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A CANONICAL DISTRIBUTION ON ISOPARAMETRIC SUBMANIFOLDS III

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Abstract. The present paper is devoted to showing that on every compact, connected homogeneous isoparametric submanifold $M = G/K$ of codimension $h \geq 2$ in a Euclidean space, there exist canonical distributions which are generated by the compact symmetric spaces associated to M (i.e., those corresponding to the group G). The central objective is to show that all these distributions are bracket generating of step 2. To that end, formulae that complement those in the first article of this series (Rev. Un. Mat. Argentina 61, no. 1 (2020), 113–130) are obtained.

1. Introduction

The present paper can be considered a sequel and extension of the papers [9] and [10]. In those papers, it was established the existence (in any compact, connected, homogeneous, isoparametric submanifold M of codimension $h \geq 2$ in a Euclidean space) of a smooth, completely non-integrable, step 2 distribution $\mathfrak{D}(\Omega)$.

Here we indicate, on the family of isoparametric submanifolds M mentioned above, the existence of new distributions having the same property as $\mathfrak{D}(\Omega)$, that is, they are all completely non-integrable of step 2. It is important to mention here that these distributions are associated to symmetric spaces of Type 1. In fact, for our isoparametric submanifold $M = K/K_E$, the symmetric spaces corresponding to the group K (which are of the form K/L) "induce" on M smooth distributions which, similarly to $\mathfrak{D}(\Omega)$, are completely non-integrable of step 2.

Recall that a distribution $\mathfrak{D}$ of r-planes ($n > r \geq 2$) in a compact, connected manifold M^n is *smooth* [12, p. 41, Def. 1.56] if for any $p \in M^n$ there are r smooth vector fields $\{X_1, \ldots, X_r\}$ defined on an open set $A \subset M^n$ containing p such that $X_j(q) \in \mathfrak{D}(q)$ and $\mathfrak{D}(q) = \mathrm{span}_{\mathbb{R}} \{X_j(q)\}$, $(1 \leq j \leq r, \forall q \in A)$. The distribution $\mathfrak{D}$ is said to be *completely non-integrable of step 2* if for every point $p \in M^n$ the above vector fields defined in A satisfy $(\forall q \in A)$:

$$\mathrm{Span}_{\mathbb{R}} \{X_j(q), [X_k, X_j](q) : 1 \leq k, j \leq r\} = T_q(M),$$

i.e., the generated real vector space coincides with the corresponding tangent space.

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The mentioned *homogeneous* isoparametric submanifolds M^n of codimension $h \geq 2$ in Euclidean spaces are obtained as principal orbits of the tangential representation (at a basic point) of a compact (or noncompact dual) symmetric space. A way to obtain explicitly these submanifolds is to consider a real simple noncompact Lie algebra $\mathfrak{g}_0$ with Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ and Cartan involution θ. Then $\mathfrak{k}_0$ is a maximal compactly embedded subalgebra of $\mathfrak{g}_0$ [5, Pr. 7.4, p. 184]. Let K be the analytic subgroup K of $\mathrm{Int}(\mathfrak{g}_0)$ corresponding to the subalgebra $\mathrm{ad}_{\mathfrak{g}_0}(\mathfrak{k}_0)$ of $\mathrm{ad}_{\mathfrak{g}_0}(\mathfrak{g}_0)$ which is compact and let B_θ be the positive definite, symmetric bilinear form on $\mathfrak{g}_0$ defined by

$$B_\theta\left(x,y\right) := \left\langle x,y\right\rangle_\theta = -B\left(x,\theta y\right), \tag{1.1}$$

where B is the Killing form of $\mathfrak{g}_0$.

The *principal orbits* of the representation of K on $\mathfrak{p}_0$ are isoparametric submanifolds M^n of $\mathbb{R}^{n+h} = \mathfrak{p}_0$. Let $\mathfrak{a}_0$ be a maximal abelian subspace of $\mathfrak{p}_0$ and consider the set $\Phi(\mathfrak{g}_0,\mathfrak{a}_0)$ of roots "restricted" to $\mathfrak{a}_0$ (see [9] for the required details and notation). Let $\Delta\left(\mathfrak{g}_0,\mathfrak{a}_0\right)$ be a corresponding system of simple roots in $\Phi\left(\mathfrak{g}_0,\mathfrak{a}_0\right)$. For $\lambda \in \Phi\left(\mathfrak{g}_0,\mathfrak{a}_0\right)$, it is usual to define the subspaces

$$\begin{aligned} \mathfrak{k}_{0,\lambda} &= \left\{x \in \mathfrak{k}_0 : \left(ad\left(h\right)\right)^2 x = \lambda^2\left(h\right)x \ \ \forall h \in \mathfrak{a}_0\right\}, \\ \mathfrak{p}_{0,\lambda} &= \left\{x \in \mathfrak{p}_0 : \left(ad\left(h\right)\right)^2 x = \lambda^2\left(h\right)x \ \ \forall h \in \mathfrak{a}_0\right\}, \end{aligned} \tag{1.2}$$

for which obviously $\mathfrak{k}_{0,\lambda} = \mathfrak{k}_{0,(-\lambda)}$, $\mathfrak{p}_{0,\lambda} = \mathfrak{p}_{0,(-\lambda)}$ and with them, respect to B_θ (1.1), we have orthogonal decompositions

$$\mathfrak{k}_0 = \mathfrak{m}_0 \oplus \sum_{\lambda\in\Phi^+(\mathfrak{g}_0,\mathfrak{a}_0)} \mathfrak{k}_{0,\lambda}, \qquad \mathfrak{p}_0 = \mathfrak{a}_0 \oplus \sum_{\lambda\in\Phi^+(\mathfrak{g}_0,\mathfrak{a}_0)} \mathfrak{p}_{0,\lambda}, \tag{1.3}$$

where $\Phi^+\left(\mathfrak{g}_0,\mathfrak{a}_0\right)$ is the set of roots written with non-negative coefficients in terms of $\Delta\left(\mathfrak{g}_0,\mathfrak{a}_0\right)$ and $\mathfrak{m}_0$ is the centralizer of $\mathfrak{a}_0$ in $\mathfrak{k}_0$.[1] As usual, the *height* of a root in $\Phi^+\left(\mathfrak{g}_0,\mathfrak{a}_0\right)$ is defined as the sum of its coefficients with respect to $\Delta\left(\mathfrak{g}_0,\mathfrak{a}_0\right)$. Let $\Omega \subset \Phi^+\left(\mathfrak{g}_0,\mathfrak{a}_0\right)$ be the set of positive roots of odd height. As in [9] and [10], associated to Ω we define the subspace

$$\mathfrak{D}(\Omega) = \sum_{\lambda\in\Omega} \mathfrak{p}_{0,\lambda} \subset \mathfrak{p}_0.$$

Let us fix a *regular* element $E \in \mathfrak{a}_0 \subset \mathfrak{p}_0$, call $M = \mathrm{Ad}(K)E \subset \mathfrak{p}_0$ its orbit and let K_E be the isotropy subgroup of K at E. The regularity of E implies that the isotropy subalgebra (corresponding to) K_E is $\mathfrak{k}_{0,E} = \mathfrak{m}_0$. Furthermore, the tangent and normal spaces of M at E are

$$T_E(M) = \sum_{\lambda\in\Phi^+(\mathfrak{g}_0,\mathfrak{a}_0)} [\mathfrak{k}_{0,\lambda}, E] = \sum_{\lambda\in\Phi^+(\mathfrak{g}_0,\mathfrak{a}_0)} \mathfrak{p}_{0,\lambda} \qquad \text{and} \qquad T_E^\perp(M) = \mathfrak{a}_0. \tag{1.4}$$

Since the subspace $\mathfrak{D}(\Omega)$ is contained in $T_E(M)$ and it is invariant by the action of K_E, by translation with K, we obtain in M a distribution which we also call $\mathfrak{D}(\Omega)$ and is contained in the tangent bundle of M. The main result of [9] and [10]

[1]The subspaces defined in (1.2) are also defined in [1, p. 57] and are related to the eigenspaces of the shape operator as in [1, pp. 70–71].

is that this distribution is completely non-integrable of step 2. The difference between [9] and [10] resides in the nature of the system of restricted roots – *reduced* in [9] and *non-reduced* in [10].

As indicated above, in the present paper we show the existence, on $M = \mathrm{Ad}(K)E$ of other distributions, all with the same property as $\mathfrak{D}(\Omega)$. These distributions are associated to two classes of compact symmetric spaces. The first one is that of *symmetric R-spaces*, i.e., *extrinsic symmetric spaces* (these are the compact Hermitian symmetric spaces and their real forms, as indicated in [1, pp. 427–428]). The way in which symmetric R-spaces are presented is well known but it may be convenient to recall it. Let $\mathfrak{g}_0$ be a real simple noncompact Lie algebra with Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ and Cartan involution θ. The subalgebra $\mathfrak{k}_0$ is a maximal compactly embedded in $\mathfrak{g}_0$. Let K be the analytic subgroup of $\mathrm{Int}(\mathfrak{g}_0)$ corresponding to the subalgebra $\mathrm{ad}_{\mathfrak{g}_0}(\mathfrak{k}_0)$ of $\mathrm{ad}_{\mathfrak{g}_0}(\mathfrak{g}_0)$ which is compact. Let us consider the Euclidean space $\mathfrak{p}_0$ with the inner product B_θ (1.1). Let $\mathfrak{a}_0 \subset \mathfrak{p}_0$, $\Phi(\mathfrak{g}_0, \mathfrak{a}_0)$ and $\Delta(\mathfrak{g}_0, \mathfrak{a}_0)$ have the same meaning as above and assume that there exists an element $H \in \mathfrak{a}_0$ such that the eigenvalues of $\mathrm{ad}(H)$ on $\mathfrak{g}_0$ are $\{(-1), 0, 1\}$ (these elements are called *extrinsically symmetric*). Then the orbit $N = \mathrm{Ad}(K)H \subset \mathfrak{p}_0$ is a symmetric R-space. On the other hand, the principal orbits of the representation of K on $\mathfrak{p}_0$ are the isoparametric submanifolds that support the associated distribution (one of them is chosen by taking a regular element $E \in \mathfrak{a}_0$ and considering its orbit $M = \mathrm{Ad}(K)E \subset \mathfrak{p}_0$ by the adjoint action of K on $\mathfrak{p}_0$). A particular subset of symmetric R-spaces is that of the Hermitian ones and they are presented as follows: Let $\mathfrak{u}_0$ be a compact simple Lie algebra and consider the real Lie algebra $\mathfrak{g}^{\mathbb{R}} = \mathfrak{u}_0 \oplus i\mathfrak{u}_0$. This is a *Cartan decomposition* of $\mathfrak{g}^{\mathbb{R}}$ [5, p. 185]. Let us take a Cartan subalgebra $\mathfrak{t}_0 \subset \mathfrak{u}_0$ so $i\mathfrak{t}_0 \subset i\mathfrak{u}_0$ is a *maximal abelian* subspace of $i\mathfrak{u}_0$ and $\mathfrak{h} = (\mathfrak{t}_0 \oplus i\mathfrak{t}_0) \subset \mathfrak{u}_0 \oplus i\mathfrak{u}_0 = \mathfrak{g}^{\mathbb{R}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{R}}$. We have the roots in $\Phi\left(\mathfrak{g}^{\mathbb{R}}, \mathfrak{h}\right)$ and the restricted roots are those in $\Phi\left(\mathfrak{g}^{\mathbb{R}}, i\mathfrak{t}_0\right)$. They are just the roots of $\mathfrak{u}_0$ with respect to $\mathfrak{t}_0$. Let us take a compact connected Lie group K (without center) corresponding to $\mathfrak{u}_0$, the compact Hermitian symmetric space can be realized (isometrically embedded) as orbit of an extrinsically symmetric element $H \in i\mathfrak{t}_0 \subset i\mathfrak{u}_0 \subset \mathfrak{g}^{\mathbb{R}}$ by the adjoint action of K on $(i\mathfrak{u}_0)$. For Hermitian symmetric spaces the associated isoparametric submanifolds are the *manifolds of complete flags* of the group K. These are the principal orbits of the adjoint representation of K.

The other set of symmetric spaces to be considered contains some of the so called *quaternionic symmetric spaces* and also the space $EVIII = E_8/\mathrm{Spin}(16)/Z_2$, which is not a quaternionic one. These *are not R-spaces.*

The symmetric spaces $\mathbb{G}\mathbf{r}_2(\mathbb{C}^{n+2}) = \mathrm{SU}(n+2)/S(U(n) \times U(2))$ $(n \geq 1)$ are *quaternionic symmetric* and *Hermitian symmetric*, so we exclude them from the present considerations and take the space $EVIII$ instead. Then they are:

- classical:

$$\begin{aligned} \mathbb{G}\mathbf{r}_4(\mathbb{R}^{n+4}) &= \mathrm{SO}(n+4)/\mathrm{SO}(n) \times \mathrm{SO}(4), \quad n \geq 3, \\ \mathbb{HP}^n &= \mathrm{Sp}(n+1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1), \quad n \geq 1, \end{aligned} \tag{1.5}$$

- exceptional:

$$
\begin{aligned}
EIX &= E_8/E_7\mathrm{Sp}(1), & FI &= F_4/\mathrm{Sp}(3)\mathrm{Sp}(1), \\
EVI &= E_7/\mathrm{Spin}(12)\mathrm{Sp}(1), & G &= G_2/\mathrm{SO}(4), \\
EII &= E_6/\mathrm{SU}(6)\mathrm{Sp}(1), & EVIII &= E_8/\mathrm{Spin}(16)/Z_2.
\end{aligned} \tag{1.6}
$$

The distributions for this class of symmetric spaces K/H are defined (as for Hermitian ones) in the manifolds of complete flags of the group K.

The paper is organized as follows. The next section contains the two results that are the objectives of the present paper; they are Theorem 2.1, which involves symmetric R-space, and Theorem 2.2 concerning the other type of symmetric spaces considered here.

The paper goes along the lines of [9] and, for that reason, notation and some results from that paper have to be recalled. They are contained in Appendix A, which is divided into five short sections recalling: basis, smooth local fields, known identities and finally, formulae (A.13), (A.16) and (A.17) obtained in [9] and corresponding to the sums of roots. Standard facts and notation from Lie theory are taken from [7, 5, 4, 8, 2], as in [9]. On the other hand, in Appendix B we get the new formulae expressing the vectors of the basis associated to the difference of roots as combination of brackets of local fields in the distribution (evaluated at the basic point E of M). The reader shall certainly notice that formulae in Appendix B are dual to those in Appendix A. Section 3 contains the construction of the distributions, required notation and the necessary lemmata. It contains two subsections, reflecting the differences of the situations considered. Section 4 contains some examples that illustrate the way in which the distributions are generated and hopefully shall clarify their meaning. Finally, Section 5 contains the proofs of Theorem 2.1 and 2.2, where the formulae given in Appendices A and B are essentially used.

2. Objectives

Here we indicate the results contained in the present paper, namely Theorems 2.1 and 2.2. Since the large majority of the compact, connected, irreducible symmetric spaces are extrinsically symmetric (nowadays called *symmetric R-spaces*) we indicate first the result associated to them, keeping the notation indicated in the previous sections.

Theorem 2.1. *Let $E \in \mathfrak{a}_0$ be a regular element and assume that there exists an element $H \in \mathfrak{a}_0$ such that the eigenvalues of $\mathrm{ad}(H)$ on $\mathfrak{g}_0$ are $\{(-1), 0, 1\}$ (we call these elements* extrinsically symmetric*). Then the orbit $N = \mathrm{Ad}(K)H \subset \mathfrak{p}_0$ is a symmetric R-space. The tangent space $T_H(N)$ of the symmetric R-space N at H "induces" a distribution $\mathfrak{D}(N)$ in $T(M)$ ($M = \mathrm{Ad}(K)E$) which is completely non-integrable of step 2.*

Proof. The proof is contained in Section 5. □

We shall describe the construction of $\mathfrak{D}(N)$ in the next section. Let us consider now the situation for those symmetric spaces in (1.5) and (1.6).

Theorem 2.2. *Let* $\mathfrak{u}_0$ *be one of the compact simple Lie algebras corresponding to the compact simple groups generating the spaces in* (1.5) *and* (1.6), *and consider the real Lie algebra* $\mathfrak{g}^{\mathbb{R}} = \mathfrak{u}_0 \oplus i\mathfrak{u}_0$. *With* $\mathfrak{k}_0 = \mathfrak{u}_0$ *and* $\mathfrak{p}_0 = i\mathfrak{u}_0$ *this* $\mathfrak{k}_0 \oplus \mathfrak{p}_0$ *is a Cartan decomposition of* $\mathfrak{g}^{\mathbb{R}}$. *Then* $\mathfrak{k}_0 = \mathfrak{u}_0$ *is a maximal compactly embedded subalgebra of* $\mathfrak{g}^{\mathbb{R}}$. *Let* K *be a compact, connected, adjoint Lie group* K *corresponding to* $\mathfrak{u}_0$. *Let us consider the Euclidean space* $i\mathfrak{u}_0 = \mathfrak{p}_0$ *with the inner product given by the Killing form* B. *Let* $i\mathfrak{t}_0 \subset i\mathfrak{u}_0 = \mathfrak{p}_0$, *while* $\Phi(\mathfrak{g}_0, \mathfrak{a}_0)$ *and* $\Delta(\mathfrak{g}_0, \mathfrak{a}_0)$ *have the above meaning. The principal orbits of the representation of* K *on* $i\mathfrak{u}_0 = \mathfrak{p}_0$ *are isoparametric submanifolds. Let us choose a regular element* $E \in \mathfrak{a}_0$ *and set* $M = \mathrm{Ad}(K)E \subset \mathfrak{p}_0$. *The symmetric space* K/H *in* (1.5) *or* (1.6) *induces a distribution* $\mathfrak{D}(\Theta)$ *in* $T(M)$ *which is completely non-integrable of step 2.*

Proof. The proof of this theorem is also contained in Section 5. □

Recall that a Lie group with trivial center is called an *adjoint* group.

3. Construction of the distributions

3.1. **Distribution generated by symmetric R-spaces.** Let us assume that there exists $H \in \mathfrak{a}_0 \subset \mathfrak{p}_0$ extrinsically symmetric (i.e., $\mathrm{ad}(H)$ has only eigenvalues $\{(-1), 0, 1\}$). Then $(\mathrm{ad}(H))^2$ has eigenvalues $\{0, 1\}$ and determines two subsets of $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$, namely

$$\begin{aligned} \Psi_0 &= \left\{\lambda \in \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0) : \lambda(H) = 0\right\}, \\ \Theta &= \left\{\lambda \in \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0) : \lambda(H) = 1\right\}. \end{aligned} \tag{3.1}$$

Note that $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0) = \Psi_0 \cup \Theta$ and consider the orbit $N = \mathrm{Ad}(K)H \subset \mathfrak{p}_0$. N is a *symmetric R-space* (see for instance [3]) whose isotropy subalgebra and tangent space at H are, respectively,

$$\begin{aligned} \mathfrak{k}_H &= \sum_{\lambda \in \Psi_0} \mathfrak{k}_{0,\lambda} \subset \mathfrak{k}_0 \\ T_H(N) = [\mathfrak{k}_0, H] &= \sum_{\lambda \in \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)} [\mathfrak{k}_{0,\lambda}, H] = \sum_{\lambda \in \Theta} \mathfrak{p}_{0,\lambda} \subset \mathfrak{p}_0. \end{aligned} \tag{3.2}$$

Now we observe the following:

Lemma 3.1. *If the system of roots* $\Phi(\mathfrak{g}_0, \mathfrak{a}_0)$ *is irreducible* [6, p. 52] *and there is an* $H \in \mathfrak{a}_0 \subset \mathfrak{p}_0$ *extrinsically symmetric, then there is one and only one simple root* $\eta \in \Delta(\mathfrak{g}_0, \mathfrak{a}_0)$ *such that* $\eta \in \Theta$ *in* (3.1).

Proof. This is clear. See Remark 3.4 below. □

The roots in Ψ_0 (written in terms of $\Delta(\mathfrak{g}_0, \mathfrak{a}_0)$) are those without the term η, while those in Θ have the term η (with coefficient 1).

Let us consider now the following:

Lemma 3.2. *Let us assume that the maximal root* $\mu \in \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$ *has a simple root term with coefficient* 1. *Then for each* $\gamma \in \Psi_0 \subset \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$ *there are two roots* φ *and* ψ *in* Θ *such that* $\gamma = \varphi - \psi$ *and* $(\varphi + \psi)$ *is not a root. The simple root systems considered in this lemma are* A_r, B_r, C_r, D_r, E_6, E_7.

Proof. The proof is by inspection on the systems of roots. See Remark 3.4 below. □

We have then two subspaces of $\mathfrak{p}_0$, namely (3.2) and (1.4). We may now define $\mathfrak{D}_E(\Theta) \subset T_E(M)$ by $\mathfrak{D}_E(\Theta) = \sum_{\lambda\in\Theta} \mathfrak{p}_{0,\lambda}$ and have the obvious inclusion

$$\mathfrak{D}_E(\Theta) = \sum_{\lambda\in\Theta} \mathfrak{p}_{0,\lambda} \subset \sum_{\lambda\in\Phi^+(\mathfrak{g}_0,\mathfrak{a}_0)} \mathfrak{p}_{0,\lambda} = T_E(M).$$

The subspace $\mathfrak{D}_E(\Theta)$ is invariant by the isotropy subgroup K_E of K at E and hence, by translation with K, we get the distribution $\mathfrak{D}(\Theta)$ on M.

3.2. **Distribution generated by the other spaces.** We indicate now how to construct the distributions associated to the spaces in (1.5) and (1.6).

Recall the notations indicated in Theorem 2.2 and take a regular element $E \in \mathfrak{a}_0 = i\mathfrak{t}_0 \subset i\mathfrak{u}_0 = \mathfrak{p}_0$, while $\Phi(\mathfrak{g}_0, \mathfrak{a}_0)$ and $\Delta(\mathfrak{g}_0, \mathfrak{a}_0)$ have the above meaning. The orbit $M = \mathrm{Ad}(K)E \subset \mathfrak{p}_0$ (a principal orbit) is a *manifold of complete flags of* K (the isotropy group of M at E is a maximal torus of K). This is our isoparametric submanifold in the present case.

Let us write the roots in $\Phi(\mathfrak{g}_0, \mathfrak{a}_0)$ in terms of $\Delta(\mathfrak{g}_0, \mathfrak{a}_0)$ as

$$\delta = \sum_{\gamma\in\Delta(\mathfrak{g}_0,\mathfrak{a}_0)} s_\gamma(\delta)\,\gamma$$

and assume that we can choose a simple root $\lambda \in \Delta(\mathfrak{g}_0, \mathfrak{a}_0)$ such that for the maximal root μ we have $s_\lambda(\mu) = 2$. The chosen root $\lambda \in \Delta(\mathfrak{u}_0, \mathfrak{t}_0)$ splits $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$ into three sets, namely

$$\begin{aligned} \Psi_0 &= \left\{\delta \in \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0) : s_\lambda(\delta) = 0\right\}, \\ \Psi_2 &= \left\{\delta \in \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0) : s_\lambda(\delta) = 2\right\}, \\ \Theta &= \left\{\delta \in \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0) : s_\lambda(\delta) = 1\right\}. \end{aligned} \tag{3.3}$$

The subspace $\mathfrak{D}_E(\Theta) = \sum_{\delta\in\Theta} \mathfrak{p}_{0,\delta}$ of $T_E(M)$ in (1.4) associated to $\Theta \subset \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$ is invariant by the maximal torus T of K which is the isotropy subgroup of K at E and so, by translation with K, defines a distribution $\mathfrak{D}(\Theta)$ on the manifold $\dot{M}$.

Let us consider the symmetric spaces in (1.5) and (1.6). All these spaces have the property that there is a *simple* root $\lambda \in \Delta(\mathfrak{u}_0, \mathfrak{t}_0)$ such that all roots in the tangent space (written in terms of $\Delta(\mathfrak{g}_0, \mathfrak{a}_0)$) have a term λ with coefficient 1 and $s_\lambda(\mu) = 2$. Then, the tangential roots in these symmetric spaces are those in Θ. In the following Tables 1 and 2 we indicate, for each one of them, the simple root that defines Θ, as in (3.3). The subscripts of the indicated roots are those in the notation from [5, pp. 477–478]:

A glance at the table in [5, pp. 477–478] shows that for these choices the coefficient $s_\lambda(\mu)$ for these simple roots is $s_\lambda(\mu) = 2$. Tables 1 and 2 indicate the existence of *at least* one such root for each of these spaces. It is important to mention that there is no orbit of the type of these symmetric spaces in the corresponding adjoint representations of their groups. The set Θ, for each of the indicated symmetric spaces, is defined by (3.3) with the roots in 1 and 2.

space	root
$\mathbb{G}\mathbf{r}_4(\mathbb{R}^{n+4}) = \mathrm{SO}(n+4)/\mathrm{SO}(n)\times\mathrm{SO}(4)$, $n\geq 3$	$\begin{cases}\alpha_m, & n=2m\\ \alpha_2, & n=2m+1\end{cases}$
$\mathbb{HP}^n = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)\times\mathrm{Sp}(1)$, $n\geq 1$	α_n

Table 1.

space	root	space	root
EII	α_2	*EIX*	α_8
EVI	α_1	*FI*	α_1
EVIII	α_1	*G*	α_2

Table 2.

We have the following lemma which replaces Lemma 3.2 in the present situation.

Lemma 3.3. *For each $\gamma\in\Psi_0\subset\Phi^+(\mathfrak{g}_0,\mathfrak{a}_0)$ there are two roots φ and ψ in Θ such that $\gamma=\varphi-\psi$ and the sum $(\varphi+\psi)$ is not a root of $\Phi(\mathfrak{g}_0,\mathfrak{a}_0)$. Also, for each $\eta\in\Psi_2\subset\Phi^+(\mathfrak{g}_0,\mathfrak{a}_0)$, there are two roots δ and ω in Θ such that $\eta=\delta+\omega$ and $|\delta-\omega|$ is not a root of $\Phi(\mathfrak{g}_0,\mathfrak{a}_0)$.*

Proof. The proof is by inspection on the systems of roots. □

Remark 3.4. The reader can find complete proofs of Lemmata 3.1, 3.2 and 3.3 in [11].

At this point, it seems convenient to present some examples to illustrate the construction of the distributions considered in Theorems 2.1 and 2.2.

4. Examples

Let us consider the extrinsic symmetric spaces which are real forms of the Hermitian symmetric space *EVII*. They are:

Hermitian: $EVII = E_7/(E_6U(1))$

real forms: $EIV = (E_6.U(1))/F_4$ $\quad AII = \mathrm{SU}(8)/\mathrm{Sp}(4)$

Each one of them is realized as orbit in the *tangential representation* of the symmetric spaces indicated in the following table.

space	dim		ambient	dim
$EIV = (E_6.U(1))/F_4$	27	$\hookrightarrow$	$EVII = E_7/(E_6U(1))$	54
$AII = \mathrm{SU}(8)/\mathrm{Sp}(4)$	27	$\hookrightarrow$	$EV = E_7/\mathrm{SU}(8)$	70

We have Cartan decompositions and *restricted root system (RRS)* for the corresponding "ambient" spaces.

	space	RRS
$EVII$	$\mathfrak{e}_7 = (\mathfrak{e}_6 \oplus \mathbb{R}) \oplus \mathfrak{p}_0$	$\mathfrak{c}_3$
EV	$\mathfrak{e}_7 = \mathfrak{su}(8) \oplus \mathfrak{p}_0$	$\mathfrak{e}_7$

4.1. **The space** $EIV = (E_6.U(1))/F_4$**.** Let us consider the symmetric space $EVII$ and the associated Cartan decomposition

$$\mathfrak{e}_7 = (\mathfrak{e}_6 \oplus \mathbb{R}) \oplus \mathfrak{p}_0$$

and the maximal abelian subspace $\mathfrak{a}_0$ in $\mathfrak{p}_0$. The Dynkin diagrams of $\mathfrak{e}_7$ and $\mathfrak{c}_3$, indicating the coefficients of the corresponding highest roots, are

$$\begin{array}{cccccc} & & & \overset{2}{\alpha_2} & & \\ & & & \circ & & \\ & & & \downarrow & & \\ \overset{1}{\circ} & -\overset{2}{\circ} & -\overset{3}{\circ} & -\overset{4}{\circ}- & \overset{3}{\circ}- & \overset{2}{\circ} \\ \alpha_7 & \alpha_6 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_1 \end{array}, \qquad \begin{array}{ccc} \overset{2}{\circ} & -\overset{2}{\circ} & \Longleftarrow \overset{1}{\circ} \\ \lambda_1 & \lambda_6 & \lambda_7 \end{array} \tag{4.1}$$

The *restriction rule* [5, p. 534] of the roots is

$$\begin{array}{cccccc} & & & \alpha_2 & & \\ & & & \bullet & & \\ & & & \downarrow & & \\ \circ & -\circ & -\bullet & -\bullet- & \bullet- & \circ \\ \alpha_7 & \alpha_6 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_1 \end{array} \longmapsto \begin{array}{ccc} \circ & -\circ & \Longleftarrow \circ \\ \lambda_1 & \lambda_6 & \lambda_7 \end{array}$$

with the notation in [5, p. 534] this is: $\alpha_j \longmapsto \lambda_j$ for $j = 1, 6, 7$ and $\alpha_j \longmapsto 0$ for $j = 2, 3, 4, 5$. The multiplicities of the simple roots are: $m(\lambda_j) = 8$ for $j = 1, 6$ and $m(\lambda_7) = 1$. For convenience, we change the names of the simple roots of $\mathfrak{c}_3$ to $\{\lambda_1, \lambda_2 := \lambda_6, \lambda_3 := \lambda_7\}$. The 9 positive roots of $\mathfrak{c}_3$ are

$$\begin{array}{lll} e_1 - e_2 = \lambda_1 & e_1 + e_2 = \lambda_1 + 2\lambda_2 + \lambda_3 & 2e_1 = 2\lambda_1 + 2\lambda_2 + \lambda_3 \\ e_1 - e_3 = \lambda_1 + \lambda_2 & e_2 + e_3 = \lambda_2 + \lambda_3 & 2e_2 = 2\lambda_2 + \lambda_3 \\ e_2 - e_3 = \lambda_2 & e_1 + e_3 = \lambda_1 + \lambda_2 + \lambda_3 & 2e_3 = \lambda_3 \end{array}$$

with maximal roots

$$\text{long: } \mu = 2(\lambda_1 + \lambda_2) + \lambda_3, \qquad \text{short: } \eta = \lambda_1 + 2\lambda_2 + \lambda_3.$$

The following table indicates the corresponding multiplicities for all the positive roots of $\mathfrak{c}_3$:

$$\begin{array}{rrr} m(\lambda_1) = 8 & m(\lambda_2) = 8 & m(\lambda_3) = 1 \\ m(\lambda_1 + \lambda_2) = 8 & m(\lambda_2 + \lambda_3) = 8 & m(2\lambda_2 + \lambda_3) = 1 \\ m(\lambda_1 + \lambda_2 + \lambda_3) = 8 & m(\lambda_1 + 2\lambda_2 + \lambda_3) = 8 & m(2\lambda_1 + 2\lambda_2 + \lambda_3) = 1. \end{array}$$

We have the subset $\Omega \subset \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$ of roots of *odd height* with respect to the simple roots $\{\lambda_1, \lambda_2, \lambda_3\}$. The set Ω has the 6 roots

$$\begin{array}{lll} \lambda_1, & \lambda_2, & \lambda_3, \\ \lambda_1 + \lambda_2 + \lambda_3, & 2\lambda_2 + \lambda_3, & 2\lambda_1 + 2\lambda_2 + \lambda_3. \end{array}$$

We see that the dimension of $\mathfrak{D}(\Omega)$ is $\dim(\mathfrak{D}(\Omega)) = 27$.

We can take the dual basis $\{v_1, v_2, v_3\}$ of $\{\lambda_1, \lambda_2, \lambda_3\}$ defined by $\lambda_k(v_j) = \delta_{k,j}$ and consider the vector $E = v_1 + v_2 + v_3$ which is clearly a *regular element* (no root vanishes on E). So the orbit of E by $E_6U(1)$ is a *principal orbit* which we can take as our isoparametric submanifold M in this example. The dimension of M is the sum of the multiplicities of all the positive roots so $\dim(M) = 51$. Then we have (see (1.3)) that $\dim(\mathfrak{m}_0) = 28$. In order to get the symmetric R-space, namely $EIV = (E_6.U(1)/F_4)$, we just have to take the vector $v_3 \in \mathfrak{a}_0$ just defined and evaluating the roots on v_3 we see that

$$\begin{array}{lll} \lambda_1(v_3) = 0 & (\lambda_1 + 2\lambda_2 + \lambda_3)(v_3) = 1 & (2\lambda_1 + 2\lambda_2 + \lambda_3)(v_3) = 1 \\ (\lambda_1 + \lambda_2)(v_3) = 0 & (\lambda_2 + \lambda_3)(v_3) = 1 & (2\lambda_2 + \lambda_3)(v_3) = 1 \\ \lambda_2(v_3) = 0 & (\lambda_1 + \lambda_2 + \lambda_3)(v_3) = 1 & \lambda_3(v_3) = 1. \end{array}$$

So v_3 is extrinsically symmetric and its orbit is in fact a symmetric R-space. We see that the orbit of v_3 by $E_6.U(1)$ has dimension 27, which is that of $E_6.U(1)/F_4$. Now Θ is the set of roots with λ_3 with coefficient 1 and it defines the subspace

$$\mathfrak{D}_E(EIV) = \sum_{\lambda \in \Theta} \mathfrak{p}_{0,\lambda}$$

of dimension 27 in the tangent space $T_E(M)$ at M at the point E, which in turn extends to a distribution of this dimension in the isoparametric submanifold M of dimension 51.

4.2. **The space $AII = \mathrm{SU}(8)/\mathrm{Sp}(4)$.**

$$AII = \mathrm{SU}(8)/\mathrm{Sp}(4)27 \quad \hookrightarrow \quad EV = E_7/\mathrm{SU}(8)70.$$

Let us consider the symmetric space

	space	dim	rank	$\Phi(\mathfrak{g}_0, \mathfrak{a}_0)$
EV	$E_7/\mathrm{SU}(8)$	70	7	$\mathfrak{e}_7$

This is an *inner split symmetric space*. We have the associated Cartan decomposition

$$\mathfrak{e}_7 = \mathfrak{su}(8) \oplus \mathfrak{p}_0.$$

The restricted root system is $\mathfrak{e}_7$. The Dynkin diagram of $\mathfrak{e}_7$ (with coefficients of the highest root) is in (4.1). Let us consider the maximal abelian subspace $\mathfrak{a}_0$ in $\mathfrak{p}_0$ of dimension 7. The orthogonal complement as in (1.3) has dimension 63, which is the dimension of the principal orbits. Now we take the duals to the simple roots of $\mathfrak{e}_7$, namely ξ_j such that $\alpha_k(\xi_j) = \delta_{k,j}$ and take the vector $\xi_7 \in \mathfrak{a}_0 \subset \mathfrak{p}_0 \subset \mathfrak{e}_7$. By looking at the table of roots in [4, p. 529], we see that evaluating each *positive* root in ξ_7 we get either 1 or 0 so this vector is *extrinsically symmetric* since *all* the roots in $\mathfrak{e}_7$ evaluated in the vector ξ_7 give either $1, 0$ or (-1).

It is important to observe that there are 27 roots with coefficient 1 in α_7 (α_7 is the extreme of the long arm of the above Dynkin diagram), and since the multiplicities of the roots are all $m = 1$ we get that the orbit of ξ_7 by the adjoint action of SU(8) has dimension 27.

We include now some examples referring to the distributions considered in Theorem 2.2.

4.3. **The space** $EII = E_6/\mathrm{SU}(6)\mathrm{SU}(2)$**.** Let us consider the *quaternionic symmetric space EII*. It generates a distribution of dimension 40 in the isoparametric submanifold $M = E_6/T^6$ of dimension 72. M is any chosen principal orbit in the adjoint representation of E_6 in its Lie algebra.

We have

	space	dim	rank	$\Phi(\mathfrak{g}_0, \mathfrak{a}_0)$
EII	$E_6/\mathrm{SU}(6)\mathrm{SU}(2)$	40	4	$\mathfrak{f}_4$

The maximal root μ of $\mathfrak{e}_6$ is $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$, so we have three simple roots with coefficient 2. But, as indicated in Table 2, we take the root α_2. Let us consider now the subsets of roots

$$\begin{aligned}\Psi_0(\alpha_2) &= \left\{\lambda \in \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0) : s_\lambda(\alpha_2) = 0\right\},\\ \Psi_2(\alpha_2) &= \left\{\lambda \in \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0) : s_\lambda(\alpha_2) = 2\right\},\\ \Theta(\alpha_2) &= \left\{\lambda \in \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0) : s_\lambda(\alpha_2) = 1\right\}.\end{aligned}$$

There are 36 *positive* roots in $\mathfrak{e}_6$. Then, ny taking a look at the corresponding table of roots, we see that

$$\begin{aligned}|\Psi_0(\alpha_2)| &= 15,\\ |\Psi_2(\alpha_2)| &= 1,\\ |\Theta(\alpha_2)| &= 20.\end{aligned}$$

Since we are considering the adjoint representation of E_6, the multiplicities of all the roots are $m(\lambda) = 2$, and we see that, by considering the roots in the set Θ, in the tangent space to the principal orbit E_6/T^6, we get the subspace $\mathfrak{D}(\Theta)$. It generates a distribution of dimension 40, which is the dimension of *EII*. If we take α_5 instead we have

$$\begin{aligned}|\Psi_0(\alpha_5)| &= 11,\\ |\Psi_2(\alpha_5)| &= 5,\\ |\Theta(\alpha_5)| &= 20,\end{aligned}$$

and similarly for α_3. With the three simple roots of $\mathfrak{e}_6$ we get distributions $\mathfrak{D}(\Theta)$ of rank (dimension) 40 in the tangent bundle of E_6/T^6.

4.4. **The space** $EVI = E_7/\mathrm{SO}(12)\mathrm{SU}(2)$**.**

	space	dim	rank	$\Phi(\mathfrak{g}_0, \mathfrak{a}_0)$
EVI	$E_7/\mathrm{SO}(12)\mathrm{SU}(2)$	64	4	$\mathfrak{f}_4$

$$\dim(E_7) = 133, \quad |\text{positive roots}| = 63.$$

For this space we take the root α_1 with the notation in [5, p. 477]. (In this notation, α_1 is the exterior root in the short arm of the diagram of E_7). We have 63 positive roots in $\mathfrak{e}_7$ and 32 of them have coefficient 1 on α_1. Since the roots have multiplicity $m = 2$, we see that we have a subspace of dimension 64 in the tangent space to the isoparametric submanifold E_7/T^7 of dimension 126. On the other hand, the set Ω generates a distribution of dimension 70.

4.5. The space $FI = F_4/\mathrm{Sp}(3)\mathrm{SU}(2)$.

	space	dim	rank	$\Phi(\mathfrak{g}_0, \mathfrak{a}_0)$
FI	$F_4/\mathrm{Sp}(3)\mathrm{SU}(2)$	28	4	$\mathfrak{f}_4$

This is a *split* symmetric space and the roots that we have to consider by the table are those involving the root α_1. Again here we consider the adjoint representation (in this case of F_4) and choose a principal orbit which is of the form F_4/T^4 and has dimension 48.

With the notation in [5, p. 477], α_1 is the first long root at the left in the diagram. The algebra $\mathfrak{f}_4$ has 24 positive roots and 14 of them have the coefficient of α_1 equal to 1. Since in the adjoint representation the roots have multiplicity $m = 2$, we see that we have a subspace of dimension 28 in the tangent space to the isoparametric submanifold F_4/T^4.

5. Proof of Theorems 2.1 and 2.2

In the present section we shall prove Theorems 2.1 and 2.2. To that end we are going to use formulae (A.13), (A.16) and (A.17), recalled in Appendix A, and also their dual versions (B.12), (B.13) and (B.14) that are obtained in Appendix B.

Let us start with a general observation. In order to prove each of the Theorems 2.1 and 2.2, it suffices to show that, for each positive root λ which *does not belong to* Θ, each vector of the basis $\Xi_{\mathfrak{p}}(\lambda)$ (A.10) of $\mathfrak{p}_{0\lambda} \subset T_E(M)$ may be computed as a linear combination of brackets (evaluated at E) of local fields defined around E that *belong to the distribution* $\mathfrak{D}(\Theta)$. It is important to mention that the vectors in (A.10) are associated to the roots in $\rho^{-1}(\lambda) = \rho^{-1}(\lambda)_{\mathbb{R}} \cup \rho^{-1}(\lambda)^*_{\mathbb{C}}$ and that in $\rho^{-1}(\lambda)^*_{\mathbb{C}}$ we have only one element of the pair $\{\alpha, \alpha^\sigma\}$ for each $\alpha \in \rho^{-1}(\lambda)_{\mathbb{C}}$.

Proof of Theorem 2.1. Here our space $N = \mathrm{Ad}(K)H \subset \mathfrak{p}_0$ is a *symmetric R-space* and the vector H is dual to a simple root η in $\Delta(\mathfrak{g}_0, \mathfrak{a}_0)$ which appears with coefficient 1 in the maximal root of $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$. We have $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0) = \Psi_0 \cup \Theta$ (see (3.1)) and the roots in Ψ_0 (written in terms of $\Delta(\mathfrak{g}_0, \mathfrak{a}_0)$) are those without the term η, while those in Θ have the term η with coefficient 1.

Let us take then $\lambda \in \Psi_0 \subset \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$ and recall the basis of $\mathfrak{p}_{0\lambda}$ given in (A.10). We start by taking $\gamma \in \left(\rho^{-1}(\lambda)^*_{\mathbb{C}}\right)$ for our λ and consider U_γ, V_γ for our chosen γ. By Lemma 3.2, there exist two roots δ and φ in $\Theta \subset \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$ such that

$$\lambda = \delta - \varphi$$

and $\delta + \varphi$ *is not a root* of $\Phi(\mathfrak{g}_0, \mathfrak{a}_0)$. Furthermore, for the root $\gamma \in \rho^{-1}(\lambda)$, there exist roots $\alpha \in \rho^{-1}(\delta)$ and $\beta \in \rho^{-1}(\varphi)$ such that $\gamma = \alpha - \beta$. So we consider $U_\gamma = U_{(\alpha-\beta)}$, $V_\gamma = V_{(\alpha-\beta)}$. Then, we are to use formulae (B.12) (for the present subscripts). We have

$$\begin{aligned}
\mathfrak{L}_{(\delta,\varphi,\alpha,\beta)} U_{(\alpha-\beta)} + \mathfrak{B}\mathfrak{1}_{(\delta,\varphi,\alpha,\beta)} &= \left[U^F_\alpha, U^F_\beta\right](E) + \left[V^F_\alpha, V^F_\beta\right](E), \\
\mathfrak{F}_{(\delta,\varphi,\alpha,\beta)} V_{(\alpha-\beta)} + \mathfrak{B}\mathfrak{2}_{(\delta,\varphi,\alpha,\beta)} &= \left[U^F_\alpha, V^F_\beta\right](E) - \left[V^F_\alpha, U^F_\beta\right](E).
\end{aligned}$$

Let us consider the terms $\mathfrak{B}\mathfrak{1}_{(\delta,\varphi,\alpha,\beta)}$ and $\mathfrak{B}\mathfrak{2}_{(\delta,\varphi,\alpha,\beta)}$ which, except by non-zero factors, are $B_1(\alpha, \beta)$ and $B_2(\alpha, \beta)$. Since $\delta + \varphi$ *is not a root* of $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$, $\alpha + \beta$ is not a root either because $\rho\left(\alpha+\beta\right)=\delta+\varphi$ and furthermore, by the same reason, neither $\alpha^{\sigma}+\beta$ nor $\alpha+\beta^{\sigma}$ are roots of $\Phi^{+}\left(\mathfrak{g},\mathfrak{h}\right)$. Thus $B_{1}\left(\alpha,\beta\right)$ and $B_{2}\left(\alpha,\beta\right)$ *vanish* and then the formulae above reduce to

$$\mathfrak{L}_{(\delta,\varphi,\alpha,\beta)}U_{(\alpha-\beta)}=\left[U_{\alpha}^{F},U_{\beta}^{F}\right](E)+\left[V_{\alpha}^{F},V_{\beta}^{F}\right](E),$$
$$\mathfrak{F}_{(\delta,\varphi,\alpha,\beta)}V_{(\alpha-\beta)}=\left[U_{\alpha}^{F},V_{\beta}^{F}\right](E)-\left[V_{\alpha}^{F},U_{\beta}^{F}\right](E),$$

so we see that $U_{(\alpha-\beta)}$, $V_{(\alpha-\beta)}$ are linear combinations of brackets (evaluated on E) of local fields defined around E and belonging to the distribution $\mathfrak{D}(\Theta)$.

It remains to consider the case of real roots. So take $\gamma\in\left(\rho^{-1}(\lambda)_{\mathbb{R}}\right)$ for $\lambda\in\Psi_{0}\subset\Phi^{+}\left(\mathfrak{g}_{0},\mathfrak{a}_{0}\right)$, then we have the vector W_{γ}. Again there exist two roots δ and φ in $\Theta\subset\Phi^{+}\left(\mathfrak{g}_{0},\mathfrak{a}_{0}\right)$ such that $\lambda=\delta-\varphi$, and roots $\alpha\in\rho^{-1}\left(\delta\right)$, $\beta\in\rho^{-1}(\varphi)$ such that $\gamma=\alpha-\beta$. Then we have the following possibilities:

$$\begin{aligned}&\text{(i) } \alpha \text{ and } \beta \text{ are both real roots of } \Phi\left(\mathfrak{g},\mathfrak{h}\right);\\&\text{(ii) } \alpha \text{ and } \beta \text{ are both complex roots of } \Phi\left(\mathfrak{g},\mathfrak{h}\right).\end{aligned}\tag{5.1}$$

Considering first the case (i) in (5.1), we have to use formulae (B.14). Here the notation (subscripts) in (B.14) should be changed as follows:

$$\lambda\longmapsto\delta,\quad\mu\longmapsto\varphi,\quad\delta\longmapsto\alpha,\quad\varphi\longmapsto\beta.\tag{5.2}$$

Then, by the same reason indicated above, $\mathfrak{B}\mathfrak{1}_{(\delta,\varphi,\alpha,\beta)}$ and $\mathfrak{B}\mathfrak{2}_{(\delta,\varphi,\alpha,\beta)}$ vanish in all cases. This shows that $W_{\gamma}=W_{(\alpha-\beta)}$ is a bracket (evaluated at E) of local fields defined around E that belong to the distribution $\mathfrak{D}(\Theta)$. On the other hand, in case (ii) of (5.1), we may apply formulae (B.13) (also with (5.2)) and here again $\mathfrak{B}\mathfrak{1}_{(\delta,\varphi,\alpha,\beta)}$ and $\mathfrak{B}\mathfrak{2}_{(\delta,\varphi,\alpha,\beta)}$ vanish. We see, again in this case, that W_{γ} is a sum of brackets (evaluated at E) of local fields that belong to the distribution $\mathfrak{D}(\Theta)$. Then we have the proof of Theorem 2.1. ☐

Proof of Theorem 2.2. Here the positive roots *not contained in* Θ are those $\omega\in\Psi_{0}$ and $\lambda\in\Psi_{2}$, defined in (3.3). Here, Ψ_{0} has the same meaning as in the previous theorem, but here we must apply Lemma 3.3, which says that for each $\omega\in\Psi_{0}\subset\Phi^{+}\left(\mathfrak{g}_{0},\mathfrak{a}_{0}\right)$ there are two roots φ and ψ in Θ such that $\omega=\varphi-\psi$ and the sum $(\varphi+\psi)$ *is not a root of* $\Phi\left(\mathfrak{g}_{0},\mathfrak{a}_{0}\right)$. By taking $\alpha\in\rho^{-1}(\varphi)$ and $\beta\in\rho^{-1}(\psi)$, we see again, as above, that $\alpha+\beta$ is not a root and neither $\alpha^{\sigma}+\beta$ nor $\alpha+\beta^{\sigma}$ are roots of $\Phi^{+}(\mathfrak{g},\mathfrak{h})$.

Then, for $\omega\in\Psi_{0}$, the proof just given yields that each vector of the basis $\Xi_{\mathfrak{p}}\left(\omega\right)$ of $\mathfrak{p}_{0\omega}\subset T_{E}(M)$ may be computed as a linear combination of brackets (evaluated at E) of local fields defined around E that belong to the distribution $\mathfrak{D}(\Theta)$.

It remains to consider the case in which $\lambda\in\Psi_{2}$. Again we start by taking $\gamma\in\left(\rho^{-1}(\lambda)_{\mathbb{C}}^{*}\right)$ for $\lambda\in\Psi_{2}\subset\Phi^{+}\left(\mathfrak{g}_{0},\mathfrak{a}_{0}\right)$ and consider U_{γ} and V_{γ} for our chosen γ. By Lemma 3.3, there exist two roots δ and φ in $\Theta\subset\Phi^{+}\left(\mathfrak{g}_{0},\mathfrak{a}_{0}\right)$ such that $\lambda=\delta+\varphi$ and $|\delta-\varphi|$ is not a root of $\Phi\left(\mathfrak{g}_{0},\mathfrak{a}_{0}\right)$. Furthermore, we have roots $\alpha\in\rho^{-1}\left(\delta\right)$ and $\beta\in\rho^{-1}(\varphi)$ such that $\gamma=\alpha+\beta$ and $|\alpha-\beta|$ is not a root and, as in the above proof of Theorem 2.1, neither $|\alpha-\beta^{\sigma}|$ nor $|\alpha^{\sigma}-\beta|$ are roots, and so $H_{1}\left(\alpha,\beta\right)$ and $T_{2}\left(\alpha,\beta\right)$ in (A.14) and (A.15) (on roots α,β) *vanish*, and we may write formulae

(A.13) as

$$\Theta_{(\lambda,\mu,\alpha,\beta)}U_{(\alpha+\beta)} = \left[U_\alpha^F, U_\beta^F\right](E) - \left[V_\alpha^F, V_\beta^F\right](E),$$
$$\Theta_{(\lambda,\mu,\alpha,\beta)}V_{(\alpha+\beta)} = \left[U_\alpha^F, V_\beta^F\right](E) + \left[V_\alpha^F, U_\beta^F\right](E).$$

Then we have that $U_\gamma = U_{(\alpha+\beta)}$, $V_\gamma = V_{(\alpha+\beta)}$ are sums of brackets (evaluated on E) of local fields defined around E and belonging to the distribution $\mathfrak{D}(\Theta)$. Now we see that the case of real roots is managed by using formulae (A.16) and/or (A.17) in similar fashion to the procedure applied above. This completes the proof of Theorem 2.2. $\square$

Appendix A. Notation and previous results

We shall use the notation in [9], and for that reason, we hope that the reader will have opportunity to take a look at the second section of [9]. On the other hand, we recall here parts of the fourth section of that paper which are needed in the proof of the theorems.

A.1. **Basis for $\mathfrak{g}_0$.** Let us take the σ and τ adapted Chevalley basis for $(\mathfrak{g},\mathfrak{h})$ from [9] (also [7]). In [9], we defined k_α for each $\alpha \in \Phi(\mathfrak{g},\mathfrak{h})$ by

$$\sigma(x_\alpha) = k_\alpha x_{\alpha^\sigma}, \qquad k_\alpha = \pm 1$$

and observed the identities

$$k_\alpha k_{\alpha^\sigma} = 1, \quad k_{\alpha^\sigma} = k_\alpha, \quad k_{-\alpha} = k_\alpha, \tag{A.1}$$

$$\theta(x_\alpha) = k_\alpha x_{-\alpha^\sigma}, \quad \theta(x_{\alpha^\sigma}) = k_\alpha x_{-\alpha}. \tag{A.2}$$

Keeping the σ and τ adapted Chevalley basis for $(\mathfrak{g},\mathfrak{h})$, let us consider, for $\alpha \in \Phi(\mathfrak{g},\mathfrak{h})$, the vectors

$$X_\alpha = x_\alpha + \sigma(x_\alpha), \quad Y_\alpha = i\left(x_\alpha - \sigma(x_\alpha)\right), \quad Z_\alpha = X_\alpha + Y_\alpha.$$

They are *fixed by* σ, so they belong to $\mathfrak{g}_0$. Now, setting

$$\begin{aligned} P_\alpha &= (X_\alpha + \theta X_\alpha), & Q_\alpha &= (Y_\alpha + \theta Y_\alpha), & R_\alpha &= (Z_\alpha + \theta Z_\alpha), \\ U_\alpha &= (X_\alpha - \theta X_\alpha), & V_\alpha &= (Y_\alpha - \theta Y_\alpha), & W_\alpha &= (Z_\alpha - \theta Z_\alpha), \end{aligned} \tag{A.3}$$

we see that the vectors in the first row of (A.3) belong to $\mathfrak{k}_0$ and those in the second row to $\mathfrak{p}_0$. Using (A.2) and the definitions, we observe that

$$\begin{aligned} P_\alpha &= (X_\alpha + \theta X_\alpha) = (x_\alpha + k_\alpha x_{\alpha^\sigma}) + (k_\alpha x_{-\alpha^\sigma} + x_{-\alpha}), \\ U_\alpha &= (X_\alpha - \theta X_\alpha) = (x_\alpha + k_\alpha x_{\alpha^\sigma}) - (k_\alpha x_{-\alpha^\sigma} + x_{-\alpha}), \\ Q_\alpha &= (Y_\alpha + \theta Y_\alpha) = i(x_\alpha - k_\alpha x_{\alpha^\sigma}) + i(k_\alpha x_{-\alpha^\sigma} - x_{-\alpha}), \\ V_\alpha &= (Y_\alpha - \theta Y_\alpha) = i(x_\alpha - k_\alpha x_{\alpha^\sigma}) - i(k_\alpha x_{-\alpha^\sigma} - x_{-\alpha}). \end{aligned} \tag{A.4}$$

On the other hand, the vectors R_α and W_α shall be considered only for α *real* (i.e., $\alpha^\sigma = \alpha$), and we have the equalities

$$\begin{aligned} &\text{if } k_\alpha = 1, & R_\alpha &= P_\alpha, \quad W_\alpha = U_\alpha; \\ &\text{if } k_\alpha = -1, & R_\alpha &= Q_\alpha, \quad W_\alpha = V_\alpha. \end{aligned} \tag{A.5}$$

For $\alpha \in \Phi_{\mathbb{C}}$ and $\beta \in \Phi_{\mathbb{R}}$ we see that P_α, Q_α, $R_\beta \in \mathfrak{k}_0$ and U_α, V_α, $W_\beta \in \mathfrak{p}_0$. We must notice also that by (A.1) we have

$$P_{-\alpha} = P_\alpha, \quad Q_{-\alpha} = -Q_\alpha, \quad U_{-\alpha} = -U_\alpha, \quad V_{-\alpha} = -V_\alpha. \tag{A.6}$$

Now as in [9], setting $\rho(\alpha) = \rho(\beta) = \lambda$, for $\alpha \in \Phi_{\mathbb{C}}$, $\beta \in \Phi_{\mathbb{R}}$, the vectors in (A.3) are such that

$$P_\alpha, Q_\alpha, R_\beta \in \mathfrak{k}_{0\lambda} \quad \text{and} \quad U_\alpha, V_\alpha, W_\beta \in \mathfrak{p}_{0\lambda}. \tag{A.7}$$

A.2. **Basis for $\mathfrak{k}_{0,\lambda}$ and $\mathfrak{p}_{0,\lambda}$, $\lambda \in \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$.** Consider now for $\lambda \in \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$ the set $\rho^{-1}(\lambda) = \{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{h}) : \rho(\alpha) = \lambda\}$ and split it separating the real roots from the complex ones. So we set $\rho^{-1}(\lambda)_{\mathbb{R}} = \rho^{-1}(\lambda) \cap \Phi_{\mathbb{R}}$ and $\rho^{-1}(\lambda)_{\mathbb{C}} = \rho^{-1}(\lambda) \cap \Phi_{\mathbb{C}}$. For a root α in $\rho^{-1}(\lambda)_{\mathbb{C}}$ we have $\alpha^\sigma \neq \alpha$; then we define, as in [7], the set $\rho^{-1}(\lambda)^*_{\mathbb{C}}$ where we place *one of the two elements* in $\{\alpha, \alpha^\sigma\}$ for each $\alpha \in \rho^{-1}(\lambda)_{\mathbb{C}}$. Now for $\lambda, \mu \in \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$ take the sets

$$\begin{aligned} \Xi_{\mathfrak{k}}(\lambda) &= \left\{R_\eta, P_\delta, Q_\gamma : \eta \in \rho^{-1}(\lambda)_{\mathbb{R}}, \delta, \gamma \in \rho^{-1}(\lambda)^*_{\mathbb{C}}\right\}, \\ \Xi_{\mathfrak{p}}(\mu) &= \left\{W_\alpha, U_\beta, V_\varphi : \alpha \in \rho^{-1}(\mu)_{\mathbb{R}}, \beta, \varphi \in \rho^{-1}(\mu)^*_{\mathbb{C}}\right\}. \end{aligned} \tag{A.8}$$

By (A.7), we have $\Xi_{\mathfrak{k}}(\lambda) \subset \mathfrak{k}_{0\lambda}$ and $\Xi_{\mathfrak{p}}(\mu) \subset \mathfrak{p}_{0\mu}$, and each set is linearly independent over $\mathbb{R}$. Since the equal cardinalities of $\Xi_{\mathfrak{k}}(\lambda)$ and $\Xi_{\mathfrak{p}}(\lambda)$ coincide with the dimensions of $\mathfrak{k}_{0,\lambda}$ and $\mathfrak{p}_{0,\lambda}$, we have a basis for each of these subspaces. Obviously, there is a one-to-one correspondence between $\Xi_{\mathfrak{k}}(\lambda)$ and $\Xi_{\mathfrak{p}}(\lambda)$. For the members of the basis $\Xi_{\mathfrak{k}}(\lambda)$ and $\Xi_{\mathfrak{p}}(\lambda)$ we have

$$[R_\eta, E] = -\eta(E) W_\eta, \quad [P_\delta, E] = -\delta(E) U_\delta, \quad [Q_\delta, E] = -\delta(E) V_\delta,$$

which is coherent with their one-to-one correspondence.

A.3. **Smooth local fields.** Proceeding as in [9], we may extend the vectors of the basis $\bigcup_{\lambda \in \Theta} \Xi_{\mathfrak{p}}(\lambda)$ to local fields defined in some open set A_E containing E in M. We use for them the same notation as in [9], that is,

$$\left\{U^F_\beta, V^F_\beta, W^F_\alpha : \beta \in \left(\rho^{-1}(\lambda)^*_{\mathbb{C}}\right), \alpha \in \left(\rho^{-1}(\lambda)_{\mathbb{R}}\right), \lambda \in \Theta\right\}.$$

At the point $E \in M$, they coincide with the vectors $\{W_\alpha, U_\beta, V_\beta\}$ of $\Xi_{\mathfrak{p}}(\lambda)$ for $\lambda \in \Theta$. We have then a local basis for $\mathfrak{D}(\Theta)$ in the open set A_E containing E and these fields are *smooth in A_E*. Now, at any other point $p \in M$ there is a $g \in K$ such that $p = \mathrm{Ad}(g)E$ and we may consider the open set $\mathrm{Ad}(g)A_E$ containing p. On such open set, we have a local basis of smooth vector fields defined by translation of those on A_E with $\mathrm{Ad}(g)$. Hence, by the usual definition ([12, p. 41, Def. 1.56]), the distribution $\mathfrak{D}(\Theta)$ on M is smooth.

In [9] we computed the brackets of the fields in $\mathfrak{D}(\Theta)$ constructed above by using the Levi-Civita connection on M. We recall the resulting formula. The bracket of the fields U^F_φ and U^F_γ at E, for $\gamma \in \rho^{-1}(\lambda)$ and $\varphi \in \rho^{-1}(\mu)$, evaluated at E, is

$$\left[U^F_\varphi, U^F_\gamma\right](E) = \left(\frac{-1}{\mu(E)}\right) Ta\left([P_\varphi, U_\gamma]\right) - \left(\frac{-1}{\lambda(E)}\right) Ta\left([P_\gamma, U_\varphi]\right). \tag{A.9}$$

In this formula, we have brackets of fields (evaluated at E) on the left side and products in $\mathfrak{g}_0$ on the right side. We use the words *brackets* for fields and *products* for vectors in $\mathfrak{g}_0$.

For λ, $\mu \in \Theta$, we have the bases $\Xi_{\mathfrak{p}}(\lambda)$ for $\mathfrak{p}_{0\lambda}$ and $\Xi_{\mathfrak{p}}(\mu)$ for $\mathfrak{p}_{0\mu}$, respectively, as indicated in (A.8). To fix notation we set them as

$$\begin{aligned}\Xi_{\mathfrak{p}}(\lambda) &= \left\{U_\gamma, V_\gamma, W_\delta : \gamma \in \rho^{-1}(\lambda)^*_{\mathbb{C}},\ \delta \in \rho^{-1}(\lambda)_{\mathbb{R}}\right\} \subset \mathfrak{p}_{0\lambda},\\ \Xi_{\mathfrak{p}}(\mu) &= \left\{U_\varphi, V_\varphi, W_\eta : \varphi \in \rho^{-1}(\mu)^*_{\mathbb{C}},\ \eta \in \rho^{-1}(\mu)_{\mathbb{R}},\right\} \subset \mathfrak{p}_{0\mu}.\end{aligned} \tag{A.10}$$

Each of these tangent vectors at E generates a corresponding field $\left\{U^F_\gamma, V^F_\gamma, W^F_\beta\right\}$ and $\left\{U^F_\varphi, V^F_\varphi, W^F_\delta\right\}$ around E. So we have nine possible brackets of these fields.

A.4. **Known identities.** We need to mention some important identities proven in [9] that are to be used in the required computations. First recall that we have

$$c_{\delta,\beta} = c_{-\delta,-\beta}. \tag{A.11}$$

Since $\sigma\left(x_{(\alpha+\beta)}\right) = k_{(\alpha+\beta)} x_{(\alpha+\beta)^\sigma}$ and $[x_\alpha, x_\beta] = c_{\alpha,\beta} x_{(\alpha+\beta)}$ with real coefficients $c_{\alpha,\beta}$, the following identities hold:

$$\begin{aligned}\sigma\left[x_\alpha, x_\beta\right] &= \sigma\left(c_{\alpha,\beta} x_{\alpha+\beta}\right) = c_{\alpha,\beta}\sigma\left(x_{\alpha+\beta}\right) = c_{\alpha,\beta} k_{(\alpha+\beta)} x_{(\alpha+\beta)^\sigma},\\ \sigma\left[x_\alpha, x_\beta\right] &= \left[\sigma x_\alpha, \sigma x_\beta\right] = \left[k_\alpha x_{\alpha^\sigma}, k_\beta x_{\beta^\sigma}\right] = k_\alpha k_\beta c_{\alpha^\sigma,\beta^\sigma} x_{(\alpha^\sigma+\beta^\sigma)}.\end{aligned}$$

By repeating this computation for $[x_\alpha, x_{-\beta}]$, $[x_{-\alpha}, x_\beta]$ and $[x_{-\alpha}, x_{-\beta}]$ (using (A.11) and (A.1)), we get the four equalities

$$\begin{aligned}k_\alpha k_\beta c_{\alpha^\sigma,\beta^\sigma} x_{\alpha^\sigma+\beta^\sigma} &= c_{\alpha,\beta} k_{(\alpha+\beta)} x_{(\alpha+\beta)^\sigma},\\ k_\alpha k_\beta c_{\alpha^\sigma,-\beta^\sigma} x_{\alpha^\sigma-\beta^\sigma} &= c_{\alpha,-\beta} k_{(\alpha-\beta)} x_{(\alpha-\beta)^\sigma},\\ k_\alpha k_\beta c_{-\alpha^\sigma,\beta^\sigma} x_{-\alpha^\sigma+\beta^\sigma} &= c_{-\alpha,\beta} k_{(-\alpha+\beta)} x_{(-\alpha+\beta)^\sigma},\\ k_\alpha k_\beta c_{-\alpha^\sigma,-\beta^\sigma} x_{-\alpha^\sigma-\beta^\sigma} &= c_{\alpha,\beta} k_{(\alpha+\beta)} x_{-(\alpha+\beta)^\sigma}.\end{aligned} \tag{A.12}$$

A.5. **Formulae for the sum of roots.** We need to recall now some formulae obtained in [9] and used there and also in [10]. Those formulae give expressions for the vectors

$$U_{(\delta+\varphi)},\ V_{(\delta+\varphi)},\ W_{(\delta+\varphi)} \qquad \text{in } T_E(M)$$

(for the roots λ, $\mu \in \Omega \subset \Phi^+\left(\mathfrak{g}_0, \mathfrak{a}_0\right)$ with $\delta \in \rho^{-1}\left(\lambda\right)^*_{\mathbb{C}}$ and $\varphi \in \rho^{-1}(\mu)^*_{\mathbb{C}}$) in terms of brackets, evaluated at E, of local fields defined around E. They are:

$$\begin{aligned}\Theta_{(\lambda,\mu,\delta,\varphi)} U_{(\delta+\varphi)} + \Lambda_{(\lambda,\mu)}(Ta(H_1)) &= [U^F_\delta, U^F_\varphi](E) - [V^F_\delta, V^F_\varphi](E),\\ \Theta_{(\lambda,\mu,\delta,\varphi)} V_{(\delta+\varphi)} + \Lambda_{(\lambda,\mu)}(Ta(T_2)) &= [U^F_\delta, V^F_\varphi](E) + [V^F_\delta, U^F_\varphi](E),\end{aligned} \tag{A.13}$$

with

$$\begin{aligned}H_1(\delta,\varphi) = 2(&k_\delta c_{\delta^\sigma,-\varphi}(x_{-\delta^\sigma+\varphi} - x_{\delta^\sigma-\varphi})\\ &- k_\varphi c_{\delta,-\varphi^\sigma}(x_{\delta-\varphi^\sigma} - x_{-\delta+\varphi^\sigma})),\end{aligned} \tag{A.14}$$

$$\begin{aligned}T_2(\delta,\varphi) = 2i(&k_\delta c_{\delta^\sigma,-\varphi}(x_{\delta^\sigma-\varphi} + x_{-\delta^\sigma+\varphi})\\ &- k_\varphi c_{\delta,-\varphi^\sigma}(x_{\delta-\varphi^\sigma} + x_{-\delta+\varphi^\sigma})).\end{aligned} \tag{A.15}$$

We have to consider also the case in which $\delta+\varphi$ is real and both δ and φ complex. Again, $\lambda, \mu \in \Omega \subset \Phi^{+}(\mathfrak{g}_0, \mathfrak{a}_0)$, $\delta \in \rho^{-1}(\lambda)^*_{\mathbb{C}}$ and $\varphi \in \rho^{-1}(\mu)^*_{\mathbb{C}}$. In this case, from (A.13) and having (A.5) in mind we obtain

$$\begin{aligned}
&k_{(\delta+\varphi)} = 1,\\
&\Theta_{(\lambda,\mu,\delta,\varphi)} W_{(\delta+\varphi)} + \Lambda_{(\lambda,\mu)}\left(Ta\left(H_1\right)\right) = \left[U_\delta^F, U_\varphi^F\right](E) - \left[V_\delta^F, V_\varphi^F\right](E),\\
&k_{(\delta+\varphi)} = (-1),\\
&\Theta_{(\lambda,\mu,\delta,\varphi)} W_{(\delta+\varphi)} + \Lambda_{(\lambda,\mu)}\left(Ta\left(T_2\right)\right) = \left[U_\delta^F, V_\varphi^F\right](E) + \left[V_\delta^F, U_\varphi^F\right](E),
\end{aligned} \tag{A.16}$$

and it is necessary to consider also the case in which both δ and φ are real. That is, $\lambda, \mu \in \Omega \subset \Phi^{+}(\mathfrak{g}_0, \mathfrak{a}_0)$, $\delta \in \rho^{-1}(\lambda)_{\mathbb{R}}$ and $\varphi \in \rho^{-1}(\mu)_{\mathbb{R}}$. Furthermore, the first line in (A.12) in the present case clearly yields $k_\delta k_\varphi = k_{(\delta+\varphi)}$ and then formulae (A.16) become

$$\begin{aligned}
&k_{(\delta+\varphi)} = 1,\ k_\delta = k_\varphi = 1,\\
&\Theta_{(\lambda,\mu,\delta,\varphi)} W_{(\delta+\varphi)} + \Lambda_{(\lambda,\mu)}\left(Ta\left(H_1\right)\right) = \left[W_\delta^F, W_\varphi^F\right](E),\\
&k_{(\delta+\varphi)} = 1,\ k_\delta = k_\varphi = -1,\\
&\Theta_{(\lambda,\mu,\delta,\varphi)} W_{(\delta+\varphi)} + \Lambda_{(\lambda,\mu)}\left(Ta\left(H_1\right)\right) = -\left[W_\delta^F, W_\varphi^F\right](E),\\
&k_{(\delta+\varphi)} = (-1),\ k_\delta = 1,\ k_\varphi = -1,\\
&\Theta_{(\lambda,\mu,\delta,\varphi)} W_{(\delta+\varphi)} + \Lambda_{(\lambda,\mu)}\left(Ta\left(T_2\right)\right) = \left[W_\delta^F, W_\varphi^F\right](E),\\
&k_{(\delta+\varphi)} = (-1),\ k_\delta = -1,\ k_\varphi = 1,\\
&\Theta_{(\lambda,\mu,\delta,\varphi)} W_{(\delta+\varphi)} + \Lambda_{(\lambda,\mu)}\left(Ta\left(T_2\right)\right) = \left[W_\delta^F, W_\varphi^F\right](E).
\end{aligned} \tag{A.17}$$

Appendix B. Formulae for the difference of roots

In this appendix, we give a proof for the new formulae (B.12), (B.13) and (B.14), used in the proofs of Theorems 2.1 and 2.2. These formulae are complementary to those in (A.13), (A.16) and (A.17) (obtained in [9]), as the latter give vectors corresponding to sums of roots, while the former give those associated to the difference of roots.

By (A.9), to get the brackets of tangent fields we take the basis $\Xi_{\mathfrak{k}}(\lambda)$ and $\Xi_{\mathfrak{p}}(\mu)$ in (A.8), for $\mathfrak{k}_{0,\lambda}$ and $\mathfrak{p}_{0,\mu}$ respectively. There are nine *possible* products, namely

$$\begin{array}{lll}
(1)\ \left[R_\eta, W_\alpha\right], & (2)\ \left[R_\eta, U_\beta\right], & (3)\ \left[R_\eta, V_\varphi\right],\\
(4)\ \left[P_\delta, W_\alpha\right], & (5)\ \left[P_\delta, U_\beta\right], & (6)\ \left[P_\delta, V_\varphi\right],\\
(7)\ \left[Q_\gamma, W_\alpha\right], & (8)\ \left[Q_\gamma, U_\beta\right], & (9)\ \left[Q_\gamma, V_\varphi\right].
\end{array} \tag{B.1}$$

However, we shall need only (1) (for η, α real) and (5), (6), (8) and (9) for complex roots (δ, φ, β and γ). Now we compute the products (5), (6), (8) and (9) mentioned in (B.1). For reasons of space, we shall not perform all these computations in full detail; however, it is straightforward to complete them. We take a pair of *complex* roots δ, β in $\Phi^{+}(\mathfrak{g}_0, \mathfrak{a}_0)$. Let us start computing the product (5) with the expressions in (A.4):

$$(5)\ [P_\delta, U_\beta] = \big[\left((x_\delta + k_\delta x_{\delta^\sigma}) + (k_\delta x_{-\delta^\sigma} + x_{-\delta})\right), \\ (x_\beta + k_\beta x_{\beta^\sigma}) - (k_\beta x_{-\beta^\sigma} + x_{-\beta})\big].$$

We obtain

$$\begin{aligned}(5)\ [P_\delta, U_\beta] = {} & c_{\delta,\beta} x_{\delta+\beta} + k_\beta c_{\delta,\beta^\sigma} x_{\delta+\beta^\sigma} + k_\delta c_{\delta^\sigma,\beta} x_{\delta^\sigma+\beta} + k_\delta k_\beta c_{\delta^\sigma,\beta^\sigma} x_{\delta^\sigma+\beta^\sigma} \\ & - k_\beta c_{\delta,-\beta^\sigma} x_{\delta-\beta^\sigma} - c_{\delta,-\beta} x_{\delta-\beta} - k_\delta k_\beta c_{\delta^\sigma,-\beta^\sigma} x_{\delta^\sigma-\beta^\sigma} \\ & - k_\delta c_{\delta^\sigma,-\beta} x_{\delta^\sigma-\beta} + k_\delta c_{-\delta^\sigma,\beta} x_{-\delta^\sigma+\beta} + k_\delta k_\beta c_{-\delta^\sigma,\beta^\sigma} x_{-\delta^\sigma+\beta^\sigma} \\ & + c_{-\delta,\beta} x_{-\delta+\beta} + k_\beta c_{-\delta,\beta^\sigma} x_{-\delta+\beta^\sigma} - k_\delta k_\beta c_{-\delta^\sigma,-\beta^\sigma} x_{-\delta^\sigma-\beta^\sigma} \\ & - k_\delta c_{-\delta^\sigma,-\beta} x_{-\delta^\sigma-\beta} - k_\beta c_{-\delta,-\beta^\sigma} x_{-\delta-\beta^\sigma} - c_{-\delta,-\beta} x_{-\delta-\beta}.\end{aligned}$$

There are four terms with the product $k_\delta k_\beta$, which can be replaced using the above identities (A.12); by doing this we get

$$\begin{aligned}(5)\ [P_\delta, U_\beta] = {} & c_{\delta,\beta} x_{\delta+\beta} + k_\beta c_{\delta,\beta^\sigma} x_{\delta+\beta^\sigma} + k_\delta c_{\delta^\sigma,\beta} x_{\delta^\sigma+\beta} + c_{\delta,\beta} k_{(\delta+\beta)} x_{(\delta+\beta)^\sigma} \\ & - k_\beta c_{\delta,-\beta^\sigma} x_{\delta-\beta^\sigma} - c_{\delta,-\beta} x_{\delta-\beta} - c_{\delta,-\beta} k_{(\delta-\beta)} x_{(\delta-\beta)^\sigma} \\ & - k_\delta c_{\delta^\sigma,-\beta} x_{\delta^\sigma-\beta} + k_\delta c_{-\delta^\sigma,\beta} x_{-\delta^\sigma+\beta} + c_{-\delta,\beta} k_{(-\delta+\beta)} x_{(-\delta+\beta)^\sigma} \\ & + c_{-\delta,\beta} x_{-\delta+\beta} + k_\beta c_{-\delta,\beta^\sigma} x_{-\delta+\beta^\sigma} - c_{\delta,\beta} k_{(\delta+\beta)} x_{-(\delta+\beta)^\sigma} \\ & - k_\delta c_{-\delta^\sigma,-\beta} x_{-\delta^\sigma-\beta} - k_\beta c_{-\delta,-\beta^\sigma} x_{-\delta-\beta^\sigma} - c_{-\delta,-\beta} x_{-\delta-\beta}.\end{aligned}$$

Let us consider now the expression for U_a in (A.4) (taking $\alpha = (\delta - \beta)$). If we multiply by $(-1)\, c_{\delta,-\beta}$, it takes the form

$$(-1) c_{\delta,-\beta} U_{(\delta-\beta)} = -c_{\delta,-\beta}\big(x_{(\delta-\beta)} + k_{(\delta-\beta)} x_{(\delta-\beta)^\sigma} - k_{(\delta-\beta)} x_{-(\delta-\beta)^\sigma} - x_{-(\delta-\beta)}\big),$$

and observe that *the four terms conforming* $(-1)c_{\delta,-\beta}U_{(\delta-\beta)}$ *are present in the product* (5). By placing those terms at the end (recalling that $c_{\delta,\beta} = c_{-\delta,-\beta}$ and $k_{(\delta-\beta)} = k_{(\beta-\delta)}$), we have

$$\begin{aligned}(5)\ [P_\delta, U_\beta] = {} & c_{\delta,\beta} x_{\delta+\beta} + k_\beta c_{\delta,\beta^\sigma} x_{\delta+\beta^\sigma} + k_\delta c_{\delta^\sigma,\beta} x_{\delta^\sigma+\beta} + c_{\delta,\beta} k_{(\delta+\beta)} x_{(\delta+\beta)^\sigma} \\ & - k_\beta c_{\delta,-\beta^\sigma} x_{\delta-\beta^\sigma} - k_\delta c_{\delta^\sigma,-\beta} x_{\delta^\sigma-\beta} + k_\delta c_{-\delta^\sigma,\beta} x_{-\delta^\sigma+\beta} \\ & + k_\beta c_{-\delta,\beta^\sigma} x_{-\delta+\beta^\sigma} - c_{\delta,\beta} k_{(\delta+\beta)} x_{-(\delta+\beta)^\sigma} - k_\delta c_{-\delta^\sigma,-\beta} x_{-\delta^\sigma-\beta} \\ & - k_\beta c_{-\delta,-\beta^\sigma} x_{-\delta-\beta^\sigma} - c_{-\delta,-\beta} x_{-\delta-\beta} - c_{\delta,-\beta} x_{\delta-\beta} \\ & - c_{\delta,-\beta} k_{(\delta-\beta)} x_{(\delta-\beta)^\sigma} + c_{-\delta,\beta} k_{(-\delta+\beta)} x_{(-\delta+\beta)^\sigma} + c_{-\delta,\beta} x_{-\delta+\beta}.\end{aligned}$$

Then, replacing them, the product (5) can be written as

$$\begin{aligned}(5)\ [P_\delta, U_\beta] = {} & c_{\delta,\beta} x_{\delta+\beta} + k_\beta c_{\delta,\beta^\sigma} x_{\delta+\beta^\sigma} + k_\delta c_{\delta^\sigma,\beta} x_{\delta^\sigma+\beta} + c_{\delta,\beta} k_{(\delta+\beta)} x_{(\delta+\beta)^\sigma} \\ & - k_\beta c_{\delta,-\beta^\sigma} x_{\delta-\beta^\sigma} - k_\delta c_{\delta^\sigma,-\beta} x_{\delta^\sigma-\beta} + k_\delta c_{-\delta^\sigma,\beta} x_{-\delta^\sigma+\beta} \\ & + k_\beta c_{-\delta,\beta^\sigma} x_{-\delta+\beta^\sigma} - c_{\delta,\beta} k_{(\delta+\beta)} x_{-(\delta+\beta)^\sigma} - k_\delta c_{-\delta^\sigma,-\beta} x_{-\delta^\sigma-\beta} \\ & - k_\beta c_{-\delta,-\beta^\sigma} x_{-\delta-\beta^\sigma} - c_{-\delta,-\beta} x_{-\delta-\beta} + (-1) c_{\delta,-\beta} U_{(\delta-\beta)}.\end{aligned}$$

We may repeat the computation just performed but with the product (9) and *the same pair of roots* δ and β.

$$\begin{aligned}(9)\ \ [Q_\delta, V_\beta] = \big[& i\left(x_\delta - k_\delta x_{\delta^\sigma}\right) + i\left(k_\delta x_{-\delta^\sigma} - x_{-\delta}\right), \\ & \qquad\qquad i\left(x_\beta - k_\beta x_{\beta^\sigma}\right) - i\left(k_\beta x_{-\beta^\sigma} - x_{-\beta}\right)\big].\end{aligned}$$

We get

$$\begin{aligned}(9)\ \ [Q_\delta, V_\beta] = & -c_{\delta,\beta} x_{\delta+\beta} + k_\beta c_{\delta,\beta^\sigma} x_{\delta+\beta^\sigma} + k_\delta c_{\delta^\sigma,\beta} x_{\delta^\sigma+\beta} - c_{\delta,\beta} k_{(\delta+\beta)} x_{(\delta+\beta)^\sigma} \\ & + k_\beta c_{\delta,-\beta^\sigma} x_{\delta-\beta^\sigma} + k_\delta c_{\delta^\sigma,-\beta} x_{\delta^\sigma-\beta} - k_\delta c_{-\delta^\sigma,\beta} x_{-\delta^\sigma+\beta} \\ & - k_\beta c_{-\delta,\beta^\sigma} x_{-\delta+\beta^\sigma} + c_{\delta,\beta} k_{(\delta+\beta)} x_{-(\delta+\beta)^\sigma} - k_\delta c_{-\delta^\sigma,-\beta} x_{-\delta^\sigma-\beta} \\ & - k_\beta c_{-\delta,-\beta^\sigma} x_{-\delta-\beta^\sigma} + c_{-\delta,-\beta} x_{-\delta-\beta} + (-1) c_{\delta,-\beta} U_{(\delta-\beta)}.\end{aligned}$$

Now, by computing *the sum* (that is, (5) + (9)) line by line of the two final expressions, we observe that the second lines in (5) and (9) cancel each other and the sum is

$$\begin{aligned}[P_\delta, U_\beta] + [Q_\delta, V_\beta] = & \, 2\left(k_\beta c_{\delta,\beta^\sigma} x_{\delta+\beta^\sigma} + k_\delta c_{\delta^\sigma,\beta} x_{\delta^\sigma+\beta}\right) \\ & + 2\left(-k_\delta c_{-\delta^\sigma,-\beta} x_{-\delta^\sigma-\beta} - k_\beta c_{-\delta,-\beta^\sigma} x_{-\delta-\beta^\sigma}\right) \\ & + (-2) c_{\delta,-\beta} U_{(\delta-\beta)},\end{aligned}$$

which we may write as

$$(5) + (9)\ \ (-2) c_{\delta,-\beta} U_{(\delta-\beta)} + B_1(\delta,\beta) = [P_\delta, U_\beta] + [Q_\delta, V_\beta], \tag{B.2}$$

where

$$\begin{aligned}B_1(\delta,\beta) = & \, 2k_\beta c_{\delta,\beta^\sigma}\big(x_{\delta+\beta^\sigma} - x_{-(\delta+\beta^\sigma)}\big) \\ & + 2k_\delta c_{\delta^\sigma,\beta}\big(x_{\delta^\sigma+\beta} - x_{-(\delta^\sigma+\beta)}\big),\end{aligned} \tag{B.3}$$

and observe that

$$B_1(\beta,\delta) = (-1) B_1(\delta,\beta). \tag{B.4}$$

Now, using the pair of roots $\{\omega,\varphi\}$ and proceeding as above, we may compute the products (6) and (8), and observe that in both of them appear (*with opposite signs*) the terms of

$$\begin{aligned}c_{\omega,-\varphi} V_{(\omega-\varphi)} = & \, i\left(c_{\omega,-\varphi} x_{(\omega-\varphi)} - k_{(\omega-\varphi)} c_{\omega,-\varphi} x_{(\omega-\varphi)^\sigma}\right) \\ & - i\left(k_{(\omega-\varphi)} c_{\omega,-\varphi} x_{-(\omega-\varphi)^\sigma} - c_{\omega,-\varphi} x_{-(\omega-\varphi)}\right).\end{aligned}$$

Then, by computing the *difference* (6) − (8), we get

$$(6) - (8)\ \ 2c_{\omega,-\varphi} V_{(\omega-\varphi)} + B_2(\omega,\varphi) = [P_\omega, V_\varphi] - [Q_\omega, U_\varphi], \tag{B.5}$$

and, similarly to (B.3), we find that

$$\begin{aligned}B_2(\omega,\varphi) = & -2ik_\varphi c_{\omega,\varphi^\sigma}\left(x_{\omega+\varphi^\sigma} + x_{-(\omega+\varphi^\sigma)}\right) \\ & + 2k_\omega c_{\omega^\sigma,\varphi}\left(x_{\omega^\sigma+\varphi} + x_{-(\omega^\sigma+\varphi)}\right)\end{aligned} \tag{B.6}$$

and $B_2(\omega,\varphi)$ has the property

$$B_2(\varphi,\omega) = B_2(\omega,\varphi). \tag{B.7}$$

In this fashion, we have formula (B.2) with its companion (B.3) for the complex roots (δ,β) and (B.5) with (B.6) for the pair of complex roots (ω,φ).

B.1. **Brackets of fields.** Recalling (B.1) and the definitions (A.3), we take the following bases of $p_{0,\lambda}$ and $p_{0,\mu}$, respectively:

$$\Xi_{\mathfrak{p}}(\lambda)=\left\{W_\eta,U_\delta,V_\gamma:\eta\in\rho^{-1}(\lambda)_{\mathbb{R}},\ \delta,\gamma\in\rho^{-1}(\lambda)^*_{\mathbb{C}}\right\},$$
$$\Xi_{\mathfrak{p}}(\mu)=\left\{W_\alpha,U_\beta,V_\varphi:\alpha\in\rho^{-1}(\mu)_{\mathbb{R}},\ \beta,\varphi\in\rho^{-1}(\mu)^*_{\mathbb{C}}\right\}.$$

With these two bases we may obtain the corresponding local fields and with them form the nine brackets corresponding to the products in (B.1). However, we shall need only those indicated in the following formulae. Using (A.9) (for η and α real and γ, δ, β and φ complex) they are

$$\{\eta,\alpha\}\text{ real }\{\delta,\varphi\}\text{ complex; }\varphi\in\rho^{-1}(\lambda),\ \delta\in\rho^{-1}(\mu)$$

$$\begin{aligned}
&(1)\ \left[W^F_\eta,W^F_\alpha\right](E)=\left(\frac{-1}{\lambda(E)}\right)Ta\left([R_\eta,W_\alpha]\right)-\left(\frac{-1}{\mu(E)}\right)Ta\left([R_\alpha,W_\eta]\right),\\
&(5)\ \left[U^F_\delta,U^F_\varphi\right](E)=\left(\frac{-1}{\lambda(E)}\right)Ta\left([P_\delta,U_\varphi]\right)-\left(\frac{-1}{\mu(E)}\right)Ta\left([P_\varphi,U_\delta]\right),\\
&(6)\ \left[U^F_\delta,V^F_\varphi\right](E)=\left(\frac{-1}{\lambda(E)}\right)Ta\left([P_\delta,V_\varphi]\right)-\left(\frac{-1}{\mu(E)}\right)Ta\left([Q_\varphi,U_\delta]\right),\\
&(8)\ \left[V^F_\delta,U^F_\varphi\right](E)=\left(\frac{-1}{\lambda(E)}\right)Ta\left([Q_\delta,U_\varphi]\right)-\left(\frac{-1}{\mu(E)}\right)Ta\left([P_\varphi,V_\delta]\right),\\
&(9)\ \left[V^F_\delta,V^F_\varphi\right](E)=\left(\frac{-1}{\lambda(E)}\right)Ta\left([Q_\delta,V_\varphi]\right)-\left(\frac{-1}{\mu(E)}\right)Ta\left([Q_\varphi,V_\delta]\right).
\end{aligned}\tag{B.8}$$

Let us consider now the following two vectors in $T_E(M)$ for $\delta\in\rho^{-1}(\mu)^*_{\mathbb{C}}$ and $\varphi\in\rho^{-1}(\lambda)^*_{\mathbb{C}}$:

$$\begin{aligned}
I(\delta,\varphi)&=\left[U^F_\delta,U^F_\varphi\right](E)+\left[V^F_\delta,V^F_\varphi\right](E),\\
J(\delta,\varphi)&=\left[U^F_\delta,V^F_\varphi\right](E)-\left[V^F_\delta,U^F_\varphi\right](E).
\end{aligned}\tag{B.9}$$

Considering equalities (5) and (9) in (B.8) and (B.2), we have

$$\begin{aligned}
I(\delta,\varphi)=&\left(\frac{-1}{\lambda(E)}\right)\{Ta\,[P_\delta,U_\varphi]+Ta\,[Q_\delta,V_\varphi]\}\\
&-\left(\frac{-1}{\mu(E)}\right)\{Ta\,[P_\varphi,U_\delta]+Ta\,[Q_\varphi,V_\delta]\},
\end{aligned}$$

and, by (B.2), we may write

$$\begin{aligned}
I(\delta,\varphi)=&\left(\frac{-1}{\lambda(E)}\right)\left\{(-2)\,c_{\delta,-\varphi}U_{(\delta-\varphi)}+B_1(\delta,\varphi)\right\}\\
&-\left(\frac{-1}{\mu(E)}\right)\left\{(-2)\,c_{\varphi,-\delta}U_{(\varphi-\delta)}+B_1(\varphi,\delta)\right\}.
\end{aligned}$$

Now, since $U_{-\alpha} = -U_\alpha$ (see (A.6)) and $c_{\delta,-\varphi} = c_{-\delta,\varphi} = -c_{\varphi,-\delta}$, to simplify notation, we may set

$$\mathfrak{L}_{(\lambda,\mu,\delta,\varphi)} = 2c_{\delta,-\varphi}\left(\frac{1}{\lambda(E)} - \frac{1}{\mu(E)}\right),$$

$$\begin{aligned}\mathfrak{B}\mathfrak{1}_{(\lambda,\mu,\delta,\varphi)} &= \left(\frac{-1}{\lambda(E)}\right) B_1(\delta,\varphi) - \left(\frac{-1}{\mu(E)}\right) B_1(\varphi,\delta)\\ &= \left(\left(\frac{-1}{\lambda(E)}\right) + \left(\frac{-1}{\mu(E)}\right)\right) B_1(\delta,\varphi).\end{aligned}$$

By (B.4) and recalling the definition of $I(\delta,\varphi)$, we may finally write

$$\begin{aligned}I(\delta,\varphi) &= \left[U_\delta^F, U_\varphi^F\right](E) + \left[V_\delta^F, V_\varphi^F\right](E)\\ &= \mathfrak{L}_{(\lambda,\mu,\delta,\varphi)} U_{(\delta-\varphi)} + \mathfrak{B}\mathfrak{1}_{(\lambda,\mu,\delta,\varphi)}.\end{aligned} \tag{B.10}$$

Proceeding similarly with (6) and (8) we see that

$$\begin{aligned}J(\delta,\varphi) = &\left(\frac{-1}{\lambda(E)}\right)\{Ta[P_\delta, V_\varphi] - Ta[Q_\delta, U_\varphi]\}\\ &- \left(\frac{-1}{\mu(E)}\right)\{Ta[Q_\varphi, U_\delta] - Ta[P_\varphi, V_\delta]\},\end{aligned}$$

and by the definition (B.9) and the equality (B.5), we see that

$$\begin{aligned}J(\delta,\varphi) = &\left(\frac{-1}{\lambda(E)}\right)\left\{2c_{\delta,-\varphi}V_{(\delta-\varphi)} + B_2(\delta,\varphi)\right\}\\ &- \left(\frac{-1}{\mu(E)}\right)\left\{2c_{\varphi,-\delta}V_{(\varphi-\delta)} + B_2(\varphi,\delta)\right\}.\end{aligned}$$

Now, again by (A.6), $V_{-\alpha} = -V_\alpha$ and since $c_{\delta,-\varphi} = -c_{\varphi,-\delta}$ we may set

$$\mathfrak{F}_{(\lambda,\mu,\delta,\varphi)} = -2c_{\delta,-\varphi}\left(\frac{1}{\lambda(E)} + \frac{1}{\mu(E)}\right)$$

and, recalling (B.7), we may also set

$$\begin{aligned}\mathfrak{B}\mathfrak{2}_{(\lambda,\mu,\delta,\varphi)} &= \left(\frac{-1}{\lambda(E)}\right) B_2(\delta,\varphi) - \left(\frac{-1}{\mu(E)}\right) B_2(\varphi,\delta)\\ &= \left(\left(\frac{-1}{\lambda(E)}\right) - \left(\frac{-1}{\mu(E)}\right)\right) B_2(\delta,\varphi).\end{aligned}$$

This notation allows us to finally write

$$\begin{aligned}J(\delta,\varphi) &= \left[U_\delta^F, V_\varphi^F\right](E) - \left[V_\delta^F, U_\varphi^F\right](E)\\ &= \mathfrak{F}_{(\lambda,\mu,\delta,\varphi)} V_{(\delta-\varphi)} + \mathfrak{B}\mathfrak{2}_{(\lambda,\mu,\delta,\varphi)}.\end{aligned} \tag{B.11}$$

B.2. **Resulting formulae for the difference of roots.** We may now write formulae (B.10) and (B.11), for the roots λ, $\mu \in \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$, $\delta \in \rho^{-1}(\lambda)^*_{\mathbb{C}}$ and $\varphi \in \rho^{-1}(\mu)^*_{\mathbb{C}}$, as

$$\begin{aligned} &\left[U^F_\delta, U^F_\varphi\right](E) + \left[V^F_\delta, V^F_\varphi\right](E) = \mathfrak{L}_{(\lambda,\mu,\delta,\varphi)} U_{(\delta-\varphi)} + \mathfrak{B}\mathfrak{1}_{(\lambda,\mu,\delta,\varphi)}, \\ &\left[U^F_\delta, V^F_\varphi\right](E) - \left[V^F_\delta, U^F_\varphi\right](E) = \mathfrak{F}_{(\lambda,\mu,\delta,\varphi)} V_{(\delta-\varphi)} + \mathfrak{B}\mathfrak{2}_{(\lambda,\mu,\delta,\varphi)}. \end{aligned} \tag{B.12}$$

It is also necessary to consider the case in which $(\delta-\varphi)$ is a real root and both δ and φ are complex. Again, $\lambda, \mu \in \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$, $\delta \in \rho^{-1}(\lambda)^*_{\mathbb{C}}$ and $\varphi \in \rho^{-1}(\mu)^*_{\mathbb{C}}$. It follows from (B.12), due to (A.5), that we have

$$\begin{aligned} &k_{(\delta-\varphi)} = 1, \\ &\left[U^F_\delta, U^F_\varphi\right](E) + \left[V^F_\delta, V^F_\varphi\right](E) = \mathfrak{L}_{(\lambda,\mu,\delta,\varphi)} W_{(\delta-\varphi)} + \mathfrak{B}\mathfrak{1}_{(\lambda,\mu,\delta,\varphi)}, \\ &k_{(\delta-\varphi)} = (-1), \\ &\left[U^F_\delta, V^F_\varphi\right](E) - \left[V^F_\delta, U^F_\varphi\right](E) = \mathfrak{F}_{(\lambda,\mu,\delta,\varphi)} W_{(\delta-\varphi)} + \mathfrak{B}\mathfrak{2}_{(\lambda,\mu,\delta,\varphi)}. \end{aligned} \tag{B.13}$$

Let us consider now the case in which both δ and φ are real, that is, λ, $\mu \in \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$, $\delta \in \rho^{-1}(\lambda)_{\mathbb{R}}$ and $\varphi \in \rho^{-1}(\mu)_{\mathbb{R}}$. The second line in (A.12) in the present case clearly yields $k_\delta k_\varphi = k_{(\delta-\varphi)}$ and then formulae (B.13) become

$$\begin{aligned} &k_{(\delta-\varphi)} = 1,\ k_\delta = k_\varphi = 1, \\ &\mathfrak{L}_{(\lambda,\mu,\delta,\varphi)} W_{(\delta-\varphi)} + \mathfrak{B}\mathfrak{1}_{(\lambda,\mu,\delta,\varphi)} = \left[W^F_\delta, W^F_\varphi\right](E), \\ &k_{(\delta-\varphi)} = 1,\ k_\delta = k_\varphi = -1, \\ &\mathfrak{L}_{(\lambda,\mu,\delta,\varphi)} W_{(\delta-\varphi)} + \mathfrak{B}\mathfrak{1}_{(\lambda,\mu,\delta,\varphi)} = \left[W^F_\delta, W^F_\varphi\right](E), \\ &k_{(\delta-\varphi)} = (-1),\ k_\delta = 1,\ k_\varphi = -1, \\ &\mathfrak{F}_{(\lambda,\mu,\delta,\varphi)} W_{(\delta-\varphi)} + \mathfrak{B}\mathfrak{2}_{(\lambda,\mu,\delta,\varphi)} = \left[W^F_\delta, W^F_\varphi\right](E), \\ &k_{(\delta-\varphi)} = (-1),\ k_\delta = -1,\ k_\varphi = 1, \\ &\mathfrak{F}_{(\lambda,\mu,\delta,\varphi)} W_{(\delta-\varphi)} + \mathfrak{B}\mathfrak{2}_{(\lambda,\mu,\delta,\varphi)} = -\left[W^F_\delta, W^F_\varphi\right](E). \end{aligned} \tag{B.14}$$

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