

CLONES FROM COMONOIDS

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ABSTRACT. We revisit the fact that the cocommutative comonoids in a symmetric monoidal category form the best possible approximation by a cartesian category, now considering the case where the original category is only braided monoidal. This leads to the question of when the endomorphism operad of a comonoid is a clone (a Lawvere theory). By giving an explicit example, we prove that this does not imply that the comonoid is cocommutative.

1. INTRODUCTION

Clones are a special type of operads. Most authors define and study the concept in the category **Set** of sets in terms of structure maps and axioms as in Definition 3.11 below, see e.g. [12]. However, if one defines operads as multicategories with a single object, then clones are simply operads which, as a multicategory, are cartesian. In particular, they are equivalent to Lawvere theories, see e.g. [6, 9, 1].

The fact that clones are firmly rooted in the world of cartesian categories explains why they occur naturally in areas such as logic, set theory, discrete mathematics and theoretical computer science. As cartesian categories can also be characterised in terms of comonoid structures on their objects (see Theorem 1.2 below), we were wondering whether the endomorphism operads of suitable comonoids in braided monoidal categories are also clones. At first, the following seems to indicate that one must restrict to cocommutative comonoids in symmetric monoidal categories:

Theorem 1.1. *Let $\mathbf{C}$ be a braided monoidal category, $\text{Com}(\mathbf{C})$ be the category of comonoids in $\mathbf{C}$, and $(X, \Delta_X, \varepsilon_X)$ be a comonoid. Then the monoidal subcategory of $\text{Com}(\mathbf{C})$ formed by the tensor powers $X^{\otimes n}$ is cartesian if and only if X is cocommutative and the braiding on X is a symmetry.*

In this case, the operation

$$\varphi \bullet (\psi_1, \dots, \psi_m) := \varphi \circ (\psi_1 \otimes \dots \otimes \psi_m) \circ \Delta_{X^{\otimes n}}^{m-1} \quad (1.1)$$

and the morphisms

$$\pi_{i,n} := \varepsilon_X \otimes \dots \otimes \varepsilon_X \otimes \text{id}_X \otimes \varepsilon_X \otimes \dots \otimes \varepsilon_X \quad (1.2)$$

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define a clone structure on the endomorphism operad of X in $\mathbf{Com}(\mathbf{C})$. Here, $\varphi: X^{\otimes m} \rightarrow X$, $\psi_j: X^{\otimes n} \rightarrow X$ are comonoid morphisms and $\Delta_{X^{\otimes n}}$ is the canonical comultiplication on $X^{\otimes n}$ which gets applied $m - 1$ times; see Theorem 3.12 for details.

However, the endomorphism operad of X can be cartesian even if the monoidal category generated by X is not. We will illustrate this by giving a simple example of a comonoid in $\mathbf{Ab}^{\text{op}}$ (i.e. of a unital associative ring) whose endomorphism operad becomes a clone in the above way although it is not cocommutative, see Theorem 4.2. So we think it is interesting to ask:

Question. *For which comonoids is the endomorphism operad a clone?*

The purpose of this paper is to raise rather than to answer this question. Approaching it from various angles has not led to any necessary or sufficient conditions for such comonoids. Even for very small ones, computing their endomorphism operad explicitly is rather intricate, as the example we give shows, and we are not aware of a general method to construct such examples.

Although Theorem 1.1 follows easily from some standard results on cartesian categories, we also felt it worthwhile to write a self-contained exposition of this fact, as examples of endomorphism clones of cocommutative comonoids (or of commutative monoids viewed in the opposite category) and their subclones have not been explored much. The main ingredient in the proof is the statement that a cartesian category is the same as a symmetric monoidal category in which every object is in a unique way a cocommutative comonoid. To the best of our knowledge, this is originally due to Fox [5] (see e.g. [8, Theorem 4.28] for a textbook that discusses the result). However, all the references we are aware of start with a symmetric monoidal category. As we were interested in a generalisation to braided monoidal categories, we reformulate this result in a way that focuses entirely on comonoids and comonoid morphisms. The symmetry of the braiding and the cocommutativity of the comonoids are viewed rather as a side-effect:

Theorem 1.2. *A monoidal category $\mathbf{D}$ is cartesian if and only if there exists a braided monoidal category $\mathbf{C}$ such that*

- (1) *$\mathbf{D}$ is a monoidal subcategory of the category $\mathbf{Com}(\mathbf{C})$ and*
- (2) *the counit and comultiplication of every object of $\mathbf{D}$ are morphisms in $\mathbf{D}$.*

In this case, the canonical symmetry of $\mathbf{D}$ is the restriction of the braiding of $\mathbf{C}$, and all comonoids in $\mathbf{D}$ are cocommutative.

The original version of this result in [5] was stated as an adjunction: passing from a symmetric monoidal category $\mathbf{C}$ to its category $\mathbf{cCom}(\mathbf{C})$ of cocommutative comonoids defines a right adjoint to the forgetful functor from the category of cartesian categories to the category of symmetric monoidal categories. The above theorem reflects the fact that the forgetful functor from cartesian to braided monoidal categories does not have a right adjoint. This can be seen for example by noticing that the direct product of cartesian categories is also a coproduct in the category of cartesian categories, but it is not a coproduct in the category of braided monoidal categories.

Note that there is also a left adjoint to the forgetful functor from cartesian to symmetric monoidal categories, which can be constructed out of the free cartesian category and the free symmetric monoidal category functors on the category **Cat** of categories. As shown by Curien [4], the corresponding monads on **Cat** extend to the category of profunctors which is self-dual, and if one considers the monads as comonads, one obtains a neat uniform characterisation of clones and symmetric operads as monoids in their co-Kleisli categories. Using the free monoidal category functor also yields an analogous description of non-symmetric operads; see also [19] for a different unified approach to clones and operads.

Here is a brief outline of the paper: the aim of Section 2 is to discuss the relation between cartesian structures on categories and comonoid structures on their objects, starting from semi-cartesian categories (monoidal categories whose unit object is terminal) and ending with the proofs of Theorems 1.1 and 1.2. The following Section 3 recalls the definition of a clone as a cartesian operad, the action of the category of finite cardinals on clones, and the clone structure on the endomorphism operad of an object in a cartesian category. Up to here, the paper is rather expository and does not contain novel results. Its main new contribution is the example discussed in detail in the final Section 4 of the paper. Here we will consider unital associative, but not necessarily commutative rings as comonoids in $\mathbf{Ab}^{\text{op}}$. In particular, we give the example of a noncocommutative comonoid whose endomorphism operad is a clone in Theorem 4.2. We also prove at the end of Section 3 that this phenomenon can never occur for comonoids that are Hopf monoids, see Proposition 3.14.

Throughout the paper we assume the reader is familiar with basic category theory including the definitions of monoidal, braided monoidal and symmetric monoidal categories as given e.g. in [8]. To shorten the presentation, we assume that all monoidal categories are strict.

2. COCOMMUTATIVE COMONIDS AND CARTESIAN CATEGORIES

Throughout this section, $(\mathbf{C}, \otimes, \mathbf{1})$ is a (strict) monoidal category. The main goal is to recall the definition of cartesian categories and their characterisation in terms of cocommutative comonoid structures on their objects.

2.1. Semi-cartesian and cartesian categories. We begin by considering semi-cartesian categories.

Definition 2.1. One calls $\mathbf{C}$ *semi-cartesian* if $\mathbf{1}$ is terminal.

Proposition 2.2. *A monoidal category is semi-cartesian if and only if it admits a natural transformation $\varepsilon_X : X \rightarrow \mathbf{1}$ such that $\varepsilon_{\mathbf{1}} = \text{id}_{\mathbf{1}}$. In particular, $\mathbf{C}$ admits at most one such natural transformation.*

Proof. See [8, Proposition 4.15]. □

Definition 2.3. Let $\mathbf{C}$ be semi-cartesian.

- (1) The natural transformation ε from Proposition 2.2 is called the *uniform deletion* in $\mathbf{C}$.

(2) For any pair of objects X and Y , we denote by

$$\pi_{XY}^1: X \otimes Y \rightarrow X, \quad \pi_{XY}^2: X \otimes Y \rightarrow Y$$

the *canonical projections* given by

$$X \xleftarrow{\text{id}_X \otimes \varepsilon_Y} X \otimes Y \xrightarrow{\varepsilon_X \otimes \text{id}_Y} Y.$$

In general, these canonical projections do not have the universal property that makes $(X \otimes Y, \pi_{XY}^1, \pi_{XY}^2)$ a product of X and Y in $\mathbf{C}$.

Definition 2.4. A *cartesian* category is a semi-cartesian category $\mathbf{C}$ in which for any objects X, Y in $\mathbf{C}$, the triple $(X \otimes Y, \pi_{XY}^1, \pi_{XY}^2)$ is a categorical product of X and Y in $\mathbf{C}$, so that for any pair of morphisms $f: Z \rightarrow X$ and $g: Z \rightarrow Y$, there is a unique morphism

$$f * g: Z \rightarrow X \otimes Y$$

making the following diagram commutative:

$$\begin{array}{ccccc} & & Z & & \\ & f \swarrow & \downarrow f * g & \searrow g & \\ X & \xleftarrow{\pi_{XY}^1} & X \otimes Y & \xrightarrow{\pi_{XY}^2} & Y \end{array}.$$

Remark 2.5. The above definition is usually referred to as *cartesian monoidal* in the literature in order to distinguish it from the other uses of the term “cartesian category”.

We now shift the perspective on this property: rather than fixing X, Y , we fix an object Z and show that the universal property of $X \otimes Y$ hinges, for morphisms with domain Z , on the existence of a counital comagma structure on Z :

Definition 2.6. A *comagma* in a monoidal category $\mathbf{C}$ is an object Z together with a morphism $\Delta: Z \rightarrow Z \otimes Z$. A *counital comagma* is a comagma together with a morphism $\varepsilon: Z \rightarrow \mathbf{1}$ rendering the following diagram commutative:

$$\begin{array}{ccccc} & & Z & & \\ & \text{id}_Z \swarrow & \downarrow \Delta & \searrow \text{id}_Z & \\ Z & \xleftarrow{\text{id}_Z \otimes \varepsilon} & Z \otimes Z & \xrightarrow{\varepsilon \otimes \text{id}_Z} & Z \end{array}$$

The following lemma addresses the existence part of the universal property of a categorical product; the uniqueness and naturality will be discussed afterwards.

Lemma 2.7. *Let $\mathbf{C}$ be a semi-cartesian category with uniform deletion ε . If a morphism $\Delta: Z \rightarrow Z \otimes Z$ is counital with respect to ε_Z , then the maps*

$$\begin{aligned} *_{X,Y}: \mathbf{C}(Z, X) \times \mathbf{C}(Z, Y) &\rightarrow \mathbf{C}(Z, X \otimes Y), \\ (f, g) &\mapsto f * g := (f \otimes g) \circ \Delta \end{aligned}$$

make the following diagram commutative:

$$\begin{array}{ccccc}
 & & Z & & \\
 & f \swarrow & \downarrow f * g & \searrow g & \\
 X & \xleftarrow{\pi_{XY}^1} & X \otimes Y & \xrightarrow{\pi_{XY}^2} & Y
 \end{array} \quad . \quad (2.1)$$

This establishes a bijection between counital comagma structures on Z and natural transformations

$$\mathbf{C}(Z, -) \times \mathbf{C}(Z, -) \rightarrow \mathbf{C}(Z, - \otimes -)$$

with this property.

Proof. If Δ is counital, then $*_{X,Y}$ has the desired property since (2.1) expands to

$$\begin{array}{ccccc}
 & & Z & & \\
 & \text{id}_Z \swarrow & \downarrow \Delta & \searrow \text{id}_Z & \\
 Z & \xleftarrow{\pi_{ZZ}^1} & Z \otimes Z & \xrightarrow{\pi_{ZZ}^2} & Z \\
 f \downarrow & & \downarrow f \otimes g & & \downarrow g \\
 X & \xleftarrow{\pi_{XY}^1} & X \otimes Y & \xrightarrow{\pi_{XY}^2} & Y
 \end{array}$$

The naturality of $*_{X,Y}$ follows immediately from its definition. Conversely, Δ is recovered from $*$ as $\Delta = \text{id}_Z * \text{id}_Z$. $\square$

Just as we did for semi-cartesian categories, we are going to characterise cartesian categories in terms of some natural transformations.

Theorem 2.8. *A semi-cartesian category $\mathbf{C}$ is cartesian if and only if there exists a counital natural transformation $\Delta_X: X \rightarrow X \otimes X$ such that for any two objects X, Y , we have*

$$(\pi_{XY}^1 \otimes \pi_{XY}^2) \circ \Delta_{X \otimes Y} = \text{id}_X \otimes \text{id}_Y. \quad (2.2)$$

If such a natural transformation exists, it is unique.

Proof. “ $\Rightarrow$ ”: Assume $\mathbf{C}$ is cartesian. For each object Z , set

$$\Delta_Z := \text{id}_Z * \text{id}_Z.$$

This morphism is counital by construction. Furthermore, Lemma 2.7 implies that $f * g = (f \otimes g) \circ \Delta_Z$ holds for any $f: Z \rightarrow X$ and $g: Z \rightarrow Y$.

Let us show that the family of morphisms $(\Delta_Z)_{Z \in \text{Ob}(\mathbf{C})}$ is a natural transformation. Given a morphism $f: X \rightarrow Y$, we deduce from the commutative diagrams

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 & & X & & \\
 & f \swarrow & \downarrow f & \searrow f & \\
 & Y & & Y & \\
 \text{id}_Y \swarrow & & \downarrow \Delta_Y & & \searrow \text{id}_Y \\
 Y & \xleftarrow{\pi_{YY}^1} & Y \otimes Y & \xrightarrow{\pi_{YY}^2} & Y
 \end{array} & \text{and} & \begin{array}{ccccc}
 & & X & & \\
 & f \swarrow & \downarrow (f \otimes f) \circ \Delta_X & \searrow f & \\
 & Y & & Y & \\
 & \xleftarrow{\pi_{YY}^1} & Y \otimes Y & \xrightarrow{\pi_{YY}^2} & Y
 \end{array}
 \end{array}$$

that $\Delta_Y \circ f = f * f = (f \otimes f) \circ \Delta_X$, hence Δ is indeed a natural transformation.

Finally, (2.2) holds, as the diagram

$$\begin{array}{ccccc}
 & & X \otimes Y & & \\
 & \swarrow \pi_{XY}^1 & \downarrow \text{id}_X \otimes \text{id}_Y & \searrow \pi_{XY}^2 & \\
 X & \xleftarrow{\pi_{XY}^1} & X \otimes Y & \xrightarrow{\pi_{XY}^2} & Y
 \end{array}$$

commutes for any two objects X, Y in $\mathbf{C}$, which means that

$$\text{id}_X \otimes \text{id}_Y = \pi_{XY}^1 * \pi_{XY}^2 = (\pi_{XY}^1 \otimes \pi_{XY}^2) \circ \Delta_{X \otimes Y}.$$

“ $\Leftarrow$ ”: Let us now prove the converse, so assume there exists a counital natural transformation Δ satisfying (2.2). By Lemma 2.7, the morphism $(f \otimes g) \circ \Delta_Z$ satisfies the universal property for any objects X, Y, Z and morphisms $f: Z \rightarrow X$ and $g: Z \rightarrow Y$. Let us show that this morphism is unique. Let $h: Z \rightarrow X \otimes Y$ be a morphism such that $\pi_{XY}^1 \circ h = f$ and $\pi_{XY}^2 \circ h = g$. Then we have

$$\begin{aligned}
 h &= (\text{id}_X \otimes \text{id}_Y) \circ h \\
 &= (\pi_{XY}^1 \otimes \pi_{XY}^2) \circ \Delta_{X \otimes Y} \circ h \\
 &= (\pi_{XY}^1 \otimes \pi_{XY}^2) \circ (h \otimes h) \circ \Delta_Z \\
 &= [(\pi_{XY}^1 \circ h) \otimes (\pi_{XY}^2 \circ h)] \circ \Delta_Z \\
 &= (f \otimes g) \circ \Delta_Z.
 \end{aligned}$$

In particular, taking $f = g = \text{id}_Z$ shows that Δ_Z is unique. $\square$

Especially in the context of theoretical computer science, this is often rephrased by saying that a cartesian category is a semi-cartesian category with a natural *uniform copying* operation Δ . For example, the no-cloning theorem (quantum computers cannot copy information [8, Theorem 4.27]) implies that the monoidal category of finite-dimensional Hilbert spaces is not cartesian.

2.2. Categories of counital comagmas. We now study the category of counital comagmas further.

Definition 2.9. A *morphism of comagmas* $f: (X, \Delta_X) \rightarrow (Y, \Delta_Y)$ is a morphism $f: X \rightarrow Y$ in $\mathbf{C}$ which makes the following diagram commutative:

$$\begin{array}{ccc}
 X & \xrightarrow{\Delta_X} & X \otimes X \\
 f \downarrow & & \downarrow f \otimes f \\
 Y & \xrightarrow{\Delta_Y} & Y \otimes Y.
 \end{array}$$

If X and Y are counital, with counits ε_X and ε_Y respectively, then f is *counital* if the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \varepsilon_X & \downarrow \varepsilon_Y \\ & & \mathbf{1}. \end{array}$$

We denote the category of all counital comagmas by $\text{Comag}(\mathbf{C})$.

Note that in general, the tensor product of two comagmas carries no canonical comagma structure. However, $\text{Comag}(\mathbf{C})$ becomes monoidal if $\mathbf{C}$ is braided monoidal (see e.g. [8, Definition 1.17] for more background on braided monoidal categories):

Definition 2.10. A *braiding* on $\mathbf{C}$ is a natural isomorphism

$$\sigma_{X,Y}: X \otimes Y \rightarrow Y \otimes X$$

which is monoidal both in X and Y , that is, for which

$$\begin{aligned} \sigma_{X \otimes Y, Z} &= (\sigma_{X,Z} \otimes \text{id}_Y) \circ (\text{id}_X \otimes \sigma_{Y,Z}) \\ \sigma_{X, Y \otimes Z} &= (\text{id}_Y \otimes \sigma_{X,Z}) \circ (\sigma_{X,Y} \otimes \text{id}_Z) \end{aligned}$$

holds. A *symmetric monoidal category* is a braided monoidal category whose braiding is a *symmetry*, meaning that for all objects X and Y , we have

$$\sigma_{Y,X} \circ \sigma_{X,Y} = \text{id}_{X \otimes Y}.$$

From now on we assume that $\mathbf{C}$ is braided monoidal.

Proposition 2.11. *The braiding σ on $\mathbf{C}$ induces a unique monoidal structure on $\text{Comag}(\mathbf{C})$ such that the forgetful functor $\text{Comag}(\mathbf{C}) \rightarrow \mathbf{C}$ is strict monoidal.*

Proof. This is well known; the tensor product $X \otimes Y$ of two counital comagmas $(X, \Delta_X, \varepsilon_X)$ and $(Y, \Delta_Y, \varepsilon_Y)$ becomes a counital comagma whose counit and comultiplication are given as follows:

$$\begin{aligned} \varepsilon_{X \otimes Y} &:= \varepsilon_X \otimes \varepsilon_Y, \quad \text{and} \\ \Delta_{X \otimes Y} &:= (\text{id}_X \otimes \sigma_{X,Y} \otimes \text{id}_Y) \otimes (\Delta_X \otimes \Delta_Y). \end{aligned} \quad \square$$

We have moved some proofs to the Appendix, where we use the standard graphical calculus of string diagrams. For example, the above counit and comultiplication on $X \otimes Y$ are given in (5.1).

Proposition 2.12. *The category $\text{Comag}(\mathbf{C})$ is semi-cartesian.*

Proof. The unit object $\mathbf{1}$ is easily seen to carry a unique structure of a counital comagma, and the counit $\varepsilon_X: X \rightarrow \mathbf{1}$ of any counital comagma is a morphism of counital comagmas. This defines a uniform deletion in $\text{Comag}(\mathbf{C})$. $\square$

Here are two properties of the tensor product of comagmas that will be used later:

Proposition 2.13. *Let X, Y be two counital comagmas in $\mathbf{C}$. The following equalities hold in $\mathbf{C}$:*

- (1) $[(\mathrm{id}_X \otimes \varepsilon_Y) \otimes (\varepsilon_X \otimes \mathrm{id}_Y)] \circ \Delta_{X \otimes Y} = \mathrm{id}_X \otimes \mathrm{id}_Y.$
- (2) $[(\varepsilon_X \otimes \mathrm{id}_Y) \otimes (\mathrm{id}_X \otimes \varepsilon_Y)] \circ \Delta_{X \otimes Y} = \sigma_{X,Y}.$

Proof. See p. 390 in the Appendix. $\square$

So we have seen that for a braided monoidal category $\mathbf{C}$, the category $\mathrm{Comag}(\mathbf{C})$ is monoidal. However, in general it is not braided monoidal:

Proposition 2.14. *Let X, Y be two counital comagmas in $\mathbf{C}$. The braiding $\sigma_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ is a morphism of counital comagmas if and only if $\sigma_{Y,X} = \sigma_{X,Y}^{-1}.$*

Proof. See p. 390 in the Appendix. $\square$

2.3. Cocommutative comonoids. In order to prepare for the characterisation of cartesian categories we collect here everything we need to know about cocommutative comonoids.

Definition 2.15. We denote by $\mathrm{Com}(\mathbf{C}) \subseteq \mathrm{Comag}(\mathbf{C})$ the full subcategory of *comonoids*, that is, of counital comagmas (X, Δ, ε) for which Δ is *coassociative*,

$$(\Delta \otimes \mathrm{id}_Z) \circ \Delta = (\mathrm{id}_Z \otimes \Delta) \circ \Delta.$$

We denote by $\mathrm{cCom}(\mathbf{C}) \subseteq \mathrm{Com}(\mathbf{C})$ the full subcategory of comonoids which are *cocommutative*,

$$\Delta = \sigma_{X,X} \circ \Delta.$$

We thus have the sequence of forgetful functors

$$\mathrm{cCom}(\mathbf{C}) \subseteq \mathrm{Com}(\mathbf{C}) \subseteq \mathrm{Comag}(\mathbf{C}) \rightarrow \mathbf{C},$$

where the two inclusions are full.

Proposition 2.16. *$\mathrm{Com}(\mathbf{C})$ is a semi-cartesian subcategory of $\mathrm{Comag}(\mathbf{C})$.*

Proof. The unit object is a comonoid, and $\mathrm{Com}(\mathbf{C})$ contains all the terminal morphisms since it is a full subcategory. Finally, it is straightforward to verify that the tensor product of two comonoids in $\mathrm{Comag}(\mathbf{C})$ is again coassociative. $\square$

The main ingredient for the proof of Theorem 1.2 is the next proposition which is known as the *Eckmann–Hilton argument*. In the introduction we said that we view cocommutativity rather as a side-effect, and this is where this statement becomes manifest:

Proposition 2.17. *Let $X_1 := (X, \Delta_1, \varepsilon_1)$ and $X_2 := (X, \Delta_2, \varepsilon_2)$ be two counital comagma structures defined on the same underlying object X . If Δ_2 is a morphism of comagmas $X_1 \rightarrow X_1 \otimes X_1$, then*

$$\varepsilon_1 = \varepsilon_2, \quad \Delta_1 = \Delta_2,$$

and $X_1 = X_2$ is a cocommutative comonoid.

Proof. See p. 391 in the Appendix. $\square$

In other words, we have

$$\text{Comag}(\text{Comag}(\mathbf{C})) = \text{cCom}(\mathbf{C}).$$

The following is a direct corollary of the Eckmann–Hilton argument:

Corollary 2.18. *Let X be a counital comagma. The comultiplication Δ_X is a morphism of counital comagmas if and only if X is cocommutative. In this case, $(X, \Delta_X, \varepsilon_X)$ is the unique counital comagma structure on X in $\text{Comag}(\mathbf{C})$.*

Proof. The “only if” direction follows directly from the Eckmann–Hilton argument. The “if” part is shown in Proposition 5.1 in the Appendix. $\square$

Thus $\text{cCom}(\mathbf{C})$ is the category of counital comagmas in a monoidal category which is not braided. Viewed from this perspective, the following extension of Proposition 2.14 from comagmas to cocommutative comonoids is rather natural, albeit surprising at first (as pointed out by Baez [2, Lemma 3]):

Proposition 2.19. *The tensor product of two cocommutative comonoids X, Y is cocommutative if and only if $\sigma_{Y,X} = \sigma_{X,Y}^{-1}$.*

Proof. This is a direct corollary of Proposition 5.2. $\square$

2.4. Recognition theorem. We will now discuss the canonical symmetry on a cartesian category and then prove Theorems 1.1 and 1.2.

Proposition 2.20. *If $\mathbf{C}$ is cartesian, then the natural transformation*

$$\sigma_{X,Y} := \pi_{XY}^2 * \pi_{XY}^1$$

is the unique braiding on $\mathbf{C}$, and is a symmetry.

Proof. The morphisms $\sigma_{X,Y}$ are defined using the composition and tensor products of natural transformations. They are therefore natural in X and Y . Using the universal property, it is straightforward to verify that σ is a symmetry.

If τ is any braiding on $\mathbf{C}$, then unit constraints force the diagram

$$\begin{array}{ccccc} & & X \otimes Y & & \\ & \swarrow \pi_{XY}^2 & \downarrow \tau_{X,Y} & \searrow \pi_{XY}^1 & \\ Y & \xleftarrow{\pi_{YX}^1} & Y \otimes X & \xrightarrow{\pi_{YX}^2} & X \end{array}$$

to commute, hence $\tau_{X,Y} = \pi_{XY}^2 * \pi_{XY}^1 = \sigma_{X,Y}$ by the universal property. $\square$

Proposition 2.21. *Let $\mathbf{C}$ be cartesian, Δ be the uniform copying, and σ be the canonical symmetry. Then, for any $X, Y \in \text{Ob}(\mathbf{C})$ we have*

$$\Delta_{X \otimes Y} = (\text{id}_X \otimes \sigma_{X,Y} \otimes \text{id}_Y) \otimes (\Delta_X \otimes \Delta_Y).$$

Proof. Consider the following diagram:

$$\begin{array}{ccccc}
 & & X \otimes Y & & \\
 & \swarrow \text{id}_{X \otimes Y} & \downarrow \Delta_X \otimes \Delta_Y & \searrow \text{id}_{X \otimes Y} & \\
 X \otimes Y & \xleftarrow{\pi_{XX}^1 \otimes \pi_{YY}^1} & (X \otimes X) \otimes (Y \otimes Y) & \xrightarrow{\pi_{XX}^2 \otimes \pi_{YY}^2} & X \otimes Y \\
 \downarrow \text{id}_{X \otimes Y} & & \downarrow \text{id}_X \otimes \sigma_{X,Y} \otimes \text{id}_Y & & \downarrow \text{id}_{X \otimes Y} \\
 X \otimes Y & \xleftarrow{\pi_{X \otimes Y, X \otimes Y}^2} & (X \otimes Y) \otimes (X \otimes Y) & \xrightarrow{\pi_{X \otimes Y, X \otimes Y}^1} & X \otimes Y
 \end{array}$$

The first row in this diagram commutes because of the universal property of Δ_X and Δ_Y , whereas the second row commutes because of the universal property of $\sigma_{X,Y}$. The morphisms along the border of this diagram actually form the diagram defining the universal property of $\Delta_{X \otimes Y}$:

$$\begin{array}{ccccc}
 & & X \otimes Y & & \\
 & \swarrow \text{id}_{X \otimes Y} & \downarrow \Delta_{X \otimes Y} & \searrow \text{id}_{X \otimes Y} & \\
 X \otimes Y & \xleftarrow{\pi_{X \otimes Y, X \otimes Y}^1} & (X \otimes Y) \otimes (X \otimes Y) & \xrightarrow{\pi_{X \otimes Y, X \otimes Y}^2} & X \otimes Y
 \end{array}$$

thus the equality

$$\Delta_{X \otimes Y} = (\text{id}_X \otimes \sigma_{X,Y} \otimes \text{id}_Y) \otimes (\Delta_X \otimes \Delta_Y).$$

□

Proposition 2.22. *A braided monoidal category is cartesian if and only if the forgetful functor $\text{Comag}(\mathbf{C}) \rightarrow \mathbf{C}$ is an isomorphism. In this case, we have*

$$\text{cCom}(\mathbf{C}) = \text{Com}(\mathbf{C}) = \text{Comag}(\mathbf{C}) \cong \mathbf{C}.$$

Proof. “ $\Rightarrow$ ” Suppose $\mathbf{C}$ is cartesian. Then the uniform copying Δ and uniform deletion ε induce a functor $G: \mathbf{C} \rightarrow \text{Comag}(\mathbf{C})$

$$G(X) := (X, \Delta_X, \varepsilon_X), \quad G(f) := f$$

which is a section of the forgetful functor $\text{Comag}(\mathbf{C}) \rightarrow \mathbf{C}$.

The previous proposition tells us that G is strict monoidal. Then for any counital comagma (X, δ, ϵ) in $\mathbf{C}$, its image $(G(X), \delta, \varepsilon)$ is a counital comagma in $\text{Comag}(\mathbf{C})$. It follows from Corollary 2.18 that the structure maps δ and ϵ must be equal to those of $G(X) = (X, \Delta_X, \varepsilon_X)$. Hence $G(X)$ is the unique counital comagma structure on X , and G is an isomorphism.

Another consequence of Corollary 2.18 is that $G(X)$ is a cocommutative comonoid for each $X \in \text{Ob}(\mathbf{C})$, therefore

$$\text{cCom}(\mathbf{C}) = \text{Com}(\mathbf{C}) = \text{Comag}(\mathbf{C}) \cong \mathbf{C}.$$

“ $\Leftarrow$ ” Suppose the forgetful functor $\text{Comag}(\mathbf{C}) \rightarrow \mathbf{C}$ is an isomorphism. Since $\text{Comag}(\mathbf{C})$ is semi-cartesian, then so is $\mathbf{C}$.

For each X in $\mathbf{C}$, let us denote by $(X, \Delta_X, \varepsilon_X)$ the image of X by the inverse of the forgetful functor $\text{Comag}(\mathbf{C}) \rightarrow \mathbf{C}$. Then, the maps $X \mapsto \Delta_X$ and $X \mapsto \varepsilon_X$

induce natural transformations which satisfy all the conditions in Theorem 2.8, thus $\mathbf{C}$ is cartesian. $\square$

As a consequence, in a cartesian category, every object is in a unique and canonical way a cocommutative comonoid, and every morphism in $\mathbf{C}$ is a morphism of comonoids.

From now on, we assume that $\mathbf{C}$ is braided monoidal with a fixed braiding σ . Our goal is to characterise the subcategories of $\text{Comag}(\mathbf{C})$ which are cartesian.

Proposition 2.23. *A monoidal subcategory $\mathbf{D}$ of $\text{Comag}(\mathbf{C})$ is cartesian if and only if for any object (X, Δ, ε) in $\mathbf{D}$, the morphisms Δ and ε are in $\mathbf{D}$.*

Proof. “ $\Leftarrow$ ” The maps which associate to each comonoid its comultiplication and counit induce a uniform copying and deletion in $\mathbf{D}$, thus $\mathbf{D}$ is cartesian.

“ $\Rightarrow$ ” Suppose $\mathbf{D}$ is cartesian, and let (X, Δ, ε) be an object in $\mathbf{D}$. Since $\mathbf{D}$ is cartesian, then (X, Δ, ε) is a counital comagma in $\mathbf{D}$. We know however from Corollary 2.18, that this comagma structure is uniquely given by Δ and ε , thus these two morphisms are in $\mathbf{D}$. $\square$

It follows that neither $\text{Comag}(\mathbf{C})$ nor $\text{Com}(\mathbf{C})$ are cartesian in general. Even though $\text{cCom}(\mathbf{C})$ satisfies the conditions in the proposition, it is not cartesian either, since it is not closed under the tensor product in general. We have however the following:

Corollary 2.24. *If $\mathbf{C}$ is symmetric monoidal, then $\text{cCom}(\mathbf{C})$ is cartesian.*

Proof. If σ is a symmetry, then $\text{cCom}(\mathbf{C})$ is a monoidal subcategory of $\text{Com}(\mathbf{C})$, and hence satisfies the conditions in Proposition 2.23. $\square$

This induces the adjunction between symmetric monoidal categories and cartesian categories described originally in [5].

We are now ready to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let X be a comonoid and $\mathbf{D}$ be the full subcategory of $\text{Com}(\mathbf{C})$ whose objects are the tensor powers of X . By construction, the category $\mathbf{D}$ is monoidal.

Proposition 2.23 tells us that $\mathbf{D}$ is cartesian if and only if $\Delta_{X^{\otimes n}}$ and $\varepsilon_{X^{\otimes n}}$ are in $\mathbf{D}$ for all $n \geq 0$. Since $\mathbf{D}$ is full, it contains all the terminal morphisms $\varepsilon_{X^{\otimes n}}$ and the previous condition becomes equivalent to $\Delta_{X^{\otimes n}}$ being a morphism of comonoids for all $n \geq 0$. This means that $\Delta_{X^{\otimes n}}$ has to be cocommutative for all $n \geq 0$ (Corollary 2.18), and this in turn is equivalent to Δ_X being cocommutative and $\sigma_{X,X} = \sigma_{X,X}^{-1}$ (Proposition 2.19). $\square$

Proof of Theorem 1.2. The “if” part follows immediately from Proposition 2.23, and the “only if” part from Proposition 2.22. $\square$

3. CARTESIAN OPERADS AND CLONES

We retain the assumption that $\mathbf{C}$ is a strict monoidal category. Our main goal in this section is to establish an equivalence between cartesian operads and clones.

3.1. Operads. We will define operads as certain multicategories. We refer to [14] for further information.

Definition 3.1. A (plain) multicategory $\mathbf{M}$ consists of

- (1) a class $\text{Ob}(\mathbf{M})$ of *objects*;
- (2) for any $A_1, \dots, A_n, B \in \text{Ob}(\mathbf{M})$ a class $\mathbf{M}(A_1, \dots, A_n; B)$ of *morphisms*;
- (3) for any $A_{11}, \dots, A_{nm_n}, B_1, \dots, B_n, C \in \text{Ob}(\mathbf{M})$ of a *composition*

$$\begin{aligned} & \mathbf{M}(B_1, \dots, B_n; C) \times \mathbf{M}(A_{11}, A_{12}, \dots, A_{1m_1}; B_1) \times \dots \\ & \dots \times \mathbf{M}(A_{n1}, A_{n2}, \dots, A_{nm_n}; B_n) \longrightarrow \mathbf{M}(A_{11}, A_{12}, \dots, A_{nm_n}; C) \\ & (\varphi, \psi_1, \dots, \psi_n) \mapsto \varphi \circ (\psi_1, \dots, \psi_n); \end{aligned}$$

- (4) for each $A \in \text{Ob}(\mathbf{M})$, an *identity morphism* $\text{id}_A \in \mathbf{M}(A; A)$

subject to the obvious associativity and identity axioms.

Definition 3.2. A (plain) operad is a multicategory $\mathbf{O}$ with a single object X . In this case, we denote $\mathbf{O}(\underbrace{X, \dots, X}_{n \text{ times}}; X)$ simply by $\mathbf{O}_n$.

Given a collection of objects $M_0 \subseteq \text{Ob}(\mathbf{C})$ (where $\mathbf{C}$ is as elsewhere a monoidal category), we define the multicategory $\mathbf{M}$ associated to M_0 as follows:

- (1) its class of objects is $\text{Ob}(\mathbf{M}) := M_0$;
- (2) for $A_1, \dots, A_n, B \in M_0$,

$$\mathbf{M}(A_1, \dots, A_n; B) := \mathbf{C}(A_1 \otimes \dots \otimes A_n, B);$$

- (3) the composition is given by $\varphi \circ (\psi_1, \dots, \psi_n) := \varphi \circ (\psi_1 \otimes \dots \otimes \psi_n)$; and
- (4) the identity morphisms are those from $\mathbf{C}$.

If $M_0 = \text{Ob}(\mathbf{C})$, then $\mathbf{M}$ is called the *underlying multicategory* of $\mathbf{C}$. If $M_0 = \{X\}$, then $\mathbf{M}$ is called the *endomorphism operad* of X .

Before we define the analogue of cartesian categories for multicategories, we will need to introduce some notions related to maps between finite sets.

3.2. The category of finite cardinals. Let $\mathbb{F}$ be the category of finite cardinals, whose objects are the finite sets $\underline{n} := \{1, \dots, n\}$, $n \in \mathbb{N}$, and whose morphisms are all possible maps between these sets.

Addition induces a strict monoidal structure on $\mathbb{F}$ that we denote by $\oplus$, where the unit object is the empty set $\underline{0} = \emptyset$. Given two morphisms $f_1: \underline{m}_1 \rightarrow \underline{n}_1$ and $f_2: \underline{m}_2 \rightarrow \underline{n}_2$, their tensor product is

$$f_1 \oplus f_2: \underline{m}_1 \oplus \underline{m}_2 \rightarrow \underline{n}_1 \oplus \underline{n}_2$$

where

$$(f_1 \oplus f_2)(i) := \begin{cases} f_1(i) & 1 \leq i \leq m_1, \\ f_2(i - m_1) + n_1 & m_1 < i \leq m_1 + m_2. \end{cases}$$

We are actually interested in the opposite category $\mathbb{F}^{\text{op}}$. We will call a morphism $f: \underline{n} \rightarrow \underline{m}$ in $\mathbb{F}^{\text{op}}$ a *selection* to clearly distinguish it from the corresponding map $\underline{m} \rightarrow \underline{n}$; we picture f as a way to select an m -tuple with entries taken from a given n -tuple, as in Figure 1.

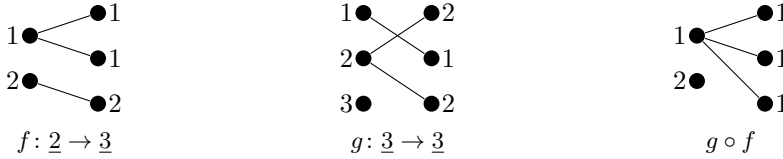


FIGURE 1. Selections

For each $n \in \mathbb{N}$, we denote by $\varepsilon_{\underline{n}}: \underline{n} \rightarrow \underline{0}$ the selection corresponding to the empty map and by $\Delta_{\underline{n}}: \underline{n} \rightarrow \underline{n} \oplus \underline{n}$ the selection which selects the tuple $(1, 2, \dots, n, 1, 2, \dots, n)$. These morphisms define a uniform deletion ε and a uniform copying Δ on the category $\mathbb{F}^{\text{op}}$, making this category cartesian.

3.3. The substitution product. We assume in this subsection that $\mathbf{C}$ is a cartesian category. For each $1 \leq i \leq n$, the i -th canonical projection $X_1 \otimes \dots \otimes X_n \rightarrow X_i$ is given by

$$\pi_{X_1, \dots, X_n}^i := \varepsilon_{X_1} \otimes \dots \otimes \varepsilon_{X_{i-1}} \otimes \text{id}_{X_i} \otimes \varepsilon_{X_{i+1}} \otimes \dots \otimes \varepsilon_{X_n}.$$

Given a selection $f: \underline{n} \rightarrow \underline{m}$, we use these projections to construct a canonical morphism

$$\pi_{X_1, \dots, X_n}^{(f)}: X_1 \otimes \dots \otimes X_n \rightarrow X_{f[1]} \otimes \dots \otimes X_{f[m]},$$

by setting

$$\pi_{X_1, \dots, X_n}^{(f)} := \pi_{X_1, \dots, X_n}^{f[1]} * \dots * \pi_{X_1, \dots, X_n}^{f[m]},$$

where $*$ is as in Definition 2.4 and $f[i]$ denotes the i -th entry of the tuple corresponding to the selection $f: \underline{n} \rightarrow \underline{m}$.

The following is verified by direct computation:

Proposition 3.3. *The operations π have the following properties:*

- (1) $\pi_{X_1, \dots, X_n}^{(\text{id}_{\underline{n}})} = \text{id}_{X_1 \otimes \dots \otimes X_n},$
- (2) $\pi_{X_1, \dots, X_n}^{(g \circ f)} = \pi_{X_{f[1]}, \dots, X_{f[m]}}^{(g)} \circ \pi_{X_1, \dots, X_n}^{(f)},$
- (3) $\pi_{X_1, \dots, X_{n_1}, Y_1, \dots, Y_{n_2}}^{(f_1 \oplus f_2)} = \pi_{X_1, \dots, X_{n_1}}^{(f_1)} \otimes \pi_{Y_1, \dots, Y_{n_2}}^{(f_2)},$
- (4) $\pi_{X_1, \dots, X_n}^{(\Delta_{\underline{n}})} = \Delta_{X_1 \otimes \dots \otimes X_n},$
- (5) $\pi_{X_1, \dots, X_n}^{(\varepsilon_{\underline{n}})} = \varepsilon_{X_1 \otimes \dots \otimes X_n},$
- (6) $\pi_{X_1, \dots, X_n}^{(\pi_{1, \dots, 1}^i)} = \pi_{X_1, \dots, X_n}^i.$

This means that the operations π define a form of action of $\mathbb{F}^{\text{op}}$:

Definition 3.4. Let $f: \underline{n} \rightarrow \underline{m}$ be a selection. Given morphisms $g_i: X_i \rightarrow Y_i$ in $\mathbf{C}$, ($1 \leq i \leq n$), we define the *substitution* of the g_i 's in f to be the morphism

$$f \wr (g_1, \dots, g_n): X_1 \otimes \dots \otimes X_n \rightarrow Y_{f[1]} \otimes \dots \otimes Y_{f[m]}$$

given by

$$f \wr (g_1, \dots, g_n) := (g_{f[1]} \otimes \dots \otimes g_{f[m]}) \circ \pi_{X_1, \dots, X_n}^{(f)}.$$

When applying this to the cartesian category $\mathbf{C} = \mathbb{F}^{\text{op}}$, the substitution $f \wr (g_1, \dots, g_n)$ becomes an internal operation on the morphisms in $\mathbb{F}^{\text{op}}$. Under the composition and this operation, the set of all morphisms in $\mathbb{F}^{\text{op}}$ is generated by the deletion ε_1 , the duplication Δ_1 and the identity id_1 .

Remark 3.5. The category $\mathbb{F}^{\text{op}}$, together with the substitution product, is a particular example of a *cartesian club*, as introduced by Kelly in [11, 10]. In [3], this club was viewed as a 2-monad whose algebras are exactly the categories with finite products.

3.4. Cartesian operads. The following material can be found for example in [6, Section 2.3], [17, Section 2.6] or [18].

Definition 3.6. Let $\mathbf{M}$ be a multicategory. A *cartesian structure* on $\mathbf{M}$ associates to a selection $f: \underline{n} \rightarrow \underline{m}$ and a choice of objects $A_1, \dots, A_n, B \in \text{Ob}(\mathbf{M})$ a map

$$- \cdot f: \mathbf{M}(A_{f[1]}, \dots, A_{f[m]}; B) \rightarrow \mathbf{M}(A_1, \dots, A_n; B),$$

such that the following properties hold:

$$\begin{aligned} (\varphi \cdot g) \cdot f &= \varphi \cdot (g \circ f), & \varphi \cdot \text{id}_{\underline{n}} &= \varphi, \\ (\varphi \cdot f) \circ (\psi_1 \cdot g_1, \dots, \psi_n \cdot g_n) &= [\varphi \circ (\psi_{f[1]}, \dots, \psi_{f[m]})] \cdot [f \wr (g_1, \dots, g_n)]. \end{aligned}$$

A *cartesian multicategory* is a multicategory with a chosen cartesian structure.

Definition 3.7. A *cartesian operad* is a cartesian multicategory with a single object.

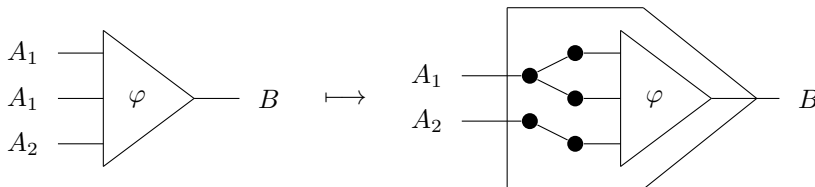


FIGURE 2. $- \cdot f: \mathbf{M}(A_1, A_1, A_2; B) \rightarrow \mathbf{M}(A_1, A_2; B)$

Proposition 3.8. If $\mathbf{C}$ is a cartesian category and $M_0 \subseteq \text{Ob}(\mathbf{C})$, then the associated multicategory is cartesian.

Proof. For any objects $A_1, \dots, A_n, B$ in $\mathbf{C}$ and selection $f: \underline{n} \rightarrow \underline{m}$, precomposing with the morphism $\pi_{A_1, \dots, A_n}^{(f)}$ induces a map

$$- \cdot f: \mathbf{C}(A_{f[1]} \otimes \dots \otimes A_{f[m]}, B) \rightarrow \mathbf{C}(A_1 \otimes \dots \otimes A_n, B)$$

with the desired properties. $\square$

The converse does not hold in general; however, we have:

Proposition 3.9. A monoidal category $\mathbf{C}$ is cartesian if and only if its underlying multicategory is cartesian.

Proof. One implication follows from the previous proposition. For the reverse implication, let us assume that the underlying multicategory is cartesian. Then, we define the uniform deletion ε and the uniform copying Δ by setting $\varepsilon_X := \text{id}_1 \cdot \varepsilon_1$ and $\Delta_X := \text{id}_{X \otimes X} \cdot \Delta_1$ for each $X \in \text{Ob}(\mathbf{C})$. $\square$

Corollary 3.10. *The endomorphism operad of an object in a cartesian category is cartesian.*

3.5. Clones. Clones are usually defined as follows:

Definition 3.11. An (abstract) clone is a collection of sets $\{C_n\}_{n \in \mathbb{N}}$, together with elements

$$\pi_{i,n} \in C_n, \quad i = 1, \dots, n,$$

and maps

$$\bullet : C_m \times (C_n)^m \rightarrow C_n$$

satisfying, for all $\varphi \in C_m$, $\psi_1, \dots, \psi_m \in C_n$, $\rho_1, \dots, \rho_n \in C_l$,

- (1) $\varphi \bullet (\pi_{1,n}, \dots, \pi_{n,n}) = \varphi$,
- (2) $\pi_{i,m} \bullet (\psi_1, \dots, \psi_m) = \psi_i$, and
- (3) $\varphi \bullet (\psi_1 \bullet (\rho_1, \dots, \rho_n), \dots, \psi_m \bullet (\rho_1, \dots, \rho_n)) = (\varphi \bullet (\psi_1, \dots, \psi_m)) \bullet (\rho_1, \dots, \rho_n)$.

As is pointed out e.g. in [6, 9], this concept is just an equivalent way to view cartesian operads:

Theorem 3.12. *Let $\mathbf{O}$ be a cartesian operad with object X . For any $m, n \in \mathbb{N}$, $1 \leq i \leq n$, $\varphi \in \mathbf{O}_m$, $\psi_1, \dots, \psi_m \in \mathbf{O}_n$, define*

$$\pi_{i,n} := \text{id}_X \cdot \pi_{\underline{1}, \dots, \underline{1}}^i, \quad \varphi \bullet (\psi_1, \dots, \psi_m) := (\varphi \circ (\psi_1, \dots, \psi_m)) \cdot \Delta_{\underline{n}}^{m-1},$$

where the map $\Delta_{\underline{n}}^{m-1} : \underline{n} \rightarrow \underline{mn} = \underline{n} \oplus \dots \oplus \underline{n}$ is the uniform copying in $\mathbb{F}^{\text{op}}$ applied $m-1$ times. With these operations, $\{\mathbf{O}_n\}_{n \in \mathbb{N}}$ becomes an abstract clone, and this defines an isomorphism between the categories of cartesian operads and of abstract clones.

Proof. The inverse of the isomorphism is as follows. Let C be an abstract clone. For each $n \in \mathbb{N}$, let $\mathbf{O}_n := C_n$; for any selection $f : \underline{n} \rightarrow \underline{m}$, let $- \cdot f : \mathbf{O}_m \rightarrow \mathbf{O}_n$ with

$$\varphi \cdot f := \varphi \bullet (\pi_{f[1],n}, \dots, \pi_{f[m],n}),$$

and for $\varphi \in \mathbf{O}_n$ and $\psi_i \in \mathbf{O}_{m_i}$ ($i = 1, \dots, n$), let

$$\varphi \circ (\psi_1, \dots, \psi_n) := \varphi \bullet (\psi_1 \cdot \pi_{\underline{m}_1, \dots, \underline{m}_n}^1, \dots, \psi_n \cdot \pi_{\underline{m}_1, \dots, \underline{m}_n}^n),$$

where the map $\pi_{\underline{m}_1, \dots, \underline{m}_n}^i : \underline{m}_1 \oplus \dots \oplus \underline{m}_n \rightarrow \underline{m}_i$ is the canonical projection in $\mathbb{F}^{\text{op}}$. We refer the reader to [6, 9] for the details of the proof. $\square$

Remark 3.13. *Lawvere theories* are yet another perspective on the same concepts. The equivalence is most directly seen when defining them as identity-on-objects functors on $\mathbb{F}^{\text{op}}$ that preserve products. We refer e.g. to [6, 9, 13] for further information.

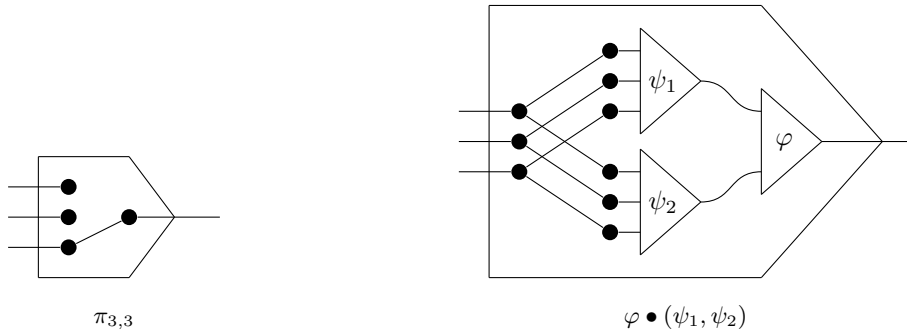


FIGURE 3. From a cartesian operad into a clone

We will now turn to our main aim to construct an object X in a monoidal category $\mathbf{C}$ for which $\{X^{\otimes n}\} \subseteq \mathbf{C}$ is not cartesian, but whose endomorphism operad is cartesian. Our last theoretical result is that for Hopf monoids [8, Definition 6.31], this phenomenon cannot occur:

Proposition 3.14. *Let H be a Hopf monoid in a braided monoidal category $\mathbf{C}$. Then the maps (1.1) and (1.2) given in the introduction turn the endomorphism operad of H in $\text{Com}(\mathbf{C})$ into a clone if and only if H is cocommutative.*

Proof. The implication “ $\Leftarrow$ ” follows from Theorem 1.1, so we prove “ $\Rightarrow$ ”. A Hopf monoid is in particular a bimonoid, that is, a monoid in the monoidal category $\text{Com}(\mathbf{C})$. Thus H comes equipped with a multiplication $\mu: H \otimes H \rightarrow H$ which is a morphism of comonoids. Using the clone product, this means that $\mu \bullet (\text{id}_H, \text{id}_H) = \mu \circ \Delta: H \rightarrow H$ is a morphism of comonoids, too.

Explicitly, this means that

$$(\mu \otimes \mu) \circ (\Delta \otimes \Delta) \circ \Delta = \Delta \circ \mu \circ \Delta.$$

Inserting the fact that μ is a comonoid morphism yields

$$(\mu \otimes \mu) \circ (\Delta \otimes \Delta) \circ \Delta = (\mu \otimes \mu) \circ (\text{id}_H \otimes \sigma_{H,H} \otimes \text{id}_H) \circ (\Delta \otimes \Delta) \circ \Delta. \quad (3.1)$$

To deduce from this the cocommutativity of Δ , we recall that the bimonoid structure on H defines a second monoid structure on the set $\mathbf{C}(H, H)$ (besides the one given by composition), whose product is the convolution

$$\varphi \diamond \psi := \mu \circ (\varphi \otimes \psi) \circ \Delta$$

and whose unit element is $\eta \circ \varepsilon$ (the composition of counit and unit morphism). The sets $\mathbf{C}(H^{\otimes m}, H^{\otimes n})$ carry commuting left and right actions of the monoid $(\mathbf{C}(H, H), \diamond, \eta \circ \varepsilon)$ given by

$$\begin{aligned} \varphi \triangleright \alpha &:= (\mu \otimes \text{id}_{H^{\otimes n-1}}) \circ (\varphi \otimes \alpha) \circ (\Delta \otimes \text{id}_{H^{\otimes m-1}}), \\ \alpha \triangleleft \psi &:= (\text{id}_{H^{\otimes n-1}} \otimes \mu) \circ (\alpha \otimes \psi) \circ (\text{id}_{H^{\otimes m-1}} \otimes \Delta). \end{aligned}$$

With this notation, (3.1) can be rewritten as

$$\text{id}_H \triangleright \Delta \triangleleft \text{id}_H = \text{id}_H \triangleright (\sigma_{H,H} \circ \Delta) \triangleleft \text{id}_H.$$

Finally, a Hopf monoid is by definition a bimonoid for which $\text{id}_H \in \mathbf{C}(H, H)$ is invertible with respect to $\diamond$, so the above implies

$$\Delta = \sigma_{H,H} \circ \Delta. \quad \square$$

4. AN EXAMPLE

Let $\mathbf{Ab}$ be the symmetric monoidal category of abelian groups with $\otimes_{\mathbb{Z}}$ as monoidal structure. The category of comonoids in the opposite category $\mathbf{Ab}^{\text{op}}$ is $\mathbf{Ring}^{\text{op}}$, the opposite of the category of unital associative rings. Our main aim is to provide an explicit example of a ring which viewed as a comonoid in $\mathbf{Ab}^{\text{op}}$ does not generate a cartesian category, yet its endomorphism operad is a clone.

4.1. Motivation. In this section we indicate why and how we were searching for such a ring. Readers who are just interested in the example itself can safely skip this material.

Recall that in algebraic geometry, the full subcategory

$$\mathbf{AffSch} := \text{cCom}(\mathbf{Ab}^{\text{op}}) \subset \mathbf{Ring}^{\text{op}}$$

of commutative rings gets interpreted as the cartesian category of affine schemes, by identifying a commutative ring A with the locally ringed space $X = \text{Spec}(A)$. A morphism of commutative rings $A \rightarrow A^{\otimes n}$ is then the same as a morphism of affine schemes $X \times \cdots \times X \rightarrow X$.

What is special about $\mathbf{Ab}^{\text{op}}$ is that it is monoidal abelian. This allows us to consider deformations of cocommutative comonoids which are not necessarily cocommutative, but are in a sense not far from being so:

Definition 4.1. An *infinitesimal deformation of order n* of a ring B is a surjective ring morphism $A \rightarrow B$ whose kernel $I \triangleleft A$ is an ideal for which $I^n = 0$, that is, for which $a_1 \cdots a_n = 0$ for all $a_1, \dots, a_n \in I$.

An infinitesimal deformation of order 2 is also known as an *abelian* (or *square zero*) *extension*, while the process of completion (taking a limit $n \rightarrow \infty$) yields *formal deformations*. For further background reading, we refer the interested reader to the literature: [16, Section 12.2] discusses the abstract theory of deformations of algebras over algebraic operads, [15, Sections 1.5.3 and 1.5.4] or [20, Section 9.3] contain more information on the specific example of associative rings and algebras, [15, Appendix E] discusses the example of commutative rings and algebras, and [7, Section II.9] provides classical background in algebraic geometry.

We felt it would be natural to test whether the endomorphism operad of such comonoids that are close to being cocommutative have a bigger chance of being cartesian. The first examples of rings that we considered failed, however; see Remark 4.3 below for an explicit one. Proposition 3.14 gave a conceptual explanation for this.

As the computation of the endomorphism operad of a given ring is rather involved, our focus was then on finding rings for which every ring morphism $A \rightarrow A^{\otimes n}$ is of the form $a \mapsto 1 \otimes \cdots \otimes 1 \otimes \sigma(a) \otimes 1 \otimes \cdots \otimes 1$ for a ring endomorphism σ , that

is, a ring whose endomorphism operad is a clone and is as such generated by the endomorphisms of A .

We also failed to find noncommutative rings with this property, but the attempt to construct such rings in terms of generators and relations has led us to the example for the original question described here.

4.2. A deformation of $\mathbb{Z}[\sqrt{q}]$. Let $q \in \mathbb{N}$ be a natural number that we assume is not a square, and let A be the universal ring with generators t, x satisfying

$$t^2 = q, \quad tx = -xt, \quad x^2 = 0.$$

So A is noncommutative (since $tx = -xt$), and the ring morphism

$$A \rightarrow \mathbb{Z}[\sqrt{q}] \cong A/I, \quad I := Ax = xA$$

turns A into a deformation of order 2 of $\mathbb{Z}[\sqrt{q}]$ as in Definition 4.1.

Let $\mathbf{O}_n$ be the set of all ring morphisms $A \rightarrow A^{\otimes n}$ and $\pi_{i,n} \in \mathbf{O}_n$ be given by

$$A \rightarrow A^{\otimes n}, \quad a \mapsto 1 \otimes \cdots \otimes 1 \otimes a \otimes 1 \otimes \cdots \otimes 1.$$

Then the ring $A^{\otimes n}$ is generated by the elements

$$t_i := \pi_{i,n}(t), \quad x_i := \pi_{i,n}(x),$$

and these generators satisfy

$$t_i t_j = t_j t_i, \quad t_i^2 = q, \quad x_i x_j = x_j x_i, \quad x_i^2 = 0,$$

$$t_i x_j = x_j t_i, \quad i \neq j, \quad t_i x_i = -x_i t_i.$$

Finally, let $\varphi \bullet (\psi_1, \dots, \psi_m)$ be given for $\varphi \in \mathbf{O}_m, \psi_j \in \mathbf{O}_n$ by

$$\mu_{A^{\otimes n}}^{m-1} \circ (\psi_1 \otimes \cdots \otimes \psi_m) \circ \varphi. \quad (4.1)$$

The main result of this section is:

Theorem 4.2. *The elements of $\mathbf{O}_n$ are given by*

$$t \mapsto \pm t_d + f x_d, \quad x \mapsto g x_d, \quad tx \mapsto \pm g t_d x_d,$$

where $d \in \{1, \dots, n\}$ and $f, g \in A^{\otimes n}$. Furthermore, $(\mathbf{O}, \bullet, \pi_{i,n})$ is a clone in $\mathbf{Ab}^{\text{op}}$.

Remark 4.3. Taking q to be 1 yields the ring generated by t, x satisfying

$$t^2 = 1, \quad tx = -xt, \quad x^2 = 0.$$

This is also a square zero extension of a commutative ring, but its endomorphism operad is not a clone. Indeed, this is a Hopf algebra, namely the integral version of Sweedler's 4-dimensional Hopf algebra, so the claim follows from Proposition 3.14. Explicitly, the comultiplication is given by

$$\Delta: A \rightarrow A \otimes A, \quad t \mapsto t \otimes t, \quad x \mapsto 1 \otimes x + x \otimes t.$$

4.3. Proof of Theorem 4.2. As an abelian group, $A^{\otimes n}$ is free with basis

$$t_S x_T := \prod_{i \in S} t_i \prod_{j \in T} x_j,$$

where S, T run through all pairs of subsets of $\{1, \dots, n\}$. The product $t_S x_T t_U x_V$ vanishes unless $T \cap V = \emptyset$. In this case,

$$t_S x_T t_U x_V = (-1)^{|T \cap U|} q^{|S \cap U|} t_{S \Delta U} x_{T \cup V}.$$

In order to describe the elements of $\mathbf{O}_n$, we first show:

Lemma 4.4. *The element $a = \sum_{S,T} a_{ST} t_S x_T \in A^{\otimes n}$ satisfies $a^2 = q$ if and only if $a = \pm t_d + f x_d$ for some $f \in A^{\otimes n}$.*

Proof. That the given elements square to q is clear. Assume conversely that

$$a^2 = \sum_{\substack{STUV \\ T \cap V = \emptyset}} a_{ST} a_{UV} (-1)^{|T \cap U|} q^{|S \cap U|} t_{S \Delta U} x_{T \cup V} = q.$$

Consider first the terms in $\mathbb{Z}$, that is, those with $T = V = \emptyset, S = U$:

$$\sum_S a_{S\emptyset}^2 q^{|S|} = q.$$

As q is assumed to be non-square, we have $a_{\emptyset\emptyset}^2 \neq q$. However, the remaining nonzero summands are each one greater or equal to q . Hence there exists some $d \in \{1, \dots, n\}$ such that $a_{\{d\}\emptyset} = \pm 1$ while $a_{S\emptyset} = 0$ for all other S .

Thus a can be uniquely written as $\pm t_d + f x_d + g$, where f, g are linear combinations of $t_S x_T$ with $T \neq \emptyset$ but $d \notin T$. It follows that $t_d f x_d + f x_d t_d = 0$ and $(f x_d)^2 = 0$, so

$$a^2 = q \pm 2t_d g + g^2 + (f x_d g + g f x_d).$$

Therefore, $a^2 = q$ implies $\pm 2t_d g + g^2 + f x_d g + g f x_d = 0$. The term $f x_d g + g f x_d$ is in the ideal generated by x_d while $\pm 2t_d g + g^2$ is a linear combination of $t_S x_T$ with $d \notin T$. Thus the two terms vanish separately.

Note that t_d and g commute and that g is nilpotent, so there exists $m \geq 0$ such that $g^m = 0$. Thus we have

$$\begin{aligned} \pm 2t_d g + g^2 &= (\pm 2t_d + g)g = 0 \\ \Rightarrow (4q - g^2)g &= (\pm 2t_d - g)(\pm 2t_d + g)g = 0 \\ \Rightarrow (16q^2 - g^4)g &= (4q + g^2)(4q - g^2)g = 0 \\ \Rightarrow \dots \\ \Rightarrow 2^{2m} q^m g &= 0, \end{aligned}$$

so $g = 0$. □

Lemma 4.5. *If $f \in A^{\otimes n}$ is arbitrary, then $T := \pm t_d + fx_d$ and $X \in A^{\otimes n}$ anti-commute if and only if $X \in A^{\otimes n}x_d$.*

Proof. Without loss of generality we may assume X is in the span of t_Ux_V with $d \notin V$. Then we have

$$\begin{aligned} 0 &= TX + XT \\ &= \pm t_dX \pm Xt_d + fx_dX + Xfx_d \\ &= \pm 2t_dX + fx_dX + Xfx_d. \end{aligned}$$

The last two terms are in the ideal generated by x_d , so by our assumption on X , the first term vanishes separately, hence $t_dX = 0$. Multiplication by t_d yields the claim. $\square$

As the elements of $A^{\otimes n}x_d$ all square to zero, the above two lemmata imply the first half of Theorem 4.2.

To prove the second half, it suffices to show that the $\mathbb{Z}$ -linear map defined in (4.1) is a ring morphism.

Let us start with the following observation:

Lemma 4.6. *Let $d \in \{1, \dots, n\}$ and $\tau_d: A^{\otimes n} \rightarrow A^{\otimes n}$ be the unique ring morphism such that $\tau_d(t_d) = -t_d$, $\tau_d(x_d) = 0$, $\tau_d(t_i) = t_i$ and $\tau_d(x_i) = x_i$ for $i \neq d$. Then for all $a \in A^{\otimes n}$, we have $x_da = \tau_d(a)x_d$ and $t_d\tau_d(a) = \tau_d(a)t_d$.*

Proof. By linearity of both expressions, it suffices to verify the claim for $a = t_Sx_T$ which is straightforward; we have $\tau_d(t_Sx_T) = 0$ if $d \in T$, have $\tau_d(t_Sx_T) = -t_Sx_T$ if $d \in S$, and $\tau_d(t_Sx_T) = t_Sx_T$ otherwise. $\square$

Lemma 4.7. *Let $d \in \{1, \dots, m\}$, $\psi_1, \dots, \psi_m: A \rightarrow A^{\otimes n}$ be ring morphisms and assume that $e \in \{1, \dots, n\}$ and $f \in A^{\otimes n}$ are such that $\psi_d(t) := t_e + fx_e$. Then,*

- (1) $\mu^{m-1} \circ (\psi_1 \otimes \dots \otimes \psi_m)(t_d) = t_e + fx_e$.
- (2) For any $h \in A^{\otimes m}$, there exists an element $h' \in A^{\otimes n}$ with

$$\begin{aligned} \mu^{m-1} \circ (\psi_1 \otimes \dots \otimes \psi_m)(hx_d) &= h'x_e \\ \mu^{m-1} \circ (\psi_1 \otimes \dots \otimes \psi_m)(ht_dx_d) &= h't_ex_e. \end{aligned}$$

Proof. (1) We have

$$\begin{aligned} (\psi_1 \otimes \dots \otimes \psi_m)(t_d) &= (\psi_1 \otimes \dots \otimes \psi_m)(1 \otimes \dots \otimes t \otimes \dots \otimes 1) \\ &= \psi_1(1) \otimes \dots \otimes \psi_d(t) \otimes \dots \otimes \psi_m(1) \\ &= 1 \otimes \dots \otimes \psi_d(t) \otimes \dots \otimes 1, \end{aligned}$$

hence

$$\mu^{m-1} \circ (\psi_1 \otimes \dots \otimes \psi_m)(t_d) = \psi_d(t) = t_e + fx_e.$$

(2) Let $g \in A^{\otimes n}$ such that $\psi_d(x) := gx_e$ and $\psi_d(tx) := gt_ex_e$. Without loss of generality, we may assume that $h = h_1 \otimes \dots \otimes h_m$ for some $h_1, \dots, h_m \in A$. We

have

$$\begin{aligned}(\psi_1 \otimes \cdots \otimes \psi_m)(hx_d) &= (\psi_1 \otimes \cdots \otimes \psi_m)(h_1 \otimes \cdots \otimes h_dx \otimes \cdots \otimes h_m) \\ &= \psi_1(h_1) \otimes \cdots \otimes \psi_d(h_dx) \otimes \cdots \otimes \psi_m(h_m) \\ &= \psi_1(h_1) \otimes \cdots \otimes \psi_d(h_d)gx_e \otimes \cdots \otimes \psi_m(h_m).\end{aligned}$$

Then

$$\mu^{m-1} \circ (\psi_1 \otimes \cdots \otimes \psi_m)(hx_d) = ax_eb = a\tau_e(b)x_e,$$

where

$$a = \psi_1(h_1) \cdots \psi_d(h_d)g \quad \text{and} \quad b = \psi_{d+1}(h_{d+1}) \cdots \psi_m(h_m).$$

Here τ_e is as in the previous lemma.

Following the same process, we obtain

$$(\psi_1 \otimes \cdots \otimes \psi_m)(ht_dx_d) = \psi_1(h_1) \otimes \cdots \otimes \psi_d(h_d)gt_ex_e \otimes \cdots \otimes \psi_m(h_m),$$

hence

$$\mu^{m-1} \circ (\psi_1 \otimes \cdots \otimes \psi_m)(ht_dx_d) = at_ex_eb = a\tau_e(b)t_ex_e. \quad \square$$

With this we are ready to prove the second half of Theorem 4.2:

End of proof of Theorem 4.2. Let $\varphi: A \rightarrow A^{\otimes m}$ and $\psi_1, \dots, \psi_m: A \rightarrow A^{\otimes n}$ be ring morphisms, and let $\chi := \mu^{m-1} \circ (\psi_1 \otimes \cdots \otimes \psi_m)$. Our goal is to show that the $\mathbb{Z}$ -linear map $\chi \circ \varphi$ is again a ring morphism. Suppose that

$$\varphi(t) = \pm t_d + fx_d, \quad \varphi(x) = gx_d, \quad \varphi(tx) = gt_dx_d$$

for some $d \in \{1, \dots, m\}$ and $f, g \in A^{\otimes m}$, and that $\psi_d(t) := t_e + hx_e$ for some $e \in \{1, \dots, n\}$ and $h \in A^{\otimes n}$. Then

$$\chi \circ \varphi(t) = \pm \chi(t_d) + \chi(fx_d) = \pm t_e + hx_e + f'x_e,$$

$$\chi \circ \varphi(x) = \chi(gx_d) = g'x_e,$$

$$\chi \circ \varphi(tx) = \chi(gt_dx_d) = g't_ex_e,$$

where $f', g' \in A^{\otimes n}$ are as in the previous lemma, hence $\chi \circ \varphi$ is a ring morphism. $\square$

5. APPENDIX: COUNITAL COMAGMAS

In this appendix we prove many of the preliminary results on comagmas using the graphical language of string diagrams when representing morphisms in $\mathbf{C}$. We read these from top to bottom, so a comultiplication $\Delta_X: X \rightarrow X \otimes X$, a counit $\varepsilon_X: X \rightarrow \mathbf{1}$ and the braiding $\sigma_{X,Y}: X \otimes Y \rightarrow Y \otimes X$ will be depicted as follows:

$$\varepsilon_X := \begin{array}{c} X \\ | \\ \circ \end{array}, \quad \Delta_X := \begin{array}{c} X \\ \diagup \quad \diagdown \\ X \quad X \end{array}, \quad \sigma_{X,Y} := \begin{array}{cc} X & Y \\ \diagdown & \diagup \\ Y & X \end{array}.$$

See e.g. [8] for more information. The comultiplication and counit on a tensor product of comagmas are depicted as follows:

$$\begin{array}{c} X \quad Y \\ | \quad | \\ \boxed{\varepsilon_{X \otimes Y}} \end{array} = \begin{array}{c} X \quad Y \\ | \quad | \\ \circ \quad \circ \end{array} \quad \text{and} \quad \begin{array}{c} X \quad Y \\ | \quad | \\ \boxed{\Delta_{X \otimes Y}} \\ | \quad | \quad | \quad | \\ X \quad Y \quad X \quad Y \end{array} := \begin{array}{c} X \quad Y \\ | \quad | \\ \text{curly brace} \quad \text{curly brace} \\ | \quad | \quad | \quad | \\ X \quad Y \quad X \quad Y \end{array}. \quad (5.1)$$

Proposition 2.13. *Let X, Y be two counital comagmas in $\mathbf{C}$. The following equalities hold in $\mathbf{C}$:*

- (1) $[(\text{id}_X \otimes \varepsilon_Y) \otimes (\varepsilon_X \otimes \text{id}_Y)] \circ \Delta_{X \otimes Y} = \text{id}_X \otimes \text{id}_Y.$
- (2) $[(\varepsilon_X \otimes \text{id}_Y) \otimes (\text{id}_X \otimes \varepsilon_Y)] \circ \Delta_{X \otimes Y} = \sigma_{X,Y}.$

Proof.

$$\begin{array}{c} X \quad Y \\ | \quad | \\ \text{curly brace} \quad \text{curly brace} \\ | \quad | \\ X \quad Y \end{array} = \begin{array}{c} X \quad Y \\ | \quad | \\ \circ \quad \circ \end{array} \quad \text{and} \quad \begin{array}{c} X \quad Y \\ | \quad | \\ \text{curly brace} \quad \text{curly brace} \\ | \quad | \\ \circ \quad \circ \end{array} = \begin{array}{c} X \quad Y \\ | \quad | \\ \text{curly brace} \quad \text{curly brace} \\ | \quad | \\ X \quad Y \end{array}. \quad \square$$

Proposition 2.14. *Let X, Y be two counital comagmas in $\mathbf{C}$. The braiding $\sigma_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ is a morphism of counital comagmas if and only if $\sigma_{Y,X} = \sigma_{X,Y}^{-1}.$*

Proof. If $\sigma_{X,Y}$ is a morphism of comagmas, then

$$\Delta_{Y \otimes X} \circ \sigma_{X,Y} = \begin{array}{c} X \quad Y \\ | \quad | \\ \text{curly brace} \quad \text{curly brace} \\ | \quad | \\ Y \quad X \quad Y \quad X \end{array} = \begin{array}{c} X \quad Y \\ | \quad | \\ \text{curly brace} \quad \text{curly brace} \\ | \quad | \\ Y \quad X \quad Y \quad X \end{array} = (\sigma_{X,Y} \otimes \sigma_{X,Y}) \circ \Delta_{X \otimes Y}.$$

This implies that

$$\begin{array}{c} \text{braiding} \end{array} = \begin{array}{c} \text{curly brace} \quad \text{curly brace} \\ | \quad | \\ \circ \quad \circ \end{array} = \begin{array}{c} \text{curly brace} \quad \text{curly brace} \\ | \quad | \\ \circ \quad \circ \end{array} = \begin{array}{c} \text{curly brace} \quad \text{curly brace} \\ | \quad | \\ \circ \quad \circ \end{array} = \begin{array}{c} | \quad | \end{array}.$$

Conversely, if $\sigma_{Y,X} = \sigma_{X,Y}^{-1}$ then

$$\begin{array}{c} \text{curly brace} \quad \text{curly brace} \\ | \quad | \\ \text{braiding} \end{array} = \begin{array}{c} \text{curly brace} \quad \text{curly brace} \\ | \quad | \\ \text{braiding} \end{array} = \begin{array}{c} \text{curly brace} \quad \text{curly brace} \\ | \quad | \\ \text{braiding} \end{array}. \quad \square$$

Proposition 2.17. Let $X_1 := (X, \Delta_1, \varepsilon_1)$ and $X_2 := (X, \Delta_2, \varepsilon_2)$ be two counital comagma structures defined on the same underlying object X . If Δ_2 is a morphism of comagmas $X_1 \rightarrow X_1 \otimes X_1$, then

$$\varepsilon_1 = \varepsilon_2, \quad \Delta_1 = \Delta_2,$$

and $X_1 = X_2$ is a cocommutative comonoid.

Proof. Let $(\Delta_1, \varepsilon_1) = (\text{[diagram]}, \text{[diagram]})$ and $(\Delta_2, \varepsilon_2) = (\text{[diagram]}, \text{[diagram]})$; then Δ_2 being a morphism of comagmas means that

The equality for the counit is obtained as follows:

From here onwards we will omit the colours for the counit. Next comes the equality for the comultiplications:

Now we shall remove the colours for the comultiplications. For the coassociativity, we have:

and for the cocommutativity:

Proposition 5.1. *If a comonoid X is cocommutative, then its comultiplication $\Delta_X: X \rightarrow X \otimes X$ is a morphism in $\text{Com}(\mathbf{C})$.*

Proof. The coassociativity implies that

$$\text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3}$$

Combined with the cocommutativity, it yields:

$$(\Delta_X \otimes \Delta_X) \circ \Delta_X = \text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3} = \text{Diagram 4} = \Delta_{X \otimes X} \otimes \Delta_X. \quad \square$$

Proposition 5.2. *Let X, Y be two comonoids. Their tensor product $X \otimes Y$ is cocommutative if and only if the three conditions in Figure 4 hold.*

(1)

(2)

(3)

FIGURE 4. Conditions for cocommutativity

Proof. Let us assume that $X \otimes Y$ is cocommutative, that is,

$$\sigma_{X \otimes Y, X \otimes Y} \circ \Delta_{X \otimes Y} = \text{Diagram 1} = \text{Diagram 2} = \Delta_{X \otimes Y}.$$

First, we obtain (1) as follows:

$$\text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3} = \text{Diagram 4}.$$

The property (2) is proven in a similar manner. Next, (3) is obtained as follows:

Conversely, if the three conditions are satisfied, then

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