REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 68, No. 2, 2025, Pages 405–422
Published online: August 18, 2025
https://doi.org/10.33044/revuma.4349

# LINEAR FUNCTIONALS AND Δ-COHERENT PAIRS OF THE SECOND KIND

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Abstract. We classify all the Δ-coherent pairs of measures of the second kind on the real line. We obtain five cases, corresponding to all the families of discrete semiclassical orthogonal polynomials of class $s \leq 1$.

## 1. Introduction

The aim of this contribution is to provide a characterization of all pairs of discrete measures $\{\rho_0, \rho_1\}$ supported on the real line, such that the corresponding sequences of *monic orthogonal polynomials* (MOPs for short) $\{P_n(\rho_0; x)\}_{n\geq 0}$ and $\{P_n(\rho_1; x)\}_{n\geq 0}$ satisfy

$$P_n(\rho_1; x) - \tau_n P_{n-1}(\rho_1; x) = \frac{1}{n+1} \Delta P_{n+1}(\rho_0; x), \quad n \geq 1, \tag{1.1}$$

where $\tau_n \neq 0$ for $n \geq 1$ and $\Delta$ denotes the *forward difference operator* defined by

$$\Delta [f(x)] = f(x+1) - f(x). \tag{1.2}$$

We will solve this problem by dealing with a more general situation concerning the characterization of pairs of quasi-definite linear functionals (with complex moments) such that the corresponding sequences of MOPs satisfy (1.1).

As we will show in this contribution, we get special pairs of linear functionals (in particular, positive definite linear functionals associated with positive measures supported on infinite subsets of the real line) and the corresponding associated

2020 *Mathematics Subject Classification.* Primary 42C05; Secondary 33C45.

*Key words and phrases.* discrete orthogonal polynomials, discrete semiclassical functionals, discrete Sobolev inner products, coherent pairs of discrete measures, coherent pairs of the second kind for discrete measures.

Diego Dominici would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme "Applicable resurgent asymptotics: towards a universal theory" (ARA2). The work was supported by EPSRC grant EP/R014604/1. Francisco Marcellán has been supported by FEDER/Ministerio de Ciencia e Innovación – Agencia Estatal de Investigación of Spain, grant PID2021-122154NB-I00, and the Madrid Government (Comunidad de Madrid, Spain) under the Multiannual Agreement with UC3M in the line of Excellence of University Professors, grant EPUC3M23 in the context of the V PRICIT (Regional Program of Research and Technological Innovation).

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sequences of orthogonal polynomials have interesting properties. These pairs are formed with *discrete semiclassical linear functionals* of class at most $s=1$ (for a definition, see (2.9)), which have been studied in [9] and [10] in the framework of the classification problem based on a hierarchy structure.

On the other hand, these sequences of orthogonal polynomials are related to problems in approximation theory. More precisely, the analysis of Fourier expansions in terms of sequences of polynomials orthogonal with respect to the Sobolev inner product

$$\langle f,g\rangle_{\mathfrak{S}}=\sum_{x=0}^{\infty}f(x)g(x)\rho_0(x)+\lambda\sum_{x=0}^{\infty}\Delta f(x)\Delta g(x)\rho_1(x),$$

defined by the pair of discrete measures $\{\rho_0,\rho_1\}$ of class $s\geq 1$. It turns out that the monic Sobolev orthogonal polynomials $S_n(\rho_0,\rho_1;x)$ with respect to the inner product $\langle\,,\,\rangle_{\mathfrak{S}}$ satisfy the connection formulas

$$\begin{aligned}&S_{n+1}(\rho_0,\rho_1;x)-\gamma_n S_n(\rho_0,\rho_1;x)=P_{n+1}(\rho_0;x),\\&\Delta S_{n+1}(\rho_0,\rho_1;x)-\gamma_n\Delta S_n(\rho_0,\rho_1;x)=(n+1)\left[P_n(\rho_1;x)-\tau_n P_{n-1}(\rho_1;x)\right],\end{aligned}$$

for all $n\geq 1$ with $S_1(\rho_0,\rho_1;x)=P_1(\rho_0;x)$.

The above connection formulas between $\{S_n(\rho_0,\rho_1;x)\}_{n\geq 0}$ and the sequences of MOPs $\{P_n(\rho_0;x)\}_{n\geq 0}$ and $\{P_n(\rho_1;x)\}_{n\geq 0}$ yield a fruitful approach to the study of algebraic and analytic properties of the polynomials $S_n(\rho_0,\rho_1;x)$. Based on historical information given below we will refer to pairs of measures $\{\rho_0,\rho_1\}$ satisfying property (1.1) as $\Delta$-*coherent pairs of measures of the second kind on the real line.*

The concept of coherence between a pair of probability measures $\{\nu_0,\nu_1\}$ supported on the real line was introduced in [18]. Indeed, a pair of probability measures supported on the real line is said to be a *coherent pair* if the corresponding sequences of MOPs $\{P_n(\nu_0;x)\}_{n\geq 0}$ and $\{P_n(\nu_1;x)\}_{n\geq 0}$, satisfy

$$P_n(\nu_1;x)=\frac{1}{n+1}\left[P'_{n+1}(\nu_0;x)-\rho_n P'_n(\nu_0;x)\right],\quad \rho_n\neq 0,\quad n\geq 1. \tag{1.3}$$

It was shown in this case that the sequence of monic polynomials $\{S_n(\nu_0,\nu_1;x)\}_{n\geq 0}$ orthogonal with respect to the inner product

$$\langle f,g\rangle_S=\int f(x)g(x)d\nu_0+\lambda\int f'(x)g'(x)d\nu_1 \tag{1.4}$$

satisfies the connection formulas

$$\begin{aligned}&S_{n+1}(\nu_0,\nu_1;x)-\gamma_n S_n(\nu_0,\nu_1;x)=P_{n+1}(\nu_0;x)-\rho_n P_n(\nu_0;x),\\&S'_{n+1}(\nu_0,\nu_1;x)-\gamma_n S'_n(\nu_0,\nu_1;x)=(n+1)P_n(\nu_1;x),\end{aligned} \tag{1.5}$$

for all $n\geq 1$. These formulas are very useful in the study of analytic properties of the corresponding Sobolev orthogonal polynomials. For more information about polynomials orthogonal with respect to Sobolev inner products see the updated survey [24].

The motivation for introducing such pairs of measures in [18] was their applications in the framework of Fourier expansions of functions in terms of sequences of polynomials orthogonal with respect to the Sobolev inner product $\langle\,,\,\rangle_{\mathfrak{S}}$.

A particular case of such Fourier series expansions based on Legendre–Sobolev orthogonal polynomials had already been considered in [19], where some numerical tests comparing these Legendre–Sobolev Fourier series expansions and the ordinary Legendre–Fourier series expansions were presented.

On the other hand, coherent pairs of measures also appear in the framework of spectral methods for boundary value problems of ordinary differential equations once you formulate the corresponding variational approach. In [14] the authors have analyzed such a kind of questions for the nonhomogeneous one dimensional Schrödinger equation with a potential $V(x) = x^2$ in the interval $[-1, 1]$.

The pairs of probability measures supported on the real line with the property that the corresponding sequences of MOPs satisfy (1.3) were completely determined by H. G. Meijer in 1997 [25]. He showed that if $(\nu_0, \nu_1)$ is a coherent pair of measures on the real line, then one of the measures must be classical (either Jacobi or Laguerre) and the other a rational perturbation of it. Note that the condition for classical linear functionals to be part of a coherent pair was deduced in [22].

Thus, what was proved in [25] is more general than what is stated above. The starting point of the classification in [25] are certain functional relations established in [21] with respect to pairs of quasi-definite linear functionals such that the corresponding sequences of MOPs satisfy a relation like (1.3). Note that if the coherence property (1.3) holds, then you get the connection formulas in (1.5).

An extension of the concept of coherence, known in the literature as $(1,1)$-coherence, is defined in terms of the corresponding MOPs as follows (see [7]):

$$P_n(\nu_1; x) - \tau_n P_{n-1}(\nu_1; x) = \frac{1}{n+1}\left[P'_{n+1}(\nu_0; x) - \xi_n P'_n(\nu_0; x)\right], \qquad \xi_n, \tau_n \neq 0, \quad n \geq 1, \quad (1.6)$$

In this case, the Sobolev orthogonal polynomials associated with the inner product $\langle \, , \, \rangle_{\mathfrak{S}}$ in (1.4) satisfy the connection formulas

$$S_{n+1}(\nu_0, \nu_1; x) - \gamma_n S_n(\nu_0, \nu_1; x) = P_{n+1}(\nu_0; x) - \xi_n P_n(\nu_0; x),$$
$$S'_{n+1}(\nu_0, \nu_1; x) - \gamma_n S'_n(\nu_0, \nu_1; x) = (n+1)\left[P_n(\nu_1; x) - \tau_n P_{n-1}(\nu_1; x)\right],$$

for all $n \geq 1$. We would like to emphasize that the characterization of probability measures satisfying the coherence property (1.6) is given in [7] assuming that $\xi_n, \tau_n \neq 0$ for $n \geq 1$. Therein you have semiclassical linear functionals of class at most 1. Its role in the study of the sequences of orthogonal polynomials with respect to the Sobolev inner product defined by a $(1,1)$ coherent pairs of measures has been emphasized in [16]. Notice that when $\tau_n = 0$ and $\xi_n \neq 0$ for $n \geq 1$, we get the results on coherent pairs presented in [25]. Thus, a natural question is to analyze the case $\xi_n = 0$, and $\tau_n \neq 0$ for $n \geq 0$.

In [17], the concept of coherent pair of the second kind was introduced. A pair of probability measures $\{\nu_0, \nu_1\}$ supported on the real line is said to be a *coherent pair of the second kind* if the corresponding sequences of MOPs $\{P_n(\nu_0; x)\}_{n\geq 0}$ and $\{P_n(\nu_1; x)\}_{n\geq 0}$ satisfy

$$\frac{1}{n+1} P'_{n+1}(\nu_0; x) = P_n(\nu_1; x) - \tau_n P_{n-1}(\nu_1; x), \quad n \geq 1,$$

where $\tau_n \neq 0$ for $n \geq 1$. The characterization of all pairs of probability measures as well as the pairs of quasi-definite linear functionals which are coherent pairs of the second kind has been given in [17]. Some illustrative examples were shown therein.

The concept of $\Delta$-coherent pair of linear functionals was introduced in [1] as well as in [3]. Indeed, a pair of linear functionals $\{L_0, L_1\}$ it is said to be a $\Delta$-*coherent pair* if the corresponding sequences of MOPs $\{P_n^{(0)}(x)\}_{n\geq 0}$ and $\{P_n^{(1)}(x)\}_{n\geq 0}$ satisfy a discrete version of (1.3):

$$P_n^{(1)}(x) = \frac{1}{n+1}\left[\Delta P_{n+1}^{(0)}(x) - \rho_n \Delta P_n^{(0)}(x)\right], \quad \rho_n \neq 0, \quad n \geq 1.$$

It was shown in this case that the sequence of monic polynomials $\{S_n(x)\}_{n\geq 0}$ orthogonal with respect to the inner product

$$\langle f, g\rangle_{\mathfrak{S}} = \langle L_0, fg\rangle + \langle L_1, \Delta f \Delta g\rangle$$

satisfies the connection formulas

$$\begin{aligned} S_{n+1}(x) - \gamma_n S_n(x) &= P_{n+1}^{(0)}(x) - \rho_n P_n^{(0)}(x), \\ \Delta S_{n+1}(x) - \gamma_n \Delta S_n(x) &= (n+1) P_n^{(1)}(x), \end{aligned}$$

for all $n \geq 1$. In [3], it was proved that one of them must be a $\Delta$-classical linear functional. The corresponding companion linear functionals were described therein. Notice that this inner product is a discretization of a Sobolev inner product (1.4), i.e., $\nu_0$ and $\nu_1$ are discrete measures.

The paper is organized as follows: Section 2 summarizes the basic concepts about linear functionals and orthogonal polynomials to be used in what follows. We emphasize a special family of linear functionals, the so-called $\Delta$-*semiclassical*, which will play a central role in the article. In Section 3, we introduce the concept of $\Delta$-coherent pair of the second kind for linear functionals and we give a characterization of them. We must point out that both functionals in the pair turn out to be discrete semiclassical functionals of class at most 1. In Section 4 we classify all $\Delta$-coherent pairs of the second kind. We show that the companions of the discrete classical functionals (Charlier, Kravchuk, Meixner and Hahn) are discrete semiclassical of class $s = 1$.

## 2. Basic background

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{N}_0$ be the set of nonnegative integers

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}.$$

We will denote by $\delta_{k,n}$ the *Kronecker delta*, defined by

$$\delta_{k,n} = \begin{cases} 1, & k = n, \\ 0, & k \neq n, \end{cases} \quad k, n \in \mathbb{N}_0,$$

and say that $\{\Lambda_n(x)\}_{n\geq 0} \subset \mathbb{P}$ is a *basis* of $\mathbb{P}$ if $\deg(\Lambda_n) = n$. We will denote by $\mathbb{P}^*$ the algebraic dual space of $\mathbb{P}$.

Suppose that $L : \mathbb{P} \to \mathbb{C}$ is a *linear functional* (i.e., $L \in P^*$), that $\{\Lambda_n(x)\}_{n\geq 0}$ is a basis of $\mathbb{P}$, and we choose a *nonzero* sequence $\{h_n\}_{n\geq 0} \subset \mathbb{C}$. If the system of linear equations

$$\sum_{i=0}^{n} L\left[\Lambda_k \Lambda_i\right] c_{n,i} = h_n \delta_{k,n}, \quad 0 \leq k \leq n, \tag{2.1}$$

has a *unique solution* $\{c_{n,i}\}_{0\leq i\leq n}$, we can define a polynomial $P_n(x)$ by $P_n(x) = \sum_{i=0}^{n} c_{n,i}\Lambda_i(x)$, $n \in \mathbb{N}_0$. We say that $\{P_n(x)\}_{n\geq 0}$ is an *orthogonal polynomial sequence* with respect to the functional $L$. The system (2.1) can be written as $L\left[\Lambda_k P_n\right] = h_n\delta_{k,n}$, $0 \leq k \leq n$, and using linearity we see that the sequence $\{P_n(x)\}_{n\geq 0}$ satisfies the *orthogonality conditions*

$$L\left[P_k P_n\right] = h_n \delta_{k,n}, \quad k, n \in \mathbb{N}_0. \tag{2.2}$$

The linear functional $L$ is said to be *quasi-definite* [6].

We define the multiplication of a linear functional by a polynomial $q(x)$ to be the linear functional $qL \in P^*$ such that

$$(qL)\left[p\right] = L\left[qp\right], \quad p \in \mathbb{P}. \tag{2.3}$$

These transformations of linear functionals are known in the literature as *Christoffel transformations* (see [29]).

On the other hand, given the forward difference operator $\Delta$ defined in (1.2), we introduce its *adjoint operator* $\Delta^* : P^* \to P^*$ by

$$(\Delta^* L)\left[p\right] = -L\left[\Delta p\right], \quad p \in \mathbb{P}. \tag{2.4}$$

We say that $L$ is a *semiclassical functional* with respect to the operator $\Delta$, ($\Delta$-semiclassical, in short) [23] if there exist polynomials $\phi(x)$, $\psi(x)$ such that $L$ satisfies the *Pearson equation*

$$\Delta^*\left(\phi L\right) + \psi L = 0, \tag{2.5}$$

where $\phi$ is monic and $\deg(\psi) \geq 1$. Note that using (2.3) and (2.4) we can rewrite the Pearson equation as

$$L\left[\phi \Delta p\right] = L\left[\psi p\right], \quad p \in \mathbb{P}. \tag{2.6}$$

**Lemma 2.1.** *If $f, g : \mathbb{Z} \to \mathbb{C}$, then we have the* summation by parts *formula*

$$\sum_{x=a}^{b} f(x)\Delta g(x) = \left[f\left(x-1\right)g(x)\right]_a^{b+1} - \sum_{x=a}^{b} g(x)\nabla f(x), \quad a, b \in \mathbb{Z}, \tag{2.7}$$

*where $\nabla$ denotes the* backward difference operator *defined by*

$$\nabla\left[f(x)\right] = f(x) - f(x-1).$$

*Proof.* The formula follows from the telescoping sum

$$\begin{aligned}\sum_{x=a}^{b}\left[f(x)\Delta g(x) + g(x)\nabla f(x)\right] &= \sum_{x=a}^{b} f(x)g\left(x+1\right) - g(x)f\left(x-1\right) \\ &= f\left(b\right)g\left(b+1\right) - g(a)f\left(a-1\right).\end{aligned}$$

□

Using summation by parts, we can write an alternative form of the Pearson equation (2.5).

**Proposition 2.2.** *Let $L$ be a linear functional associated with a discrete measure $\rho$ and defined by*

$$L[p] = \sum_{x=0}^{\infty} p(x)\rho(x), \quad p \in \mathbb{P},$$

*where $\rho(-1) = 0$ and*

$$\lim_{x\to\infty} p(x)\rho(x) = 0, \quad p \in \mathbb{P}.$$

*Then, $L$ satisfies the Pearson equation* (2.5) *if and only if*

$$\nabla(\phi\rho) + \psi\rho = 0. \tag{2.8}$$

*Proof.* Since (2.5) is equivalent to (2.6), we can use (2.7) and conclude that $L$ satisfies (2.5) if and only if

$$\sum_{x=0}^{\infty} \psi(x)p(x)\rho(x) = -\sum_{x=0}^{\infty} p(x)\nabla(\phi\rho)(x), \quad p \in \mathbb{P}.$$

If the last equation holds for all $p \in \mathbb{P}$, then we must have

$$\psi\rho = -\nabla(\phi\rho),$$

and the result follows. □

The *class* of a discrete semiclassical linear functional $L$ is defined by

$$s = \min\{\max\{\deg(\phi) - 2, \deg(\psi)\} - 1\}, \tag{2.9}$$

where the minimum is taken among all pairs of polynomials $(\phi, \psi)$ such that the Pearson equation (2.5) holds for the linear functional $L$. Functionals of class $s = 0$ are called $\Delta$-*classical* (see [15], [20], and [27], among others). The $\Delta$-semiclassical linear functionals of class $s = 1$ have been described in [9] and [10].

**Remark 2.3.** Let $\deg(\phi) = d_1$, $\deg(\psi) = d_2$. If $d_1 = d_2 + 1 = s + 2$, we will always assume the admissibility condition

$$\psi_{d_2} \neq n - s, \quad n \in \mathbb{N}_0, \tag{2.10}$$

where $\psi_{d_2}$ is the leading coefficient of $\psi(x)$.

We will denote by $\{\varphi_n(x)\}_{n\geq 0}$ the basis of *falling factorial polynomials* defined by $\varphi_0(x) = 1$ and

$$\varphi_n(x) = \prod_{k=0}^{n-1}(x - k), \quad n \in \mathbb{N}.$$

The polynomials $\varphi_n(x)$ satisfy the basic identities

$$x\varphi_n(x) = \varphi_{n+1}(x) + n\varphi_n(x), \quad n \in \mathbb{N}_0, \tag{2.11}$$

and

$$\Delta\varphi_n = n\varphi_{n-1}, \quad n \in \mathbb{N}_0. \tag{2.12}$$

**Lemma 2.4.** *We have the representation*

$$\varphi_n\left(x-1\right)=\sum_{k=0}^{n}\left(-1\right)^k\varphi_k(n)\varphi_{n-k}(x),\quad n\in\mathbb{N}_0. \tag{2.13}$$

*Proof.* We use induction. The identity is true for $n=0$ since $\varphi_0(x)=1$. Assuming (2.13) to be true for all $0\le k\le n$ and using the recurrence (2.11), we have

$$\begin{aligned}\varphi_{n+1}\left(x-1\right)=\left(x-1-n\right)\varphi_n\left(x-1\right)&=\sum_{k=0}^{n}\left(-1\right)^k\varphi_k(n)\left(x-1-n\right)\varphi_{n-k}(x)\\&=\sum_{k=0}^{n}\left(-1\right)^k\varphi_k(n)\left[\varphi_{n+1-k}(x)-\left(k+1\right)\varphi_{n-k}(x)\right].\end{aligned}$$

Since $\varphi_{n+1}(n)=0$, we can rewrite the sum above as

$$\varphi_{n+1}\left(x-1\right)=\sum_{k=0}^{n+1}\left(-1\right)^k\varphi_k(n)\varphi_{n+1-k}(x)+\sum_{k=0}^{n}\left(-1\right)^{k+1}\left(k+1\right)\varphi_k(n)\varphi_{n-k}(x),$$

or, shifting the index in the second sum,

$$\varphi_{n+1}\left(x-1\right)=\sum_{k=0}^{n+1}\left(-1\right)^k\varphi_k(n)\varphi_{n+1-k}(x)+\sum_{k=1}^{n+1}\left(-1\right)^k k\varphi_{k-1}(n)\varphi_{n+1-k}(x).$$

But from (2.12) we see that

$$\varphi_k\left(n+1\right)=\varphi_k(n)+k\varphi_{k-1}(n),$$

and we conclude that

$$\varphi_{n+1}\left(x-1\right)=\sum_{k=0}^{n+1}\left(-1\right)^k\varphi_k\left(n+1\right)\varphi_{n+1-k}(x). \qquad \square$$

**Proposition 2.5.** *Let $\{p_n(x)\}_{n\ge0}$ be a sequence of polynomials orthogonal with respect to a $\Delta$-semiclassical functional $L$ of class $s$. Then,*

$$L\left[\phi\Delta q\Delta p_n\right]=0,\quad q\in\mathbb{P},\quad \deg(q)<n-s. \tag{2.14}$$

*Proof.* The equation (2.14) is obviously true if $q=1$, so we can assume that $\deg(q)\ge1$. Using (2.13) with $k\ge1$, we have

$$\Delta\varphi_k=-\sum_{j=1}^{k}\left(-1\right)^j\varphi_j(k)\varphi_{k-j}\left(x+1\right),$$

and multiplying by $\Delta p_n$ we get

$$\Delta\varphi_k\Delta p_n=\sum_{j=1}^{k}\left(-1\right)^j\varphi_j(k)\left[\Delta\varphi_{k-j}p_n-\Delta\left(\varphi_{k-j}p_n\right)\right],$$

where we have used the identity

$$\Delta\left(fg\right)=g\left(x+1\right)\Delta f+f\Delta g=g\Delta f+f\Delta g+\Delta f\Delta g. \tag{2.15}$$

Using the Pearson equation (2.5) and (2.12), we obtain

$$\begin{aligned} L\left[\phi\Delta\varphi_k\Delta p_n\right] &= \sum_{j=1}^{k}(-1)^j\,\varphi_j(k)\left(L\left[\phi\Delta\varphi_{k-j}p_n\right]-L\left[\psi\varphi_{k-j}p_n\right]\right)\\ &= \sum_{j=1}^{k}(-1)^j\,\varphi_j(k)\left\{(k-j)\,L\left[\phi\varphi_{k-j-1}p_n\right]-L\left[\psi\varphi_{k-j}p_n\right]\right\}, \end{aligned}$$

and since the sequence of polynomials $\{p_n(x)\}_{n\geq 0}$ is orthogonal with respect to $L$ we conclude that

$$L\left[\phi\Delta\varphi_k\Delta p_n\right]=0, \quad k<\min\left\{n+2-\deg(\phi),\, n+1-\deg(\psi)\right\}=n-s.$$

Using the fact that $\{\varphi_k(x)\}_{k\geq 0}$ is a polynomial basis, the result follows. □

Using Proposition 2.5, we can prove the following characterization of discrete semiclassical polynomials.

**Proposition 2.6.** *Let $\{p_n(x)\}_{n\geq 0}$ be a sequence of monic polynomials orthogonal with respect to a functional $L$. The following statements are equivalent:*

(i) *$L$ is $\Delta$-semiclassical of class $s$.*
(ii) *There exists a monic polynomial $\phi(x)$ such that the polynomials $p_n$ satisfy the* structure equation

$$\phi(x)\Delta p_{n+1}=\sum_{k=n-s}^{n+d_1}\varepsilon_{n,k}p_k, \quad n>s, \quad \varepsilon_{n,n-s}\neq 0. \tag{2.16}$$

*Proof.* (i) ⇒ (ii) Since the sequence $\{p_n\}_{n\geq 0}$ is a polynomial basis in the linear space of polynomials, we can write

$$\phi(x)\Delta p_{n+1}=\sum_{k=0}^{n+d_1}\varepsilon_{n,k}p_k.$$

From the orthogonality relation (2.2) and (2.15), we have

$$h_k\varepsilon_{n,k}=L\left[\phi p_k\Delta p_{n+1}\right]=L\left[\phi\Delta\left(p_kp_{n+1}\right)\right]-L\left[\phi\Delta p_kp_{n+1}\right]-L\left[\phi\Delta p_k\Delta p_{n+1}\right].$$

Using (2.6) and orthogonality, we get

$$L\left[\phi\Delta\left(p_kp_{n+1}\right)\right]=L\left[\psi p_kp_{n+1}\right]=0, \quad k<n+1-d_2,$$

and

$$L\left[\phi\Delta\varphi_kp_{n+1}\right]=kL\left[\phi\varphi_{k-1}p_{n+1}\right]=0, \quad k<n+2-d_1,$$

from which it follows that

$$L\left[\phi\Delta\left(p_kp_{n+1}\right)\right]-L\left[\phi\Delta p_kp_{n+1}\right]=0, \quad k<n-s.$$

Using (2.14) we see that

$$L\left[\phi\Delta p_k\Delta p_{n+1}\right]=0, \quad k<n+1-s,$$

and therefore

$$\varepsilon_{n,k}=0, \quad k<n-s.$$

For $k = n - s$, we have $\varepsilon_{n,n-s} = h_{n+1}\left(\frac{\psi_{d_2}}{h_{n+1-d_2}}\delta_{d_2,s+1} - \frac{n+2-d_1}{h_{n+2-d_1}}\delta_{d_1,s+2}\right)$, and it follows that $\varepsilon_{n,n-s} \neq 0$ as long as $d_1 - 2 \neq d_2 - 1$. If $d_1 = d_2 + 1$, we need the additional condition (2.10).

(ii) ⇒ (i) Since $\left\{\frac{p_n}{h_n}L\right\}_{n\geq 0}$ is a basis of $\mathbb{P}^*$ satisfying

$$\frac{p_n}{h_n}L\left[p_k\right] = \frac{1}{h_n}L\left[p_k p_n\right] = \delta_{k,n},$$

we have the representation

$$\Delta^*\left(\phi L\right) = \sum_{n=0}^{\infty} c_n \frac{p_n}{h_n}L.$$

Taking into account the structure relation (2.16) and orthogonality, we get

$$\begin{aligned} c_{n+1} &= \Delta^*\left(\phi L\right)\left[p_{n+1}\right] = -L\left[\phi \Delta p_{n+1}\right] \\ &= -\sum_{k=n-s}^{n+d_1} \varepsilon_{n,k} L\left[p_k\right] = \begin{cases} 0, & n > s, \\ -\mu_0 \varepsilon_{n,0}, & n \leq s, \end{cases} \end{aligned}$$

where $\mu_0 = L\left[1\right]$. Thus, $\Delta^*\left(\phi L\right) + \psi L = 0$, with

$$\psi(x) = \mu_0 \sum_{n=1}^{s+1} \frac{\varepsilon_{n-1,0}}{h_n} p_n(x).$$

Note that $\deg(\psi) - 1 \leq s$, in agreement with (2.9). □

For more information about this kind of structure relations for Δ-semiclassical linear functionals, check [23]. Notice that in the Δ-classical case ($s = 0$) two structure relations characterizing such linear functionals are given in [15].

## 3. Coherent pairs

Let $\{L_0, L_1\}$ be a pair of quasi-definite functionals and let $\left\{P_n^{(0)}(x)\right\}_{n\geq 0}$ and $\left\{P_n^{(1)}(x)\right\}_{n\geq 0}$ be the corresponding sequences of monic orthogonal polynomials. We say that $\{L_0, L_1\}$ is a Δ-*coherent pair of the second kind* (abbreviated Δc2) if there exists $\{\tau_n\}_{n\geq 0} \subset \mathbb{C} \setminus \{0\}$ such that

$$Q_n(x) = \frac{\Delta P_{n+1}^{(0)}(x)}{n+1} = P_n^{(1)}(x) - \tau_n P_{n-1}^{(1)}(x), \quad n \in \mathbb{N}_0, \tag{3.1}$$

where $P_{-1}^{(0)}(x) = P_{-1}^{(1)}(x) = 0$. The sequences of polynomials $\left\{P_n^{(i)}(x)\right\}_{n\geq 0}$, $i = 0, 1$, satisfy the orthogonality conditions

$$L_i\left[P_k^{(i)} P_n^{(i)}\right] = h_n^{(i)} \delta_{n,k}, \quad i = 0, 1, \quad n, k \in \mathbb{N}_0. \tag{3.2}$$

**Proposition 3.1.** *If* $\{L_0, L_1\}$ *is a* Δc2 *and the linear functionals* $\mathbf{v}_n^{(i)}$ *are defined by*

$$\mathbf{v}_n^{(i)} = \frac{P_n^{(i)}}{h_n^{(i)}} L_i, \quad i = 0, 1, \quad n \in \mathbb{N}_0, \tag{3.3}$$

*then*

$$\Delta^* \mathbf{v}_n^{(1)} = \tau_{n+1} (n+2) \mathbf{v}_{n+2}^{(0)} - (n+1) \mathbf{v}_{n+1}^{(0)}, \quad n \in \mathbb{N}_0. \tag{3.4}$$

*Proof.* From (3.2) and (3.3), we have

$$\mathbf{v}_n^{(i)} \left[ P_k^{(i)} \right] = \delta_{n,k}, \quad i = 0, 1, \quad n, k \in \mathbb{N}_0. \tag{3.5}$$

Let $\mathbf{u}_n \in \mathbb{P}^*$ be defined by

$$\mathbf{u}_n \left[ Q_k \right] = \delta_{n,k}, \quad n, k \in \mathbb{N}_0.$$

Since the sequence of linear functionals $\{\mathbf{u}_n\}_{n \geq 0}$ is a basis of $\mathbb{P}^*$, we can write

$$\mathbf{v}_n^{(1)} = \sum_{k=0}^{\infty} a_{n,k} \mathbf{u}_n.$$

Using (3.1) and (3.5), we get

$$a_{n,k} = \mathbf{v}_n^{(1)} \left[ Q_k \right] = \mathbf{v}_n^{(1)} \left[ P_k^{(1)} - \tau_k P_{k-1}^{(1)} \right] = \delta_{n,k} - \tau_k \delta_{n,k-1},$$

and therefore

$$\mathbf{v}_n^{(1)} = \mathbf{u}_n - \tau_{n+1} \mathbf{u}_{n+1}, \quad n \in \mathbb{N}_0. \tag{3.6}$$

If we write

$$\Delta^* \mathbf{u}_n = \sum_{k=0}^{\infty} b_{n,k} \mathbf{v}_k^{(0)},$$

then

$$b_{n,k} = \Delta^* \mathbf{u}_n \left[ P_k^{(0)} \right] = -\mathbf{v}_n^{(0)} \left[ \Delta P_k^{(0)} \right] = -k \mathbf{v}_n^{(0)} \left[ Q_{k-1} \right] = -k \delta_{n,k-1},$$

and, as a consequence,

$$\Delta^* \mathbf{u}_n = -(n+1) \mathbf{v}_{n+1}^{(0)}. \tag{3.7}$$

Using (3.7) in (3.6), we obtain (3.4). □

With the help of Proposition 3.1, we can prove a characterization of $\Delta$-coherent pairs of the second kind.

**Theorem 3.2.** *The following statements are equivalent:*

(i) $\{L_0, L_1\}$ *is a* $\Delta$c2.

(ii) *There exist* $\Lambda_2, \Lambda_3 \in \mathbb{P}$ *defined by*

$$\Lambda_k(x) = \sum_{i=0}^{k} \lambda_i^{(k)} x^i, \quad k = 2, 3,$$

*and satisfying*

$$\lambda_2^{(2)} + (n-1) \lambda_3^{(3)} \neq 0, \quad n \in \mathbb{N}, \tag{3.8}$$

*such that*

$$\Delta^* L_1 = \Lambda_2 L_0, \quad L_1 = \Lambda_3 L_0. \tag{3.9}$$

*Proof.* (i) ⇒ (ii) Setting $n = 0$ in (3.4), we have

$$\frac{1}{h_0^{(1)}}\Delta^* L_1 = 2\tau_1 \frac{P_2^{(0)}}{h_2^{(0)}} L_0 - \frac{P_1^{(0)}}{h_1^{(0)}} L_0,$$

and defining $\Lambda_2(x) = \frac{2\tau_1 h_0^{(1)}}{h_2^{(0)}} P_2^{(0)}(x) - \frac{h_0^{(1)}}{h_1^{(0)}} P_1^{(0)}(x)$, we get $\Delta^* L_1 = \Lambda_2 L_0$. Since $\tau_1 \neq 0$, we see that $\deg(\Lambda_2) = 2$.

Similarly, setting $n = 1$ in (3.4) gives

$$\frac{1}{h_1^{(1)}}\Delta^*\left(P_1^{(1)} L_1\right) = 3\tau_2 \frac{P_3^{(0)}}{h_3^{(0)}} L_0 - 2\frac{P_2^{(0)}}{h_2^{(0)}} L_0,$$

and using (2.15) we have

$$\Delta^* P_1^{(1)} L_1 + P_1^{(1)} \Delta^* L_1 + \Delta^* P_1^{(1)} \Delta^* L_1 = \left( \frac{3\tau_2 h_1^{(1)}}{h_3^{(0)}} P_3^{(0)} - \frac{2h_1^{(1)}}{h_2^{(0)}} P_2^{(0)} \right) L_0.$$

Since $\Delta^* P_1^{(1)} = 1$ and $\Delta^* L_1 = \Lambda_2 L_0$, we conclude that $L_1 = \Lambda_3 L_0$ with

$$\Lambda_3(x) = \frac{3\tau_2 h_1^{(1)}}{h_3^{(0)}} P_3^{(0)}(x) - \frac{2h_1^{(1)}}{h_2^{(0)}} P_2^{(0)}(x) - \Lambda_2(x)\left[P_1^{(1)}(x) + 1\right].$$

Note that $\deg(\Lambda_3) \leq 3$.

(ii) ⇒ (i) Since the polynomial sequence $\{P_n^{(1)}(x)\}_{n\geq 0}$ is a basis of $\mathbb{P}$, we have

$$Q_n(x) = P_n^{(1)} + \sum_{k=0}^{n-1} c_{n,k} P_k^{(1)}(x).$$

Orthogonality and (2.15) yield

$$\begin{aligned}
(n+1)\, h_k^{(1)} c_{n,k} &= (n+1)\, L_1\left[Q_n P_k^{(1)}\right] = L_1\left[\Delta P_{n+1}^{(0)} P_k^{(1)}\right] \\
&= L_1\left[\Delta\left(P_{n+1}^{(0)} P_k^{(1)}\right)\right] - L_1\left[P_{n+1}^{(0)} \Delta P_k^{(1)}\right] - L_1\left[\Delta P_{n+1}^{(0)} \Delta P_k^{(1)}\right] \\
&= -L_0\left[\Lambda_2 P_{n+1}^{(0)} P_k^{(1)}\right] - L_0\left[\Lambda_3 P_{n+1}^{(0)} \Delta P_k^{(1)}\right] - L_0\left[\Lambda_3 \Delta P_{n+1}^{(0)} \Delta P_k^{(1)}\right],
\end{aligned}$$

and hence, for all $0 \leq k \leq n-1$,

$$\begin{aligned}
L_0\left[\Lambda_2 P_{n+1}^{(0)} P_k^{(1)}\right] &= \lambda_2^{(2)} h_{n+1}^{(0)} \delta_{k,n-1}, \\
L_0\left[\Lambda_3 P_{n+1}^{(0)} \Delta P_k^{(1)}\right] &= (n-1)\, \lambda_3^{(3)} h_{n+1}^{(0)} \delta_{k,n-1}.
\end{aligned}$$

From (3.9) it follows that

$$\Delta^*(\Lambda_3 L_0) = \Lambda_2 L_0.$$

Therefore, $L_0$ is $\Delta$-semiclassical of class at most 1. Using (2.14), we conclude that

$$L_0\left[\Lambda_3 \Delta P_{n+1}^{(0)} \Delta P_k^{(1)}\right] = 0, \quad k < n.$$

Thus, for all $0 \leq k \leq n-1$,

$$(n+1)\, h_k^{(1)} c_{n,k} = -\left[\lambda_2^{(2)} + (n-1)\, \lambda_3^{(3)}\right] h_{n+1}^{(0)} \delta_{k,n-1},$$

and we obtain

$$Q_n(x) = P_n^{(1)}(x) - \tau_n P_{n-1}^{(1)}(x),$$

where

$$\tau_n = \frac{\lambda_2^{(2)} + (n-1)\, \lambda_3^{(3)}}{n+1} \frac{h_{n+1}^{(0)}}{h_{n-1}^{(1)}}, \quad n \in \mathbb{N}.$$

Since the polynomials $\Lambda_2, \Lambda_3$ satisfy (3.8), we see that $\tau_n \neq 0$. □

## 4. Classification

In this section, we find all $\Delta$-coherent pairs of the second kind. According to Theorem 3.2, it is enough to consider discrete semiclassical functionals of class $s \leq 1$ satisfying the Pearson equation

$$\Delta^* (\Lambda_3 L_0) = \Lambda_2 L_0, \quad \deg(\Lambda_2) = 2, \quad \deg(\Lambda_3) \leq 3.$$

In [9], we classified the discrete semiclassical functionals of class $s \leq 1$ satisfying (2.8), and obtained the following cases:

(1) Charlier (class $s = 0$):

$$\rho(x) = \frac{z^x}{x!}, \quad z > 0. \tag{4.1}$$

This discrete weight is associated to the Poisson distribution:

$$\phi(x) = 1, \quad \psi(x) = \frac{x}{z} - 1. \tag{4.2}$$

(2) Meixner (class $s = 0$):

$$\rho(x) = (a)_x \frac{z^x}{x!}, \quad a > 0, \quad 0 < z < 1. \tag{4.3}$$

This discrete weight is associated to the Pascal distribution:

$$\phi(x) = x + a, \quad \psi(x) = \frac{x}{z} - (x + a)\,. \tag{4.4}$$

Subcase: Kravchuk polynomials:

$$\rho(x) = (-N)_x \frac{z^x}{x!}, \quad N \in \mathbb{N}, \quad z > 0. \tag{4.5}$$

This discrete weight is associated to the binomial distribution:

$$\phi(x) = x - N, \quad \psi(x) = \frac{x}{z} - (x - N)\,. \tag{4.6}$$

(3) Hahn (class $s = 0$):

$$\rho(x) = \frac{(-N)_x (a)_x}{(b+1)_x} \frac{1}{x!}, \quad N \in \mathbb{N}, \quad a > 0, \quad b > -1. \tag{4.7}$$

This discrete weight is associated to the hypergeometric distribution:

$$\phi(x) = (x - N)(x + a)\,, \quad \psi(x) = (N - a + b)\, x + aN. \tag{4.8}$$

(4) Generalized Charlier (class $s=1$):

$$\rho(x)=\frac{1}{(b+1)_x}\frac{z^x}{x!},\quad z>0,\quad b>-1, \tag{4.9}$$

$$\phi(x)=1,\quad \psi(x)=\frac{x\,(x+b)}{z}-1. \tag{4.10}$$

(5) Generalized Meixner (class $s=1$):

$$\rho(x)=\frac{(a)_x}{(b+1)_x},\frac{z^x}{x!},\quad z,a>0,\quad b>-1, \tag{4.11}$$

$$\phi(x)=x+a,\quad \psi(x)=\frac{x\,(x+b)}{z}-(x+a)\,. \tag{4.12}$$

(6) Generalized Kravchuk (class $s=1$):

$$\rho(x)=(-N)_x\,(a)_x\frac{z^x}{x!},\quad N\in\mathbb{N},\quad z,a>0, \tag{4.13}$$

$$\phi(x)=(x-N)\,(x+a)\,,\quad \psi(x)=\frac{x}{z}-(x-N)\,(x+a)\,. \tag{4.14}$$

(7) Generalized Hahn of type I (class $s=1$):

$$\rho(x)=\frac{(a_1)_x\,(a_2)_x}{(b+1)_x}\frac{z^x}{x!},\quad a_1,a_2>0,\quad b>-1, \tag{4.15}$$

$$\phi(x)=(x+a_1)\,(x+a_2)\,,\quad \psi(x)=\frac{x\,(x+b)}{z}-(x+a_1)\,(x+a_2)\,. \tag{4.16}$$

Note that if we set $z=1$, $a_1=a$, $a_2=-N$, then we obtain the Hahn polynomials.

(8) Generalized Hahn of type II (class $s=1$):

$$\rho(x)=\frac{(-N)_x\,(a_1)_x\,(a_2)_x}{(b_1+1)_x\,(b_2+1)_x}\frac{1}{x!},\quad N\in\mathbb{N},\quad a_1,a_2>0,\quad b_1,b_2>-1, \tag{4.17}$$

$$\begin{aligned}\phi(x)&=(x-N)\,(x+a_1)\,(x+a_2)\,,\\ \psi(x)&=(N-a_1-a_2+b_1+b_2)\,x^2\\ &\qquad+(Na_1+Na_2-a_1a_2+b_1b_2)\,x+Na_1a_2.\end{aligned} \tag{4.18}$$

If we define $L_0,L_1\in\mathbb{P}^*$ by

$$L_i\,[p]=\sum_{x=0}^{\infty}p(x)\rho_i(x),\quad i=0,1,\quad p\in\mathbb{P},$$

with $\rho_i\,(-1)=0$, $i=0,1$, then we can use (2.2) and rewrite (3.9) as

$$\nabla\,(\Lambda_3\rho_0)+\Lambda_2\rho_0=0,\quad \rho_1=\Lambda_3\rho_0. \tag{4.19}$$

Before we start the classification of the Δc2 pairs, we need the following result.

**Proposition 4.1.** *Let $q, \phi_0, \psi_0 \in \mathbb{P}$. If $L_0 \in \mathbb{P}^*$ satisfies*

$$\nabla\left(\phi_0 \rho_0\right) + \psi_0 \rho_0 = 0 \tag{4.20}$$

*and*

$$\Lambda_2 = \left(q - \nabla q\right) \psi_0 - \phi_0 \nabla q, \quad \Lambda_3 = q \phi_0, \tag{4.21}$$

*then*

$$\nabla\left(\Lambda_3 \rho_0\right) + \Lambda_2 \rho_0 = 0. \tag{4.22}$$

*Proof.* From the identity

$$\nabla\left(fg\right) = g \nabla f + f \nabla g - \nabla f \nabla g$$

and (4.21), we have

$$\nabla\left(\Lambda_3 \rho_0\right) = \left(\nabla q\right) \phi_0 \rho_0 + \left(q - \nabla q\right) \nabla\left(\phi_0 \rho_0\right).$$

Using (4.20), we get

$$\nabla\left(\Lambda_3 \rho_0\right) = \left(\nabla q\right) \phi_0 \rho_0 - \left(q - \nabla q\right) \psi_0 \rho_0 = -\Lambda_2 \rho_0,$$

and (4.22) follows. □

We will now find all $\Delta$c2 pairs with $\deg\left(\Lambda_2\right) = 2$.

4.1. **Case I:** $\deg(\Lambda_3) = 0$**.** If we take $\Lambda_3 = 1$, we see from (4.19) that

$$\nabla \rho_0 + \Lambda_2 \rho_0 = 0, \quad \rho_1 = \rho_0,$$

and if $\rho_0(x)$ is the weight function corresponding to the generalized Charlier polynomials (4.9), using (4.10) we get

$$\Lambda_2(x) = \frac{x\left(x+b\right)}{z} - 1, \quad z \neq 0.$$

Thus, the generalized Charlier polynomials are *self-coherent* of the second kind. Note that this fact was already observed in [11].

4.2. **Case II (a):** $\deg(\Lambda_3) = 1$**.** If we take $\Lambda_3 = x + \omega$, $\omega \in \mathbb{C}$, then we see from (4.19) that

$$\nabla\left[\left(x+\omega\right) \rho_0\right] + \Lambda_2 \rho_0 = 0, \quad \rho_1 = \left(x+\omega\right) \rho_0.$$

If $\rho_0(x)$ is the weight function corresponding to the Charlier polynomials (4.1), we can use Proposition 4.1 with $q = x + \omega$ and obtain

$$\Lambda_2(x) = \frac{x}{z}\left(x + \omega - 1\right) - \left(x + \omega\right), \quad z \neq 0,$$

where we have also used (4.2). The functional $L_1$ is a *Christoffel transformation* of $L_0$ satisfying

$$\nabla\left[\left(x+\omega\right) \rho_1\right] + \Lambda_2 \rho_1 = 0,$$

and using (4.12) we see that $\rho_1(x)$ is the weight function corresponding to the generalized Meixner polynomials (4.11) with

$$a = \omega, \quad b = \omega - 1.$$

4.3. **Case II (b):** $\deg(\Lambda_3) = 1$**.** If we take $\Lambda_3 = x+\omega$, $\omega \in \mathbb{C}$, then from (4.19) we get that $\nabla\left[(x+\omega)\,\rho_0\right] + \Lambda_2\rho_0 = 0$, $\rho_1 = (x+\omega)\,\rho_0$. If $\rho_0(x)$ is the weight function corresponding to the Kravchuk polynomials (4.5), we can use Proposition 4.1 with $q = x+\omega$ and obtain $\Lambda_2(x) = \frac{x}{z}\,(x+\omega-1) - (x+\omega)\,(x-N)$, $z \neq 0$, where we have also used (4.6). The functional $L_1$ is a Christoffel transformation of $L_0$ satisfying

$$\nabla\left[(x+\omega)\,\rho_1\right] + \Lambda_2\rho_1 = 0,$$

and using (4.14) we see that $\rho_1(x)$ is the weight function corresponding to the generalized Kravchuk polynomials (4.13) with

$$a = \omega, \quad b = \omega - 1.$$

4.4. **Case III:** $\deg(\Lambda_3) = 2$**.** If we take $\Lambda_3 = (x+\omega)\,(x+a)$, $\omega \in \mathbb{C}$, then from (4.19) it follows that

$$\nabla\left[(x+\omega)\,(x+a)\,\rho_0\right] + \Lambda_2\rho_0 = 0, \quad \rho_1 = (x+\omega)\,(x+a)\,\rho_0.$$

If $\rho_0(x)$ is the weight function corresponding to the Meixner polynomials (4.3), we can use Proposition 4.1 with $q = x+\omega$ and obtain

$$\Lambda_2(x) = \frac{x}{z}\,(x+\omega-1) - (x+\omega)\,(x+a)\,, \quad z \neq 0,$$

were we have also used (4.4). The functional $L_1$ is a Christoffel transformation of $L_0$ satisfying

$$\nabla\left[(x+\omega)\,(x+a)\,\rho_1\right] + \Lambda_2\rho_1 = 0,$$

and using (4.16) we see that $\rho_1(x)$ is the weight function corresponding to the generalized Hahn polynomials of type I (4.15) with

$$a_1 = a, \quad a_2 = \omega, \quad b = \omega - 1.$$

4.5. **Case IV:** $\deg(\Lambda_3) = 3$**.** If we take $\Lambda_3 = (x+\omega)\,(x-N)\,(x+a)$, $\omega \in \mathbb{C}$, $N \in \mathbb{N}$, from (4.19) we deduce that

$$\begin{gathered}\nabla\left[(x+\omega)\,(x-N)\,(x+a)\,\rho_0\right] + \Lambda_2\rho_0 = 0,\\ \rho_1 = (x+\omega)\,(x-N)\,(x+a)\,\rho_0.\end{gathered}$$

If $\rho_0(x)$ is the weight function corresponding to the Hahn polynomials (4.7), we can use Proposition 4.1 with $q = x+\omega$ and obtain

$$\Lambda_2 = (N-a+b-1)\,x^2 + (N\omega + Na - a\omega + b\omega - b)\,x + Na\omega,$$

where we have also used (4.8). The functional $L_1$ is a Christoffel transformation of $L_0$ satisfying

$$\nabla\left[(x+\omega)\,(x-N)\,(x+a)\,\rho_1\right] + \Lambda_2\rho_1 = 0,$$

and using (4.18) we see that $\rho_1(x)$ is the weight function corresponding to the generalized Hahn polynomials of type II (4.17) with

$$a_1 = a, \quad a_2 = \omega, \quad b_1 = b, \quad b_2 = \omega - 1.$$

Note that Christoffel transforms of Δ-classical linear functionals have been analyzed in [8] and [28].

## 5. Conclusions and future directions

We have classified all $\Delta$-coherent pairs of the second kind $\{L_0, L_1\}$ and derived the following results:

| $L_0$ | $L_1$ |
|---|---|
| generalized Charlier | generalized Charlier |
| Charlier | generalized Meixner |
| Kravchuk | generalized Kravchuk |
| Meixner | generalized Hahn of type I |
| Hahn | generalized Hahn of type II |

In all cases, the functional $L_1$ is a Christoffel transformation of $L_0$, in agreement with the general results obtained by Meijer [25] for the continuous case. Notice that Christoffel transformations of discrete classical linear functionals play a key role in the study of bispectral problems, as pointed out in some recent contributions by A. J. Durán and co-workers (see [12] and [13]).

In a work in progress we are focussing our attention on the asymptotic formulas for the polynomials orthogonal with respect to a Sobolev inner product associated with a $\Delta$c2 pair of measures. In particular, we are interested on the outer relative asymptotics of such polynomials and the polynomials $P_n^{(0)}$ with the convenient scaling in the variable and the analogue of the Mehler–Heine formula. The location of their zeros is another interesting problem.

We must point out that in [5] the authors deal with asymptotic properties and the location of zeros of discrete polynomials associated with a Sobolev inner product

$$\langle p, q\rangle_{\mathfrak{S}} = \langle u_0, pq\rangle + \lambda\langle u_1, \Delta p \Delta q\rangle,$$

where $\lambda \geq 0$, $(u_0, u_1)$ is a $\Delta$-coherent pair of positive-definite linear functionals and $u_1$ is the Meixner linear functional. A limit relation between these orthogonal polynomials and the Laguerre–Sobolev orthogonal polynomials which is analogous to the one existing between Meixner and Laguerre polynomials in the Askey scheme is deduced. Notice that Meixner polynomials are $\Delta$ self-coherent, Thus, Mehler–Heine type formulas and zeros of such polynomials when $u_0 = u_1$ is the Meixner functional have been studied in [26]. Outer relative asymptotics and Plancherel–Rotach asymptotics as well as limit relations are analyzed in [4]. Algebraic properties of such polynomials as well as the behavior of their zeros appear in [2].

## Acknowledgements

We thank the valuable comments and suggestions by the referee in order to improve the presentation of the manuscript.

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*Received: July 22, 2023*
*Accepted: November 14, 2023*
*Early view: August 26, 2024*