
- $\mathcal{S}(\mathbb{R}^d)^W$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$ that are W-invariant;
- $\mathcal{S}_2(\mathbb{R}^d)^W$ the space of C^∞-functions on $\mathbb{R}^d$ that are W-invariant and such that for all $\ell, n \in \mathbb{N}$, we have
$$\sup_{\substack{|\mu|\leq n\\ x\in\mathbb{R}^d}} (1+\|x\|)^\ell F_0^{-1}(x)|D^\mu f(x)| < \infty,$$
where
$$D^\mu = \frac{\partial^{|\mu|}}{\partial x_1^{\mu_1}\dots\partial x_d^{\mu_d}}, \quad \mu = (\mu_1,\dots,\mu_d)\in\mathbb{N}^d;$$
- $PW(\mathbb{C}^d)^W$ the space of entire functions on $\mathbb{C}^d$ that are W-invariant, rapidly decreasing, and of exponential type;
- $L^p_{A_k}(\mathbb{R}^d)^W$, $1\leq p\leq\infty$, the space of measurable functions f on $\mathbb{R}^d$ that are W-invariant and satisfy
$$\|f\|_{L^p_{A_k}(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d}|f(x)|^p A_k(x)\,dx\right)^{1/p} < \infty, \quad \text{if } 1\leq p<\infty,$$
$$\|f\|_{L^\infty_{A_k}(\mathbb{R}^d)} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^d}|f(x)| < \infty;$$
- $L^p_{\nu_k}(\mathbb{R}^d)^W$, $1\leq p\leq\infty$, the space of measurable functions f on $\mathbb{R}^d$ that are W-invariant and satisfy
$$\|f\|_{L^p_{\nu_k}(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d}|f(x)|^p d\nu_k(x)\right)^{1/p} < \infty, \quad \text{if } 1\leq p<\infty,$$
$$\|f\|_{L^\infty_{\nu_k}(\mathbb{R}^d)} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^d}|f(x)| < \infty,$$
where
$$\begin{aligned} d\nu_k(\lambda) &:= C_k(\lambda)\,d\lambda \\ &= c\prod_{\alpha\in R_+}\frac{\Gamma(-i\langle\lambda,\alpha^\vee\rangle+k(\alpha)+\frac12k(\frac{\alpha}{2}))\Gamma(i\langle\lambda,\alpha^\vee\rangle+k(\alpha)+\frac12k(\frac{\alpha}{2}))}{\Gamma(-i\langle\lambda,\alpha^\vee\rangle+\frac12k(\frac{\alpha}{2}))\Gamma(i\langle\lambda,\alpha^\vee\rangle+\frac12k(\frac{\alpha}{2}))}\,d\lambda,\end{aligned}$$
with c a normalizing constant and $k(\frac{\alpha}{2})=0$ if $\frac{\alpha}{2}\notin R_+$. The measure $d\nu_k(\lambda)$ is called the symmetric Plancherel measure or Harish-Chandra measure (cf. [33, 37]).

Remark 2.4. The function C_k is positive, continuous on $\mathbb{R}^d$, and satisfies the estimate
$$\forall\,\lambda\in\mathbb{R}^d, \quad |C_k(\lambda)| \leq \text{const.}\,\|\lambda\|^{|\mathcal{R}_+^0|}(1+\|\lambda\|)^{2\gamma-|\mathcal{R}_+^0|}.$$

Definition 2.5. The hypergeometric Fourier transform of a function f in $D(\mathbb{R}^d)^W$ is given by

$$\mathcal{H}_k^W(f)(\lambda) = \int_{\mathbb{R}^d} f(x) F_\lambda(x) A_k(x)\, dx, \quad \text{for all } \lambda \in \mathbb{R}^d.$$

Proposition 2.6. *The transform $\mathcal{H}_k^W$ is a topological isomorphism from*

i) $D(\mathbb{R}^d)^W$ *onto* $PW(\mathbb{C}^d)^W$.

ii) $\mathcal{S}_2(\mathbb{R}^d)^W$ *onto* $\mathcal{S}(\mathbb{R}^d)^W$.

The inverse transform is given by

$$\forall\, x \in \mathbb{R}^d, \quad (\mathcal{H}_k^W)^{-1}(h)(x) = \int_{\mathbb{R}^d} h(\lambda) F_\lambda(-x)\, d\nu_k(\lambda).$$

Proposition 2.7. *For f in $L^1_{A_k}(\mathbb{R}^d)^W$ the function $\mathcal{H}_k^W(f)$ is continuous on $\mathbb{R}^d$ and we have*

$$\|\mathcal{H}_k^W(f)\|_{L^\infty_{\nu_k}(\mathbb{R}^d)} \leq \|f\|_{L^1_{A_k}(\mathbb{R}^d)}.$$

Proposition 2.8. *i)(Parseval's formula) For all f, g in $D(\mathbb{R}^d)^W$ (resp. in $\mathcal{S}_2(\mathbb{R}^d)^W$) we have*

$$\int_{\mathbb{R}^d} f(x)\overline{g(x)} A_k(x)\, dx = \int_{\mathbb{R}^d} \mathcal{H}_k^W(f)(\lambda)\overline{\mathcal{H}_k^W(g)(\lambda)}\, d\nu_k(\lambda).$$

ii) (Plancherel's theorem) The transform $\mathcal{H}_k^W$ extends uniquely to an isomorphism from $L^2_{A_k}(\mathbb{R}^d)^W$ onto $L^2_{\nu_k}(\mathbb{R}^d)^W$.

Proposition 2.9. *For all f in $L^2_{A_k}(\mathbb{R}^d)^W$ such that $\mathcal{H}_k^W(f)$ belongs to $L^1_{\nu_k}(\mathbb{R}^d)^W$, we have the inversion formula*

$$f(x) = \int_{\mathbb{R}^d} \mathcal{H}_k^W(f)(\lambda) F_\lambda(-x)\, d\nu_k(\lambda), \quad a.e.\ x \in \mathbb{R}^d.$$

Definition 2.10. Let x be in $\mathbb{R}^d$. The generalized translation operator $f \mapsto \tau_x^W f$ is defined on $L^2_{A_k}(\mathbb{R}^d)^W$ by

$$\mathcal{H}_k^W(\tau_x^W f)(\lambda) = F_\lambda(x)\mathcal{H}_k^W(f)(\lambda), \quad \lambda \in \mathbb{R}^d. \tag{2.4}$$

Using the generalized translation operator, we define the generalized convolution product of functions as follows.

Definition 2.11. The generalized convolution product of f and g in $L^2_{A_k}(\mathbb{R}^d)^W$ is the function $f *_k g$ defined by

$$f *_k g(x) = \int_{\mathbb{R}^d} \tau_x^W f(-y) g(y) A_k(y)\, dy, \quad x \in \mathbb{R}^d. \tag{2.5}$$

Proposition 2.12. *Let f and g be in $L^2_{A_k}(\mathbb{R}^d)^W$. Then the function $f *_k g$ belongs to $L^2_{A_k}(\mathbb{R}^d)^W$ if and only if the function $\mathcal{H}_k^W(f).\mathcal{H}_k^W(g)$ is in $L^2_{\nu_k}(\mathbb{R}^d)^W$, and we have*

$$\mathcal{H}_k^W(f *_k g) = \mathcal{H}_k^W(f).\mathcal{H}_k^W(g)$$

in the L^2-case.

2.4. **Basic generalized wavelet theory.**

Definition 2.13. A generalized wavelet on $\mathbb{R}^d$ is a measurable function h that is W-invariant on $\mathbb{R}^d$ and satisfies, for almost all $\lambda \in \mathbb{R}^d$, the condition

$$0 < C_h = \int_0^\infty |\mathcal{H}_k^W(h)(\lambda a)|^2 \frac{da}{a} < \infty.$$

Example 2.14. Let E_t, $t > 0$, be the heat kernel defined on $\mathbb{R}^d$ by

$$\forall\, x \in \mathbb{R}^d, \quad E_t(x) = \left(\mathcal{H}_k^W\right)^{-1}(e^{-t\|\lambda\|^2})(x).$$

The function $h(x) = -\dfrac{d}{dt}E_t(x)$ is a generalized wavelet on $\mathbb{R}^d$ in $\mathcal{S}_2(\mathbb{R}^d)^W$, and $C_h = \dfrac{1}{8t^2}$.

Proposition 2.15. *Let $a > 0$ and let h be a generalized wavelet in $L^2_{A_k}(\mathbb{R}^d)^W$. Then there exists a function h_a in $L^2_{A_k}(\mathbb{R}^d)^W$ such that*

$$\forall\, \lambda \in \mathbb{R}^d, \quad \mathcal{H}_k^W(h_a)(\lambda) = \mathcal{H}_k^W(h)(a\lambda).$$

This function is given by the relation

$$h_a = \frac{1}{a^{\frac{d}{2}}}\left(\mathcal{H}_k^W\right)^{-1} \circ D_{a^{-1}} \circ \mathcal{H}_k^W(h)$$

and satisfies

$$\|h_a\|_{L^2_{A_k}(\mathbb{R}^d)} \leq \frac{s(a)}{a^{\frac{d}{2}}}\|h\|_{L^2_{A_k}(\mathbb{R}^d)},$$

where

$$s(a) = \sup_{\lambda\in\mathbb{R}^d} \frac{|C_k(\lambda)|}{|C_k(\frac{\lambda}{a})|} \quad \textit{and} \quad D_a(f)(x) = \frac{1}{a^{\frac{d}{2}}} f\Big(\frac{x}{a}\Big).$$

Let $a > 0$ and h be in $L^2_{A_k}(\mathbb{R}^d)^W$. We consider the family $h_{a,x}$, $x \in \mathbb{R}^d$, of functions on $\mathbb{R}^d$ in $L^2_{A_k}(\mathbb{R}^d)$ defined by

$$h_{a,x}(y) = \frac{a^{\frac{d}{2}}}{s(a)}\tau_x^W(h_a)(-y), \quad y \in \mathbb{R}^d,$$

where τ_x^W, $x \in \mathbb{R}^d$, are the generalized translation operators given by (2.4).

We note that we have

$$\forall\, a > 0, \forall\, x \in \mathbb{R}^d, \quad \|h_{a,x}\|_{L^2_{A_k}(\mathbb{R}^d)} \leq \|h\|_{L^2_{A_k}(\mathbb{R}^d)}. \tag{2.6}$$

Notation. We denote by

- $\mathbb{R}^{d+1}_+ = \big\{(a, x) = (a, x_1, \ldots, x_d) \in \mathbb{R}^{d+1} : a > 0\big\}$;

- $L^p_{\mu_k}(\mathbb{R}^{d+1}_+)^W$, $p \in [1,\infty]$, the space of measurable functions $f(a,x)$ on $\mathbb{R}^{d+1}_+$ that are W-invariant with respect to the variable x and satisfy

$$\|f\|_{\mu_k,p} := \left(\int_{\mathbb{R}^{d+1}_+} |f(a,x)|^p \, d\mu_k(a,x) \right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty,$$
$$\|f\|_{\mu_k,\infty} := \operatorname*{ess\,sup}_{(a,x)\in\mathbb{R}^{d+1}_+} |f(a,x)| < \infty,$$

where the measure μ_k is defined by

$$\forall\,(a,x) \in \mathbb{R}^{d+1}_+, \quad d\mu_k(a,x) = \frac{s(a)A_k(x)\,dx\,da}{a^{\frac{d}{2}+1}}.$$

Definition 2.16. Let h be a generalized wavelet on $\mathbb{R}^d$ in $L^2_{A_k}(\mathbb{R}^d)^W$. The generalized continuous wavelet transform Φ^W_h on $\mathbb{R}^d$ is defined for suitable functions f on $\mathbb{R}^d$ by

$$\Phi^W_h(f)(a,x) = \int_{\mathbb{R}^d} f(y)\overline{h_{a,x}(y)}A_k(y)\,dy, \quad (a,x) \in \mathbb{R}^{d+1}_+. \tag{2.7}$$

The adjoint of Φ^W_h is $(\Phi^W_h)^* : L^2_{\mu_k}(\mathbb{R}^{d+1}_+)^W \to L^2_{A_k}(\mathbb{R}^d)^W$ defined by

$$(\Phi^W_h)^*(F)(t) = \frac{1}{C_h}\int_{\mathbb{R}^{d+1}_+} F(a,x)h_{a,x}(t)\,d\mu_k(a,x), \quad t \in \mathbb{R}^d.$$

Remark 2.17. i) The generalized continuous wavelet transform can also be written in the form

$$\Phi^W_h(f)(a,x) = \frac{a^{\frac{d}{2}}}{s(a)} f *_k \overline{h_a}(x),$$

where $*_k$ is the Heckman–Opdam convolution product given by (2.5).

ii) Let h be a generalized wavelet. Then for all f in $L^2_{A_k}(\mathbb{R}^d)^W$ we have

$$\|\Phi^W_h f\|_{\mu_k,\infty} \leq \|f\|_{L^2_{A_k}(\mathbb{R}^d)}\|h\|_{L^2_{A_k}(\mathbb{R}^d)}. \tag{2.8}$$

Theorem 2.18 (Plancherel's formula for Φ^W_h)**.** *Let h be a generalized wavelet on $\mathbb{R}^d$ in $L^2_{A_k}(\mathbb{R}^d)^W$. For all f in $L^2_{A_k}(\mathbb{R}^d)^W$ we have*

$$\int_{\mathbb{R}^d} |f(x)|^2 A_k(x)\,dx = \frac{1}{C_h}\int_{\mathbb{R}^{d+1}_+} |\Phi^W_h(f)(a,x)|^2\,d\mu_k(a,x). \tag{2.9}$$

Corollary 2.19 (Parseval's formula for Φ^W_h)**.** *Let h be a generalized wavelet on $\mathbb{R}^d$ in $L^2_{A_k}(\mathbb{R}^d)^W$ and f_1, f_2 in $L^2_{A_k}(\mathbb{R}^d)^W$. Then we have*

$$\int_{\mathbb{R}^d} f_1(x)\overline{f_2(x)}A_k(x)\,dx = \frac{1}{C_h}\int_{\mathbb{R}^{d+1}_+} \Phi^W_h(f_1)(a,x)\overline{\Phi^W_h(f_2)(a,x)}\,d\mu_k(a,x).$$

Theorem 2.20 (Inversion formula for Φ^W_h)**.** *Let h be a generalized wavelet on $\mathbb{R}^d$ in $L^2_{A_k}(\mathbb{R}^d)^W$. For all f in $L^1_{A_k}(\mathbb{R}^d)^W$ (resp. $L^2_{A_k}(\mathbb{R}^d)^W$) such that $\mathcal{H}^W_k(f)$ belongs to $L^1_{A_k}(\mathbb{R}^d)^W$ (resp. $L^1_{A_k}(\mathbb{R}^d)^W \bigcap L^\infty_{A_k}(\mathbb{R}^d)$) we have*

$$f(y) = \frac{1}{C_h}\int_0^\infty \int_{\mathbb{R}^d} \Phi^W_h(f)(a,x)h_{a,y}(x)\,d\mu_k(a,x), \quad a.e.,$$

where for each $y \in \mathbb{R}^d$, both the inner integral and the outer integral are absolutely convergent, but possibly not the double integral.

3. Localization operators for the generalized continuous wavelet transform

3.1. Preliminaries.

Notation. We denote by:

- $l^p(\mathbb{N})$ the set of all infinite sequences of real (or complex) numbers $x := (x_j)_{j\in\mathbb{N}}$ such that
$$\|x\|_p := \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{\frac{1}{p}} < \infty, \quad \text{if } 1 \leq p < \infty,$$
$$\|x\|_\infty := \sup_{j\in\mathbb{N}} |x_j| < \infty.$$
For $p = 2$, we provide this space $l^2(\mathbb{N})$ with the scalar product
$$\langle x, y\rangle_{L^2_{A_k}(\mathbb{R}^d)} := \sum_{j=1}^{\infty} x_j \overline{y_j}\,;$$
- $B(L^2_{A_k}(\mathbb{R}^d))$ the space of bounded operators from $L^2_{A_k}(\mathbb{R}^d)$ into itself.

Definition 3.1. (i) The singular values $(s_n(A))_{n\in\mathbb{N}}$ of a compact operator A in $B(L^2_{A_k}(\mathbb{R}^d))$ are the eigenvalues of the positive self-adjoint operator $|A| = \sqrt{A^*A}$.

(ii) For $1 \leq p < \infty$, the Schatten class S_p is the space of all compact operators whose singular values lie in $l^p(\mathbb{N})$. The space S_p is equipped with the norm
$$\|A\|_{S_p} := \left(\sum_{n=1}^{\infty} (s_n(A))^p\right)^{\frac{1}{p}}.$$

Remark 3.2. We note that the space S_2 is the space of Hilbert–Schmidt operators, and S_1 is the space of trace class operators.

Definition 3.3. The trace of an operator A in S_1 is defined by
$$\operatorname{tr}(A) = \sum_{n=1}^{\infty} \langle Av_n, v_n\rangle_{L^2_{A_k}(\mathbb{R}^d)}, \tag{3.1}$$
where $(v_n)_n$ is any orthonormal basis of $L^2_{A_k}(\mathbb{R}^d)$.

Remark 3.4. If A is positive, then
$$\operatorname{tr}(A) = \|A\|_{S_1}.$$
Moreover, a compact operator A on the Hilbert space $L^2_{A_k}(\mathbb{R}^d)$ is a Hilbert–Schmidt operator if the positive operator A^*A is in the space of trace class S_1. Then
$$\|A\|^2_{HS} := \|A\|^2_{S_2} = \|A^*A\|_{S_1} = \operatorname{tr}(A^*A) = \sum_{n=1}^{\infty} \|Av_n\|^2_{L^2_{A_k}(\mathbb{R}^d)}$$
for any orthonormal basis $(v_n)_n$ of $L^2_{A_k}(\mathbb{R}^d)$.

Definition 3.5. We define $S_\infty := B(L^2_{A_k}(\mathbb{R}^d))$, equipped with the norm

$$\|A\|_{S_\infty} := \sup_{v\in L^2_{A_k}(\mathbb{R}^d):\|v\|_{L^2_{A_k}(\mathbb{R}^d)}=1} \|Av\|_{L^2_{A_k}(\mathbb{R}^d)}.$$

In this section, h will be a generalized wavelet on $\mathbb{R}^d$ such that

$$\|h\|_{L^2_{A_k}(\mathbb{R}^d)} = 1.$$

3.2. **Boundedness.** In this subsection we define the localization operators for the generalized continuous wavelet transform and we show that they are bounded.

Definition 3.6. The localization operator with symbol σ associated with the generalized continuous wavelet transform, denoted by $\mathcal{L}_h(\sigma)$, is defined on $L^2_{A_k}(\mathbb{R}^d)$ by

$$\mathcal{L}_h(\sigma)(f)(y) = \frac{1}{C_h}\int_{\mathbb{R}^{d+1}_+} \sigma(a,x)\Phi_h^W(f)(a,x)h_{a,x}(y)\,d\mu_k(a,x), \quad y\in\mathbb{R}^d.$$

Often it is more convenient to interpret the definition of $\mathcal{L}_h(\sigma)$ in a weak sense, that is, for f,g in $L^2_{A_k}(\mathbb{R}^d)$ we have

$$\langle\mathcal{L}_h(\sigma)(f),g\rangle_{L^2_{A_k}(\mathbb{R}^d)} = \frac{1}{C_h}\int_{\mathbb{R}^{d+1}_+} \sigma(a,x)\Phi_h^W(f)(a,x)\overline{\Phi_h^W(g)(a,x)}\,d\mu_k(a,x). \tag{3.2}$$

In this section we prove that the linear operators

$$\mathcal{L}_h(\sigma): L^2_{A_k}(\mathbb{R}^d)\to L^2_{A_k}(\mathbb{R}^d)$$

are bounded for all symbols σ in $L^p_{\mu_k}(\mathbb{R}^{d+1}_+)^W$, $1\le p\le\infty$. We consider first this problem for σ in $L^1_{\mu_k}(\mathbb{R}^{d+1}_+)^W$ and next in $L^\infty_{\mu_k}(\mathbb{R}^{d+1}_+)^W$ and we conclude by using interpolation theory.

Proposition 3.7. *Let σ be in $L^1_{\mu_k}(\mathbb{R}^{d+1}_+)^W$. Then the localization operator $\mathcal{L}_h(\sigma)$ is in S_∞ and we have*

$$\|\mathcal{L}_h(\sigma)\|_{S_\infty} \leqslant \frac{1}{C_h}\|\sigma\|_{\mu_k,1}.$$

Proof. For all functions f and g in $L^2_{A_k}(\mathbb{R}^d)^W$, we have from the relations (3.2) and (2.8):

$$\begin{aligned}\left|\langle\mathcal{L}_h(\sigma)(f),g\rangle_{L^2_{A_k}(\mathbb{R}^d)}\right| &\leqslant \frac{1}{C_h}\int_{\mathbb{R}^{d+1}_+}\left|\sigma(a,x)\,\|\Phi_h^W(f)(a,x)\|\,\overline{\Phi_h^W(g)(a,x)}\right|d\mu_k(a,x)\\ &\leqslant \frac{1}{C_h}\|\Phi_h^W(f)\|_{\mu_k,\infty}\|\Phi_h^W(g)\|_{\mu_k,\infty}\|\sigma\|_{\mu_k,1}\\ &\leqslant \frac{1}{C_h}\|f\|_{L^2_{A_k}(\mathbb{R}^d)}\|g\|_{L^2_{A_k}(\mathbb{R}^d)}\|\sigma\|_{\mu_k,1}.\end{aligned}$$

Thus,

$$\|\mathcal{L}_h(\sigma)\|_{S_\infty} \leqslant \frac{1}{C_h}\|\sigma\|_{\mu_k,1}.$$

□

Proposition 3.8. *Let σ be in $L^\infty_{\mu_k}(\mathbb{R}^{d+1}_+)^W$. Then the localization operator $\mathcal{L}_h(\sigma)$ is in S_∞ and we have*

$$\|\mathcal{L}_h(\sigma)\|_{S_\infty} \leqslant \|\sigma\|_{\mu_k,\infty}. \tag{3.3}$$

Proof. For all functions f and g in $L^2_{A_k}(\mathbb{R}^d)^W$, we have from Hölder's inequality:

$$\begin{aligned}\left|\langle \mathcal{L}_h(\sigma)(f), g\rangle_{L^2_{A_k}(\mathbb{R}^d)}\right| &\leqslant \frac{1}{C_h}\int_{\mathbb{R}^{d+1}_+} \left|\sigma(a,x)\, \|\Phi^W_h(f)(a,x)\| \,\overline{\Phi^W_h(g)(a,x)}\right| d\mu_k(a,x)\\ &\leqslant \frac{1}{C_h}\|\sigma\|_{\mu_k,\infty}\|\Phi^W_h(f)\|_{\mu_k,2}\|\Phi^W_h(g)\|_{\mu_k,2}.\end{aligned}$$

Using Plancherel's formula for Φ^W_h, given by the relation (2.9), we get

$$|\langle \mathcal{L}_h(\sigma)(f), g\rangle_{L^2_{A_k}(\mathbb{R}^d)}| \leqslant \|\sigma\|_{\mu_k,\infty}\|f\|_{L^2_{A_k}(\mathbb{R}^d)}\|g\|_{L^2_{A_k}(\mathbb{R}^d)}.$$

Thus,

$$\|\mathcal{L}_h(\sigma)\|_{S_\infty} \leqslant \|\sigma\|_{\mu_k,\infty}.$$ □

We can now associate a localization operator

$$\mathcal{L}_h(\sigma) : L^2_{A_k}(\mathbb{R}^d)^W \to L^2_{A_k}(\mathbb{R}^d)^W$$

to every function σ in $L^p_{\mu_k}(\mathbb{R}^{d+1}_+)^W$, $1 \le p \le \infty$, and prove that $\mathcal{L}_h(\sigma)$ is in S_∞. The precise result is the following theorem.

Theorem 3.9. *Let σ be in $L^p_{\mu_k}(\mathbb{R}^{d+1}_+)^W$, $1 \le p \le \infty$. Then there exists a unique bounded linear operator $\mathcal{L}_h(\sigma) : L^2_{A_k}(\mathbb{R}^d)^W \to L^2_{A_k}(\mathbb{R}^d)^W$ such that*

$$\|\mathcal{L}_h(\sigma)\|_{S_\infty} \leqslant (\frac{1}{C_h})^{\frac{1}{p}}\|\sigma\|_{\mu_k,p}.$$

Proof. Let f be in $L^2_{A_k}(\mathbb{R}^d)^W$. We consider the operator

$$\mathcal{T} : L^1_{\mu_k}(\mathbb{R}^{d+1}_+)^W \cap L^\infty_{\mu_k}(\mathbb{R}^{d+1}_+)^W \to L^2_k(\mathbb{R}^d)^W,$$

given by

$$\mathcal{T}(\sigma) := \mathcal{L}_h(\sigma)(f).$$

Then, by Proposition 3.7 and Proposition 3.8,

$$\|\mathcal{T}(\sigma)\|_{L^2_{A_k}(\mathbb{R}^d)} \le \frac{1}{C_h}\|f\|_{L^2_{A_k}(\mathbb{R}^d)}\|\sigma\|_{\mu_k,1} \tag{3.4}$$

and

$$\|\mathcal{T}(\sigma)\|_{L^2_{A_k}(\mathbb{R}^d)} \le \|f\|_{L^2_{A_k}(\mathbb{R}^d)}\|\sigma\|_{\mu_k,\infty}. \tag{3.5}$$

Therefore, by (3.4), (3.5), and the Riesz–Thorin interpolation theorem (see [38, Theorem 2] and [41, Theorem 2.11]), $\mathcal{T}$ may be uniquely extended to a linear transformation on $L^p_{\mu_k}(\mathbb{R}^{d+1}_+)^W$, and we have

$$\|\mathcal{L}_h(\sigma)(f)\|_{L^2_{A_k}(\mathbb{R}^d)} = \|\mathcal{T}(\sigma)\|_{L^2_{A_k}(\mathbb{R}^d)} \le \left(\frac{1}{C_h}\right)^{\frac{1}{p}}\|f\|_{L^2_{A_k}(\mathbb{R}^d)}\|\sigma\|_{\mu_k,p}. \tag{3.6}$$

Since (3.6) is true for arbitrary functions f in $L^2_{A_k}(\mathbb{R}^d)^W$, we obtain the desired result. □

3.3. **Schatten–von Neumann properties for $\mathcal{L}_h(\sigma)$.** In this subsection we will prove that the localization operator

$$\mathcal{L}_h(\sigma) : L^2_{A_k}(\mathbb{R}^d)^W \to L^2_{A_k}(\mathbb{R}^d)^W$$

is in the Schatten class S_p. The first result on the Schatten property of localization operators is given in the following theorem.

Theorem 3.10. *Let σ be in $L^1_{\mu_k}(\mathbb{R}^{d+1}_+)^W$. Then the bounded localization operator*

$$\mathcal{L}_h(\sigma) : L^2_{A_k}(\mathbb{R}^d)^W \to L^2_{A_k}(\mathbb{R}^d)^W$$

is in S_1 and we have

$$\|\mathcal{L}_h(\sigma)\|_{S_1} \leqslant \frac{4}{C_h}\|\sigma\|_{\mu_k,1}.$$

Proof. First let us assume that σ is a nonnegative real-valued symbol, thus the localization operator $\mathcal{L}_h(\sigma)$ is positive. Let $\{u_j,\ j = 1, 2, \dots\}$ be any orthonormal basis for $L^2_{A_k}(\mathbb{R}^d)^W$. Then from Fubini's theorem, the Parseval identity, and relations (2.6) and (2.7), we get

$$\begin{aligned}\sum_{j=1}^{\infty}\langle \mathcal{L}_h(\sigma)(u_j), u_j\rangle_{L^2_{A_k}(\mathbb{R}^d)} &= \sum_{j=1}^{\infty}\frac{1}{C_h}\int_{\mathbb{R}^{d+1}_+}\sigma(a,x)|\Phi^W_h(u_j)(a,x)|^2\,d\mu_k(a,x)\\ &= \frac{1}{C_h}\int_{\mathbb{R}^{d+1}_+}\sigma(a,x)\bigg(\sum_{j=1}^{\infty}|\Phi^W_h(u_j)(a,x)|^2\bigg)\,d\mu_k(a,x).\end{aligned}$$

Thus we get

$$\sum_{j=1}^{\infty}\langle \mathcal{L}_h(\sigma)(u_j), u_j\rangle_{L^2_{A_k}(\mathbb{R}^d)} = \frac{1}{C_h}\int_{\mathbb{R}^{d+1}_+}\sigma(a,x)\|h_{a,x}\|^2_{L^2_{A_k}(\mathbb{R}^d)}\,d\mu_k(a,x). \tag{3.7}$$

Using now the relation (2.6), we deduce that

$$\begin{aligned}\sum_{j=1}^{\infty}\langle \mathcal{L}_h(\sigma)(u_j), u_j\rangle_{L^2_{A_k}(\mathbb{R}^d)} &\leqslant \sup_{(a,x)\in\mathbb{R}^{d+1}_+}\|h_{a,x}\|^2_{L^2_{A_k}(\mathbb{R}^d)}\frac{1}{C_h}\|\sigma\|_{\mu_k,1}\\ &= \frac{1}{C_h}\|\sigma\|_{\mu_k,1}.\end{aligned}$$

Then, by [41, Proposition 2.4], the operator $\mathcal{L}_h(\sigma)$ is in S_1.

We have $\sqrt{\mathcal{L}_h(\sigma)^*\mathcal{L}_h(\sigma)} = \mathcal{L}_h(\sigma)$, so if we consider $\{u_j,\ j = 1, 2, \dots\}$ an orthonormal basis for $L^2_{A_k}(\mathbb{R}^d)^W$ consisting of eigenvectors of the positive compact operator $\sqrt{\mathcal{L}_h(\sigma)^*\mathcal{L}_h(\sigma)}$ and let $s_j,\ j = 1, 2, \dots,$ be the eigenvalues of $|\mathcal{L}_h(\sigma)|$ corresponding to u_j, then

$$\begin{aligned}\|\mathcal{L}_h(\sigma)\|_{S_1} &= \sum_{j=1}^{\infty}s_j = \sum_{j=1}^{\infty}\Big\langle\sqrt{\mathcal{L}_h(\sigma)^*\mathcal{L}_h(\sigma)}(u_j), u_j\Big\rangle_{L^2_{A_k}(\mathbb{R}^d)}\\ &= \sum_{j=1}^{\infty}\langle \mathcal{L}_h(\sigma)(u_j), u_j\rangle_{L^2_{A_k}(\mathbb{R}^d)} \leqslant \frac{1}{C_h}\|\sigma\|_{\mu_k,1}.\end{aligned}$$

For σ a real-valued function, we write $\sigma = \sigma_+ - \sigma_-$, with

$$\sigma_+ = \max(\sigma, 0), \quad \sigma_- = -\min(\sigma, 0);$$

then $\mathcal{L}_h(\sigma)$ is in S_1 and we have

$$\|\mathcal{L}_h(\sigma)\|_{S_1} \leqslant \|L_h(\sigma_+)\|_{S_1} + \|L_h(\sigma_-)\|_{S_1} \leqslant \frac{2}{C_h}\|\sigma\|_{\mu_k,1}.$$

Finally, when $\sigma = \sigma_1 + i\sigma_2$ is a complex-valued function with σ_1 and σ_2 the real and imaginary parts of σ, we have that $\mathcal{L}_h(\sigma)$ is in S_1 and

$$\|\mathcal{L}_h(\sigma)\|_{S_1} \leqslant \|L_h(\sigma_1)\|_{S_1} + \|L_h(\sigma_2)\|_{S_1} \leqslant \frac{4}{C_h}\|\sigma\|_{\mu_k,1}. \qquad \square$$

Corollary 3.11. *For σ in $L^1_{\mu_k}(\mathbb{R}^{d+1}_+)^W$, we have the trace formula*

$$\operatorname{tr}(\mathcal{L}_h(\sigma)) = \frac{1}{C_h}\int_{\mathbb{R}^{d+1}_+} \sigma(a,x)\|h_{a,x}\|^2_{L^2_{A_k}(\mathbb{R}^d)}\, d\mu_k(a,x).$$

Proof. From the previous theorem, the localization operator $\mathcal{L}_h(\sigma)$ belongs to S_1; then by the definition of trace given by the relation (3.1), we have

$$\operatorname{tr}(\mathcal{L}_h(\sigma)) = \sum_{j=1}^{\infty}\langle \mathcal{L}_h(\sigma)(u_j), u_j\rangle_{L^2_{A_k}(\mathbb{R}^d)}.$$

The result is obtained by the relation (3.7). $\square$

Proposition 3.12. *Let σ be a symbol in $L^p_{\mu_k}(\mathbb{R}^{d+1}_+)^W$, $1 \leqslant p < \infty$. Then the localization operator $\mathcal{L}_h(\sigma)$ is compact.*

Proof. Let σ be in $L^p_{\mu_k}(\mathbb{R}^{d+1}_+)^W$ and let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^1_{\mu_k}(\mathbb{R}^{d+1}_+)^W \bigcap L^p_{\mu_k}(\mathbb{R}^{d+1}_+)^W$ such that $\sigma_n \to \sigma$ in $L^p_{\mu_k}(\mathbb{R}^{d+1}_+)^W$ as $n \to \infty$. Then by Theorem 3.9

$$\|\mathcal{L}_h(\sigma_n) - \mathcal{L}_h(\sigma)\|_{S_\infty} \leq \left(\frac{1}{C_h}\right)^{\frac{1}{p}} \|\sigma_n - \sigma\|_{\mu_k,p}.$$

Hence $\mathcal{L}_h(\sigma_n) \to \mathcal{L}_h(\sigma)$ in S_∞ as $n \to \infty$. On the other hand, as by Theorem 3.10 $\mathcal{L}_h(\sigma_n)$ is in S_1, hence compact, it follows that $\mathcal{L}_h(\sigma)$ is compact. $\square$

In the following theorem we improve the constant given in Theorem 3.10. First, we begin by investigating the case σ in $L^1_{\mu_k}(\mathbb{R}^{d+1}_+)^W$ and we give, in addition, a lower bound of the norm $\|\mathcal{L}_h(\sigma)\|_{S_1}$.

Theorem 3.13. *Let σ be in $L^1_{\mu_k}(\mathbb{R}^{d+1}_+)^W$. Then,*

$$\frac{1}{C_h}\|\widetilde{\sigma}\|_{\mu_k,1} \leqslant \|\mathcal{L}_h(\sigma)\|_{S_1} \leqslant \frac{1}{C_h}\|\sigma\|_{\mu_k,1},$$

where $\widetilde{\sigma}$ is given by

$$\widetilde{\sigma}(a,x) = \langle \mathcal{L}_h(\sigma)(h_{a,x}), h_{a,x}\rangle_{L^2_{A_k}(\mathbb{R}^d)}, \quad (a,x) \in \mathbb{R}^{d+1}_+.$$

Proof. Since σ is in $L^1_{\mu_k}(\mathbb{R}^{d+1}_+)^W$, by Theorem 3.10 $\mathcal{L}_h(\sigma)$ is in S_1. Using [41, Theorem 2.2], there exists an orthonormal basis $\{u_j,\ j=1,2,\dots\}$ for $N(\mathcal{L}_h(\sigma))^\perp$, the orthogonal complement of the kernel of $\mathcal{L}_h(\sigma)$, consisting of eigenvectors of $|\mathcal{L}_h(\sigma)|$, and $\{v_j,\ j=1,2,\dots\}$ an orthonormal set in $L^2_{A_k}(\mathbb{R}^d)^W$, such that

$$\mathcal{L}_h(\sigma)(f)=\sum_{j=1}^{\infty} s_j\langle f,u_j\rangle_{L^2_{A_k}(\mathbb{R}^d)}v_j, \tag{3.8}$$

where $s_j,\ j=1,2,\dots$, are the positive singular values of $\mathcal{L}_h(\sigma)$ corresponding to u_j. Then we get

$$\|\mathcal{L}_h(\sigma)\|_{S_1}=\sum_{j=1}^{\infty}s_j=\sum_{j=1}^{\infty}\langle\mathcal{L}_h(\sigma)(u_j),v_j\rangle_{L^2_{A_k}(\mathbb{R}^d)}.$$

Thus, by Fubini's theorem, Schwarz's inequality, Bessel's inequality, and the relations (2.6) and (2.7), we get

$$\begin{aligned}
\|\mathcal{L}_h(\sigma)\|_{S_1}&=\sum_{j=1}^{\infty}\langle\mathcal{L}_h(\sigma)(u_j),v_j\rangle_{L^2_{A_k}(\mathbb{R}^d)}\\
&=\sum_{j=1}^{\infty}\frac{1}{C_h}\int_{\mathbb{R}^{d+1}_+}\sigma(a,x)\Phi^W_h(u_j)(a,x)\overline{\Phi^W_h(v_j)(a,x)}\,d\mu_k(a,x)\\
&\leqslant\frac{1}{C_h}\int_{\mathbb{R}^{d+1}_+}|\sigma(a,x)|\Big(\sum_{j=1}^{\infty}|\Phi^W_h(u_j)(a,x)|^2\Big)^{\frac12}\Big(\sum_{j=1}^{\infty}|\Phi^W_h(v_j)(a,x)|^2\Big)^{\frac12}\,d\mu_k(a,x)\\
&\leqslant\frac{1}{C_h}\int_{\mathbb{R}^{d+1}_+}|\sigma(a,x)|\|h_{a,x}\|^2_{L^2_{A_k}(\mathbb{R}^d)}\,d\mu_k(a,x)\\
&\leqslant\frac{1}{C_h}\|\sigma\|_{\mu_k,1}.
\end{aligned}$$

It is easy to see that $\widetilde{\sigma}$ belongs to $L^1_{A_k}(\mathbb{R}^d)$, and using formula (3.8) we obtain

$$\begin{aligned}
|\widetilde{\sigma}(a,x)|&=\Big|\langle\mathcal{L}_h(\sigma)(h_{a,x}),h_{a,x}\rangle_{L^2_{A_k}(\mathbb{R}^d)}\Big|\\
&=\Big|\sum_{j=1}^{\infty}s_j\langle h_{a,x},u_j\rangle_{L^2_{A_k}(\mathbb{R}^d)}\langle v_j,h_{a,x}\rangle_{L^2_{A_k}(\mathbb{R}^d)}\Big|\\
&\leqslant\frac12\sum_{j=1}^{\infty}s_j\big(|\langle h_{a,x},u_j\rangle_{L^2_{A_k}(\mathbb{R}^d)}|^2+|\langle h_{a,x},v_j\rangle_{L^2_{A_k}(\mathbb{R}^d)}|^2\big).
\end{aligned}$$

Then using Plancherel's identity for Φ_h^W and Fubini's theorem, we get

$$\begin{aligned}\int_{\mathbb{R}_+^{d+1}} |\widetilde{\sigma}(a,x)|\, d\mu_k(a,x) &= \frac{1}{2}\sum_{j=1}^{\infty} s_j \Bigg(\int_{\mathbb{R}_+^{d+1}} |\langle h_{a,x}, u_j\rangle_{L^2_{A_k}(\mathbb{R}^d)}|^2\, d\mu_k(a,x) \\ &\quad + \int_{\mathbb{R}_+^{d+1}} |\langle h_{a,x}, v_j\rangle_{L^2_{A_k}(\mathbb{R}^d)}|^2\, d\mu_k(a,x) \Bigg) \\ &\leqslant C_h \sum_{j=1}^{\infty} s_j = C_h \|\mathcal{L}_h(\sigma)\|_{S_1}.\end{aligned}$$

The proof is complete. □

In the following theorem we give the main result of this section.

Theorem 3.14. *Let σ be in $L^p_{\mu_k}(\mathbb{R}_+^{d+1})^W$, $1 \leqslant p \leqslant \infty$. Then the localization operator*

$$\mathcal{L}_h(\sigma) : L^2_{A_k}(\mathbb{R}^d)^W \to L^2_{A_k}(\mathbb{R}^d)^W$$

is in S_p and we have

$$\|\mathcal{L}_h(\sigma)\|_{S_p} \leqslant \left(\frac{1}{C_h}\right)^{\frac{1}{p}} \|\sigma\|_{\mu_k,p}.$$

Moreover, $\mathcal{L}_h(\sigma)$ satisfies the relation (3.2).

Proof. The result follows from Proposition 3.8, Theorem 3.13 and by interpolation (see [41, Theorem 2.10 and Theorem 2.11]). □

4. Generalized wavelet scalograms

4.1. **The range of the wavelet transform.** We denote by

- $P_h : L^2_{\mu_k}(\mathbb{R}_+^{d+1})^W \to L^2_{\mu_k}(\mathbb{R}_+^{d+1})^W$ the orthogonal projection from $L^2_{\mu_k}(\mathbb{R}_+^{d+1})^W$ onto $\Phi_h^W(L^2_{A_k}(\mathbb{R}^d)^W)$;
- $P_U : L^2_{\mu_k}(\mathbb{R}_+^{d+1})^W \to L^2_{\mu_k}(\mathbb{R}_+^{d+1})^W$ the orthogonal projection from $L^2_{\mu_k}(\mathbb{R}_+^{d+1})^W$ onto the subspace of functions supported in the subset $U \subset \mathbb{R}_+^{d+1}$. In other words, we can write

  $$P_U F = \chi_U F, \quad F \in L^2_{\mu_k}(\mathbb{R}_+^{d+1})^W,$$

  where χ_U denotes the characteristic function of the subset U of $\mathbb{R}_+^{d+1}$.

Let h be a generalized wavelet on $\mathbb{R}^d$ in $L^1_{A_k}(\mathbb{R}^d)^W \cap L^2_{A_k}(\mathbb{R}^d)^W$. In this section we shall keep our focus on localization operators $\mathcal{L}_h(\sigma)$ with symbol $\sigma = \chi_U$, where U is a subset of $\mathbb{R}_+^{d+1}$ with finite measure $\mu_k(U)$ given by

$$\mu_k(U) := \int_U d\mu_k(a,x).$$

In what follows, such operator will be denoted $\mathcal{L}_h(U)$ for the sake of simplicity.

Proposition 4.1. *The space $\Phi_h^W(L^2_{A_k}(\mathbb{R}^d)^W)$ is a reproducing kernel Hilbert space with kernel*
$$\mathcal{K}_h(a',x';a,x) := \frac{1}{C_h}\int_{\mathbb{R}^d} h_{a',x'}(y)\overline{h_{a,x}(y)}A_k(y)\,dy,$$
which satisfies
$$\forall\,(a',x'),\,(a,x)\in\mathbb{R}^{d+1}_+,\quad |\mathcal{K}_h(a',x';a,x)|\le \frac{\|h\|^2_{L^2_{A_k}(\mathbb{R}^d)}}{C_h}.$$

Proof. Let f be in $L^2_{A_k}(\mathbb{R}^d)^W$. We have
$$\Phi_h^W(f)(a,x)=\int_{\mathbb{R}^d} f(y)\overline{h_{a,x}(y)}A_k(y)\,dy,\quad (a,x)\in\mathbb{R}^{d+1}_+.$$
Using the relation (2.9), we obtain
$$\Phi_h^W(f)(a,x)=\frac{1}{C_h}\int_{\mathbb{R}^{d+1}_+}\Phi_h^W(f)(a',x')\overline{\Phi_h^W(h_{a,x})(a',x')}\,d\mu_k(a',x').$$
On the other hand, by using Proposition 2.12, one can see that for every $a,a'>0$, $x,x'\in\mathbb{R}^d$ the function
$$x'\mapsto \Phi_h^W(h_{a,x})(a',x')=\frac{1}{C_h}\int_{\mathbb{R}^d} h_{a',x'}(y)\overline{h_{a,x}(y)}A_k(y)\,dy$$
belongs to $L^2_{A_k}(\mathbb{R}^d)^W$. Therefore, we obtain the result. □

Remark 4.2. i) We note that
$$\|P_UP_h\|_{HS}:=\left(\int_{\mathbb{R}^{d+1}_+\times\mathbb{R}^{d+1}_+}|\chi_U(a,x)|^2\,|\mathcal{K}_h(a',x';a,x)|^2\,d\mu_k(a',x')\,d\mu_k(a,x)\right)^{\frac12}$$
$$\le \frac{\|h\|_{L^2_{A_k}(\mathbb{R}^d)^W}}{\sqrt{C_h}}\sqrt{\mu_k(U)}<\infty.$$
Hence, P_UP_h is a Hilbert–Schmidt operator and therefore it is a compact operator.

ii) We note that $P_h=\Phi_h^W(\Phi_h^W)^*$. Explicitly, P_h is the integral operator
$$P_hF(z)=\int_{\mathbb{R}^{d+1}_+}F(a,x)\mathcal{K}_h(z;a,x)\,d\mu_k(a,x),\quad z=(a',x')\in\mathbb{R}^{d+1}_+,$$
with integral kernel $\mathcal{K}_h$.

iii) As $\mathcal{K}_h$ is the integral kernel of an orthogonal projection, it satisfies
$$\mathcal{K}_h(z;z')=\overline{\mathcal{K}_h(z';z)},\quad \text{for all } z,z'\in\mathbb{R}^{d+1}_+,$$
and
$$\mathcal{K}_h(z;z')=\int_{\mathbb{R}^{d+1}_+}\mathcal{K}_h(z;z'')\mathcal{K}_h(z'';z')\,d\mu_k(z''),\quad z,z'\in\mathbb{R}^{d+1}_+. \tag{4.1}$$

iv) If $\{v_n : n\in\mathbb{N}\}$ is an orthonormal basis of $\Phi_h^W(L^2_{A_k}(\mathbb{R}^d)^W)$, $\mathcal{K}_h$ can be expanded as
$$\mathcal{K}_h(z;z')=\sum_{n=1}^{\infty}v_n(z)\overline{v_n(z')},\quad z,z'\in\mathbb{R}^{d+1}_+.$$

Definition 4.3. Let h be a generalized wavelet on $\mathbb{R}^d$ in $L^2_{A_k}(\mathbb{R}^d)^W$. We define the generalized wavelet scalogram of f as

$$\mathbf{S}_h^W(f)(a,x) = \frac{1}{C_h}|\Phi_h^W(f)(a,x)|^2, \quad (a,x) \in \mathbb{R}_+^{d+1}.$$

Remark 4.4. From the Plancherel formula of Φ_h^W, we have

$$\int_{\mathbb{R}_+^{d+1}} \mathbf{S}_h^W(f)(a,x)\, d\mu_k(a,x) = \|f\|^2_{L^2_{A_k}(\mathbb{R}^d)}.$$

It justifies the interpretation of a scalogram as a time-frequency energy density. Note that also by (3.2) we have

$$\langle \mathcal{L}_h(\sigma)f, f\rangle_{L^2_{A_k}(\mathbb{R}^d)} = \int_{\mathbb{R}_+^{d+1}} \sigma(a,x)\mathbf{S}_h^W(f)(a,x)\, d\mu_k(a,x).$$

Definition 4.5. We define the Calderón–Toeplitz operator

$$T_{h,U} : \Phi_h^W(L^2_{A_k}(\mathbb{R}^d)^W) \to \Phi_h^W(L^2_{A_k}(\mathbb{R}^d)^W)$$

by

$$T_{h,U}\, F = P_h P_U F.$$

Proposition 4.6. *The operator* $T_{h,U} : L^2_{A_k}(\mathbb{R}^d)^W \to L^2_{A_k}(\mathbb{R}^d)^W$ *is trace class and satisfies*

$$0 \le T_{h,U} \le P_U \le I \tag{4.2}$$

and

$$T_{h,U} = \Phi_h^W \mathcal{L}_h(U)(\Phi_h^W)^*. \tag{4.3}$$

Proof. For all $F \in \Phi_h^W(L^2_{A_k}(\mathbb{R}^d)^W)$,

$$\langle T_{h,U}\, F, F\rangle_{\mu_k,2} = \langle P_h(P_U F), F\rangle_{\mu_k,2} = \langle P_U F, F\rangle_{\mu_k,2} = \int_U |F(a,x)|^2\, d\mu_k(a,x).$$

Thus we deduce (4.2), and $T_{h,U}$ is bounded and positive.

Now, we want to prove (4.3). Indeed, using Φ_h^W and $(\Phi_h^W)^*$, the time-frequency localization operator

$$\mathcal{L}_h(U) : L^2_{A_k}(\mathbb{R}^d)^W \to L^2_{A_k}(\mathbb{R}^d)^W$$

can be written as

$$\mathcal{L}_h(U)(f) = (\Phi_h^W)^*(P_U \Phi_h^W f), \quad f \in L^2_{A_k}(\mathbb{R}^d)^W.$$

Therefore

$$(\Phi_h^W \mathcal{L}_h(U)(\Phi_h^W)^*)F = P_h P_U F = T_{h,U}\, F, \quad F \in \Phi_h^W(L^2_{A_k}(\mathbb{R}^d)^W).$$

Consequently, the time-frequency operator $\mathcal{L}_h(U)$ and the Calderón–Toeplitz operator $T_{h,U}$ are related by

$$T_{h,U} = \Phi_h^W \mathcal{L}_h(U)(\Phi_h^W)^*. \qquad \square$$

Remark 4.7. From the above proposition we deduce that $T_{h,U}$ and $\mathcal{L}_h(U)$ enjoy the same spectral properties; in particular, we have the following proposition.

Proposition 4.8. *The Calderón–Toeplitz operator $T_{h,U}$ is compact and even trace class with*
$$\operatorname{tr}(T_{h,U}) = \operatorname{tr}(\mathcal{L}_h(U)) = M_k(h,U),$$
where
$$M_k(h,U) := \frac{1}{C_h^2}\int_U \|h_{a,x}\|^2_{L^2_{A_k}(\mathbb{R}^d)} d\mu_k(a,x).$$

Proof. We know that the operator $T_{h,U} : \Phi_h^W(L^2_{A_k}(\mathbb{R}^d)^W) \to \Phi_h^W(L^2_{A_k}(\mathbb{R}^d)^W)$ is bounded and positive. Now, let $\{\phi_n\}_{n=1}^\infty$ be an arbitrary orthonormal basis for $\Phi_h^W(L^2_{A_k}(\mathbb{R}^d)^W)$. If we denote by $\varphi_n = \sqrt{C_h}(\Phi_h^W)^*(\phi_n)$, then $\{\varphi_n\}_{n=1}^\infty$ is an orthonormal basis for $L^2_{A_k}(\mathbb{R}^d)^W$.

Thus by (3.2) and Fubini's theorem,
$$\begin{aligned}
\sum_{n=1}^\infty \langle T_{h,U}(\phi_n),\phi_n\rangle_{\mu_k,2} &= \sum_{n=1}^\infty \left\langle \mathcal{L}_h(U)(\Phi_h^W)^*(\phi_n),(\Phi_h^W)^*(\phi_n)\right\rangle_{L^2_{A_k}(\mathbb{R}^d)}\\
&= \frac{1}{C_h^2}\sum_{n=1}^\infty\int_U |\Phi_h^W(\varphi_n)(a,x)|^2\,d\mu_k(a,x)\\
&= \frac{1}{C_h^2}\int_U\sum_{n=1}^\infty |\Phi_h^W(\varphi_n)(a,x)|^2\,d\mu_k(a,x)\\
&= \frac{1}{C_h^2}\int_U\sum_{n=1}^\infty |\langle\varphi_n,h_{a,x}\rangle_{L^2_{A_k}(\mathbb{R}^d)}|^2\,d\mu_k(a,x)\\
&= \frac{1}{C_h^2}\int_U \|h_{a,x}\|^2_{L^2_{A_k}(\mathbb{R}^d)}\,d\mu_k(a,x)\\
&= M_k(h,U).
\end{aligned}$$
Therefore, by Definition 3.3 and Remark 3.4, the operator $T_{h,U}$ is trace class with
$$\|T_{h,U}\|_{S_1} = \operatorname{tr}(T_{h,U}) = M_k(h,U). \qquad \square$$

Let $\mathbf{V}_{h,U} : L^2_{\mu_k}(\mathbb{R}_+^{d+1})^W \to L^2_{\mu_k}(\mathbb{R}_+^{d+1})^W$ be the operator defined by $\mathbf{V}_{h,U} = P_hP_UP_h$. The advantage of $\mathbf{V}_{h,U}$ compared to $T_{h,U}$ is that it is defined on $L^2_{\mu_k}(\mathbb{R}_+^{d+1})^W$ and consequently its spectral properties can be easily related to its integral kernel.

Since $T_{h,U}$ is positive and trace class, using the decomposition
$$L^2_{\mu_k}(\mathbb{R}_+^{d+1})^W = \Phi_h^W(L^2_{A_k}(\mathbb{R}^d)^W)\oplus\left(\Phi_h^W(L^2_{A_k}(\mathbb{R}^d)^W)\right)^\perp$$
we deduce that $\mathbf{V}_{h,U}$ is also positive and trace class with
$$\operatorname{tr}(\mathbf{V}_{h,U}) = \operatorname{tr}(T_{h,U}) = M_k(h,U).$$
In addition, we have the following result.

Proposition 4.9. *The trace of $T^2_{h,U}$ is given by*
$$\operatorname{tr}(T^2_{h,U}) = \int_U\int_U |\mathcal{K}_h(a,x;a',x')|^2\,d\mu_k(a,x)\,d\mu_k(a',x').$$

Proof. As $\mathbf{V}_{h,U}$ is positive, we have

$$\mathrm{tr}(T_{h,U}^2) = \mathrm{tr}(\mathbf{V}_{h,U}^2).$$

On the other hand, using the fact that the space $\Phi_h^W(L^2_{A_k}(\mathbb{R}^d)^W)$ is a reproducing kernel Hilbert space with kernel $\mathcal{K}_h$, we get that, for $F \in L^2_{\mu_k}(\mathbb{R}_+^{d+1})^W$,

$$\begin{aligned}&\mathbf{V}_{h,U}F(a,x)\\ &= \int_{\mathbb{R}_+^{d+1}} F(a',x') \int_{\mathbb{R}_+^{d+1}} \chi_U(b,y)\mathcal{K}_h(a,x;b,y)\mathcal{K}_h(b,y;a',x')\, d\mu_k(b,y)\, d\mu_k(a',x').\end{aligned}$$

That is, $\mathbf{V}_{h,U}$ has integral kernel

$$\mathbf{N}_{h,U}(a,x;a',x') = \int_{\mathbb{R}_+^{d+1}} \chi_U(b,y)\mathcal{K}_h(a,x;b,y)\mathcal{K}_h(b,y;a',x')\, d\mu_k(b,y).$$

Therefore,

$$\begin{aligned}\mathrm{tr}(\mathbf{V}_{h,U}^2) &= \int_{\mathbb{R}_+^{d+1}} \int_{\mathbb{R}_+^{d+1}} |\mathbf{N}_{h,U}(a,x;a',x')|^2\, d\mu_k(a,x)\, d\mu_k(a',x')\\ &= \int_{\mathbb{R}_+^{d+1}} \int_{\mathbb{R}_+^{d+1}} \mathbf{N}_{h,U}(a,x;a',x')\overline{\mathbf{N}_{h,U}(a,x;a',x')}\, d\mu_k(a,x)\, d\mu_k(a',x')\\ &= \int_{\mathbb{R}_+^{d+1}} \int_{\mathbb{R}_+^{d+1}} \chi_U(z_1)\chi_U(z_2)\mathbf{K}_h(z_1;z_2)\, d\mu_k(z_1)\, d\mu_k(z_2),\end{aligned}$$

where by using the properties of the kernel of the reproducing kernel Hilbert space

$$\mathbf{K}_h(z_1;z_2) = \mathcal{K}_h(z_2;z_1)\mathcal{K}_h(z_1;z_2).$$

Using (4.1), we get

$$\mathbf{K}_h(z_1;z_2) = |\mathcal{K}_h(z_1;z_2)|^2,$$

and we conclude the proof. □

4.2. **Eigenvalues and eigenfunctions.** Since the localization operator $\mathcal{L}_h(U) = (\Phi_h^W)^*\chi_U\Phi_h^W$ that we consider is a compact and self-adjoint operator, the spectral theorem gives the following spectral representation:

$$\mathcal{L}_h(U)(f) = \sum_{n=1}^{\infty} s_n(U) \left\langle f, \varphi_n^U \right\rangle_{L^2_{A_k}(\mathbb{R}^d)} \varphi_n^U, \quad f \in L^2_{A_k}(\mathbb{R}^d)^W,$$

where $\{s_n(U)\}_{n=1}^{\infty}$ are the positive eigenvalues arranged in a nonincreasing manner and $\{\varphi_n^U\}_{n=1}^{\infty}$ is the corresponding orthonormal set of eigenfunctions. Note that $s_n(U) \searrow 0$ and by (3.3) we have, for all $n \geq 1$,

$$s_n(U) \leq s_1(U) \leq 1.$$

By using this, together with (4.3), we can deduce that the Toeplitz operator

$$T_{h,U} : \Phi_h^W(L^2_{A_k}(\mathbb{R}^d)^W) \to \Phi_h^W(L^2_{A_k}(\mathbb{R}^d)^W)$$

can be diagonalized as

$$T_{h,U}\, F = C_h \sum_{n=1}^{\infty} s_n(U) \left\langle F, \phi_n^U \right\rangle_{\mu_k,2} \phi_n^U, \quad F \in \Phi_h^W(L^2_{A_k}(\mathbb{R}^d)^W),$$

where $\phi_n^U = \frac{1}{\sqrt{C_h}} \Phi_h^W(\varphi_n^U)$.

Lemma 4.10. *For all $z = (a, x) \in \mathbb{R}_+^{d+1}$, we have*

$$\Theta(z) := \int_{\mathbb{R}_+^{d+1}} \chi_U(\omega) |\mathcal{K}_h(\omega; z)|^2 \, d\mu_k(\omega) = C_h \sum_{n=1}^{\infty} s_n(U) \mathbf{S}_h^W(\varphi_n^U)(z).$$

Proof. From Proposition 4.1, we have that for all $z = (a, x) \in \mathbb{R}_+^{d+1}$ the function $\mathcal{K}_h(.; z)$ is in $\Phi_h^W(L^2_{A_k}(\mathbb{R}^d)^W)$. Therefore using the properties of the kernel of the reproducing kernel Hilbert space, we get

$$\begin{aligned}
\langle T_{h,U}\, \mathcal{K}_h(.; z), \mathcal{K}_h(.; z)\rangle_{\mu_k,2} &= \langle P_U \mathcal{K}_h(.; z), \mathcal{K}_h(.; z)\rangle_{\mu_k,2} \\
&= \int_{\mathbb{R}_+^{d+1}} \chi_U(\omega)\, \mathcal{K}_h(\omega; z)\, \overline{\mathcal{K}_h(\omega; z)}\, d\mu_k(\omega) \\
&= \int_{\mathbb{R}_+^{d+1}} \chi_U(\omega)\, |\mathcal{K}_h(\omega; z)|^2 \, d\mu_k(\omega).
\end{aligned}$$

Let $\{w_n^U\}_{n=1}^{\infty} \subset \Phi_h^W(L^2_{A_k}(\mathbb{R}^d)^W)$ be an orthonormal basis of $\mathrm{Ker}(T_{h,U})$ (possibly empty). Hence, $\{\phi_n^U\}_{n=1}^{\infty} \cup \{w_n^U\}_{n=1}^{\infty}$ is an orthonormal basis of $\Phi_h^W(L^2_{A_k}(\mathbb{R}^d)^W)$ and therefore the reproducing kernel $\mathcal{K}_h$ can be written as

$$\mathcal{K}_h(a, x; a', x') = \overline{\mathcal{K}_h(a', x'; z)} = \sum_{n=1}^{\infty} \phi_n^U(z) \overline{\phi_n^U(a', x')} + \sum_{n=1}^{\infty} w_n^U(z)\, \overline{w_n^U(a', x')}.$$

Using this, we compute again

$$\begin{aligned}
\langle T_{h,U}\, \mathcal{K}_h(.; z), \mathcal{K}_h(.; z)\rangle_{\mu_k,2} &= \left\langle T_{h,U} \sum_{n=1}^{\infty} \overline{\phi_n^U(z)} \phi_n^U, \sum_{k=1}^{\infty} \overline{\phi_k^U(z)} \phi_k^U \right\rangle_{\mu_k,2} \\
&= \sum_{n,k} \overline{\phi_n^U(z)} \phi_k^U(z) \left\langle T_{h,U}\, \phi_n^U, \phi_k^U \right\rangle_{\mu_k,2} \\
&= C_h \sum_{n=1}^{\infty} s_n(U) \left| \phi_n^U(z) \right|^2,
\end{aligned}$$

and the conclusion follows. □

Let $\varepsilon \in (0, 1)$ and define the quantity

$$n(\varepsilon, U) := \mathrm{card}\, \{n : s_n(U) \geq 1 - \varepsilon\}.$$

Then by an easy adaptation of the proof of Lemma 3.3 in [1] we obtain the following estimate for the eigenvalue distribution.

Proposition 4.11. *Let $\varepsilon \in (0,1)$. We have*

$$|n(\varepsilon,U) - M_k(h,U)| \\ \leq \max\left\{\frac{1}{\varepsilon}, \frac{1}{1-\varepsilon}\right\} \left| \int_U \int_U |\mathcal{K}_h(z';z)|^2 \, d\mu_k(z) d\mu_k(z') - M_k(h,U) \right|.$$

4.3. **Scalogram of a subspace.** Given an N-dimensional subspace V of $L^2_{A_k}(\mathbb{R}^d)$, P_V is the orthogonal projection onto V with projection kernel $\mathcal{G}_V$, i.e.,

$$P_V f(\cdot) = \int_{\mathbb{R}^d} \mathcal{G}_V(\cdot,t) f(t) A_k(t)\, dt.$$

Recall that if $\{v_n\}_{n=1}^N$ is an orthonormal basis of V, then

$$\mathcal{G}_V(x,t) = \sum_{n=1}^N v_n(x)\overline{v_n(t)}.$$

The kernel $\mathcal{G}_V$ is independent of the choice of orthonormal basis for V.

Definition 4.12. The scalogram of the space V with generalized wavelet h is defined as

$$\mathbf{SCAL}_h^k V(a,x) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{G}_V(t,y)\overline{h_{a,x}(t)} h_{a,x}(y) A_k(t) A_k(y)\, dt\, dy.$$

Then we have the following result.

Lemma 4.13. *The scalogram $\mathbf{SCAL}_h^k V$ is given by*

$$\mathbf{SCAL}_h^k V = C_h \sum_{n=1}^N \mathbf{S}_h^W(v_n).$$

Proof. We have

$$\begin{aligned} \mathbf{SCAL}_h^k V(a,x) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sum_{n=1}^N v_n(t)\overline{v_n(y) h_{a,x}(t)} h_{a,x}(y) A_k(t) A_k(y)\, dt\, dy \\ &= \sum_{n=1}^N \langle v_n, h_{a,x}\rangle_{L^2_{A_k}(\mathbb{R}^d)} \overline{\langle v_n, h_{a,x}\rangle}_{L^2_{A_k}(\mathbb{R}^d)} \\ &= \sum_{n=1}^N \Phi_h^W(v_n)(a,x)\overline{\Phi_h^W(v_n)(a,x)} \\ &= \sum_{n=1}^N |\Phi_h^W(v_n)(a,x)|^2. \end{aligned}$$ □

Definition 4.14. We define the time-frequency concentration of a subspace V in U as

$$\xi_{U,h}(V) := \frac{1}{N} \int_U \mathbf{SCAL}_h^k V(a,x)\, d\mu_k(a,x).$$

Then, from Lemma 4.13,

$$\xi_{U,h}(V) := \frac{C_h}{N} \sum_{n=1}^{N} \int_U \mathbf{S}_h^W(v_n)(a,x)\, d\mu_k(a,x).$$

Theorem 4.15. *The N-dimensional signal space $V_N = \operatorname{span}\{\varphi_n^U\}_{n=1}^N$ consisting of the first N eigenfunctions of $\mathcal{L}_h(U)$ corresponding to the N largest eigenvalues $\{s_n(U)\}_{n=1}^N$ maximize the regional concentration $\xi_{U,h}(V)$ and*

$$\xi_{U,h}(V_N) := \frac{C_h}{N} \sum_{n=1}^{N} s_n(U).$$

Proof. We have

$$\xi_{U,h}(V_N) := \frac{C_h}{N} \sum_{n=1}^{N} \int_U \mathbf{S}_h^W(\varphi_n^U)(a,x)\, d\mu_k(a,x).$$

Moreover, the min-max lemma for self-adjoint operators states that (cf. [36])

$$\begin{aligned} s_n(U) &= \int_U \mathbf{S}_h^W(\varphi_n^U)(z)\, d\mu_k(a,x) \\ &= \max\left\{ \langle \mathcal{L}_h(U)(f), f\rangle_{L^2_{A_k}(\mathbb{R}^d)} : \|f\|_{L^2_{A_k}(\mathbb{R}^d)} = 1, f \perp \varphi_1^U, \dots, \varphi_{n-1}^U \right\}. \end{aligned}$$

So the eigenvalues of $\mathcal{L}_h(U)$ determine the number of orthogonal functions that have a well-concentrated scalogram in U. Thus,

$$\xi_{U,h}(V_N) = \frac{C_h}{N} \sum_{n=1}^{N} s_n(U).$$

The min-max characterization of the eigenvalues of compact operators implies that the first N eigenfunctions of the time-frequency operator $\mathcal{L}_h(U)$ have optimal cumulative time-frequency concentration inside U, in the sense that

$$\begin{aligned} &\sum_{n=1}^{N} \left\langle \mathcal{L}_h(U)(\varphi_n^U), \varphi_n^U \right\rangle_{L^2_{A_k}(\mathbb{R}^d)} \\ &\qquad = \max\left\{ \sum_{n=1}^{N} \langle \mathcal{L}_h(U) v_n, v_n\rangle_{L^2_{A_k}(\mathbb{R}^d)} : \{v_n\}_{n=1}^N \text{ orthonormal} \right\}. \end{aligned}$$

Therefore no N-dimensional subset V of $L^2_{A_k}(\mathbb{R}^d)$ can be better concentrated in U than V_N, i.e.,

$$\xi_{U,h}(V) \le \xi_{U,h}(V_N).$$

The proof is complete. □

Remark 4.16. The time-frequency concentration of a subspace V_N in U satisfies

$$s_N(U) \le C_h^{-1} \xi_{U,h}(V_N) \le s_1(U) \le 1.$$

4.4. **Accumulated scalogram.** Let $\rho_{(h,U)} := \mathbf{SCAL}_h^k V_{N_k(h,U)}$, called the accumulated scalogram, where we assume that $N_k(h,U) = [M_k(h,U)]$ is the smallest integer greater than or equal to $M_k(h,U)$ and

$$V_{N_k(h,U)} = \operatorname{span}\{v_n^U\}_{n=1}^{N_k(h,U)}.$$

Then

$$\rho_{(h,U)}(a,x) = \sum_{n=1}^{N_k(h,U)} |\Phi_h^W(v_n^U)(a,x)|^2 = \sum_{n=1}^{N_k(h,U)} |\phi_n^U(a,x)|^2.$$

Note that

$$\|\rho_{(h,U)}\|_{1,\mu_k} = C_h N_k(h,U) = C_h M_k(h,U) + O(1).$$

Moreover, since

$$\sum_{n=1}^{N_k(h,U)} s_n(U) \leq \operatorname{tr}(\mathcal{L}_h(U)) = M_k(h,U)$$

then we can define the quantity

$$E(h,U) := 1 - \frac{\sum_{n=1}^{N_k(h,U)} s_n(U)}{M_k(h,U)},$$

which satisfies

$$0 \leq E(h,U) \leq 1.$$

More precisely, we have the following result.

Lemma 4.17. *Let $\varepsilon \in (0,1)$. We have*

$$0 \leq E(h,U) \leq 1 - (1-\varepsilon)\min\left(1, \frac{n(\varepsilon,U)}{M_k(h,U)}\right).$$

Proof. Let $\varepsilon \in (0,1)$ and define $l_k(\varepsilon,U) = \min(N_k(h,U), n(\varepsilon,U))$. It follows that

$$s_n(U) \geq 1-\varepsilon, \quad 1 \leq n \leq l_k(\varepsilon,U).$$

As $N_k(h,U) \geq l_k(h,U)$, we get

$$\sum_{n=1}^{N_k(h,U)} s_n(U) \geq \sum_{n=1}^{l_k(\varepsilon,U)} s_n(U) \geq (1-\varepsilon) l_k(\varepsilon,U).$$

Therefore

$$0 \leq E(h,U) \leq 1 - (1-\varepsilon)\frac{l_k(\varepsilon,U)}{M_k(h,U)}.$$

As $N_k(\varepsilon,U) \geq M_k(h,U)$, we obtain the desired result. ☐

Consequently, when the eigenvalues $\{s_n(U)\}_{n=0}^{n(\varepsilon,U)}$ are close to 1, $E(h,U) \to 0$. Moreover, we have the following result bounding the error between $\rho_{(h,U)}$ and Θ.

Proposition 4.18. *We have*

$$\frac{1}{M_k(h,U)}\|\rho_{(h,U)} - \Theta\|_{1,\mu_k} \leq \frac{C_h}{M_k(h,U)} + 2C_h E(h,U). \tag{4.4}$$

Proof. From Lemma 4.10, we have, for all $z = (a, x) \in U$,

$$\rho_{(h,U)}(z) - \Theta(z) = \sum_{n=1}^{\infty}(t_n - s_n(U))|\phi_n^U(z)|^2,$$

where $t_n = 1$ if $n \leq N_k(h, U)$ and 0 otherwise. As

$$\||\phi_n^U|^2\|_{1,\mu_k} = C_h \quad \text{and} \quad \sum_{n=1}^{\infty} s_n(U) = M_k(h, U),$$

we get

$$\begin{aligned}
\|\rho_{(h,U)} - \Theta\|_{1,\mu_k} &\leq C_h \sum_{n=1}^{\infty} |t_n - s_n(U)| \\
&= C_h \sum_{n=1}^{N_k(h,U)} (1 - s_n(U)) + C_h \sum_{n > N_k(h,U)} s_n(U) \\
&= C_h N_k(h, U) + C_h \sum_{n=1}^{\infty} s_n(U) - 2C_h \sum_{n=1}^{N_k(h,U)} s_n(U) \\
&= C_h N_k(h, U) + C_h M_k(h, U) - 2C_h \sum_{n=1}^{N_k(h,U)} s_n(U) \\
&= C_h \Big(N_k(h, U) - M_k(h, U)\Big) + 2C_h \Big(M_k(h, U) - \sum_{n=1}^{N_k(h,U)} s_n(U)\Big) \\
&\leq C_h + 2C_h \Big(M_k(h, U) - \sum_{n=1}^{N_k(h,U)} s_n(U)\Big),
\end{aligned}$$

and the estimate (4.4) follows. □

Acknowledgements

The authors are deeply indebted to the referees for providing constructive comments and help in improving the contents of this article. The first author thanks professor M. W. Wong for his assistance.

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Received: July 14, 2018
Accepted: June 27, 2019