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NEW HARMONIC-MEASURE DISTRIBUTION FUNCTIONS OF SOME SIMPLY CONNECTED PLANAR REGIONS IN THE COMPLEX PLANE

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ABSTRACT. Consider a Brownian particle released from a fixed point z_0 in a region Ω. The harmonic-measure distribution function, or h-function, $h(r)$, expresses the probability that the Brownian particle first hits the boundary $\partial\Omega$ of the region Ω within distance r of z_0. In this paper, we compute the h-function of several new planar simply connected two-dimensional regions by using two different methods, both involving conformal maps. We also explain the asymptotic behaviour at certain values of r where two different regimes meet. Moreover, for some regions, we examine how the behaviour of $h(r)$ changes when part of the boundary changes.

1. Introduction and overview

This paper is a continuation of the study of *harmonic-measure distribution functions* or *h-functions*, which, broadly speaking, are functions that encapsulate geometric information about a domain in the complex plane. These functions were first studied in [8], motivated by Stephenson's questions about the function $w(r)$ (see [8, Problem 6.116]), which is closely related to the h-function $h(r)$, but not exactly the same. h-functions have been studied with the overall goals of understanding what geometric features of a domain can be recovered from an h-function, what functions are h-functions of some domain-base point pair, and ultimately, reconstruction of domains whose domain has some specified h-function. A summary of this progress can be found in the survey article [7], which traces the progress of h-functions studies from the initial paper [8] up to the time of the survey's publication. In particular, explicit formulae for h-functions for some simply connected domains can be found by conformal mapping techniques, enabling the study of how the specific features in a domain will affect the h-function.

Recently, in [4] and [5], Mahenthiram presented the h-functions of many new simply connected planar regions, including some whose boundaries feature special structures such as junctions, corners, or cusps. Also, the progress of the study on h-functions of multiply connected domains can be found in [2] and [5].

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Now, we focus on the definition of the h-function: For a domain Ω in the complex plane $\mathbb{C}$, and a basepoint z_0 in Ω, the h-function $h_{\Omega,z_0}(r)$, for short $h(r)$, is given by the harmonic measure of the set $E_r := \partial\Omega \cap \overline{B(z_0, r)}$ with respect to the domain Ω at the basepoint z_0, where $B(z_0, r)$ is the open ball of radius r centred at z_0, and $\partial\Omega$ is the boundary of Ω. In other words,

$$h_{\Omega,z_0}(r) = h(r) := \omega(z_0, E_r, \Omega),$$

where ω is the harmonic measure. Also, we can describe the h-function via the solution of the following Dirichlet problem:

$$\Delta u(z) = 0 \quad \text{if } z \in \Omega,$$

$$u(z) = \begin{cases} 1 & \text{if } z \in E_r; \\ 0 & \text{if } z \in \partial\Omega \backslash E_r. \end{cases}$$

The h-function is equal to the solution of the above Dirichlet problem at z_0. That is, $h(r) = u(z_0)$.

An alternative way of explaining the h-function is the physical intuition of the Brownian motion. For a domain Ω and a basepoint $z_0 \in \Omega$, release a Brownian particle from z_0, and allow it to wander; eventually it will hit the boundary $\partial\Omega$. We are interested where this particle hits the boundary $\partial\Omega$ within distance r of z_0. From this information, the h-function is given by the probability that the particle first exits Ω within distance r of z_0; see [3].

The h-function always lies in the unit interval $[0, 1]$ and it is a non-decreasing, right-continuous function.

Throughout this paper, we use the notation d to denote the shortest distance from the basepoint z_0 to the boundary $\partial\Omega$ of the given domain Ω.

The h-function of the upper halfplane $\Omega = \mathbb{C}\backslash(-\infty, \infty)$ with basepoint $z_0 = id$, $d > 0$, is given by

$$h(r) = \begin{cases} 0 & \text{for } 0 < r < d; \\ \frac{2}{\pi} \arctan \frac{\sqrt{r^2 - d^2}}{d} & \text{for } r \geq d. \end{cases} \tag{1.1}$$

See [8, page 292]. This h-function is defined for two different ranges of r, such as $r \in (0, d)$ and $r \in [d, \infty)$, which we call the *regimes* of r. This h-function has only two regimes of r. In the first regime, we always exclude both endpoints of the interval. However, in the later regimes, we exclude one endpoint and include the other. In the first regime $r \in (0, d)$, the h-function is zero as no part of the boundary is encloseed by the closed ball of radius r centred at z_0. Hereafter, we call the boundary of this ball as the *capture circle*. The h-function is always zero for the first regime of r. In the expression (1.1), for the second regime $r \in [d, \infty)$, the h-function is positive, but is not equal to 1 since the boundary $\partial\Omega$ of the halfplane is unbounded. Therefore, the capture circle cannot enclose the whole boundary for any value of r. However, when the boundary of a domain is bounded (and even if the domain is unbounded), the h-function of this domain will take the value 1 for the last regime of r. In the expression (1.1), the h-function $h(r) \to 1$ as $r \to \infty$, but will not take the value 1.

This paper focuses on two different methods for computing the h-functions of certain new simply connected planar domains with a given basepoint z_0. It also explains the asymptotics of some of these h-functions at certain values of r where two regimes meet. Through the investigation of the h-functions of these domains, we try to understand the relationship between the h-function and the shape of the respective domain. Also, through the study of asymptotics, we try to answer for the longstanding open problem: Is there any simply connected domain with exponent $\beta = 1$? (See Definition 2.1 for the explanation of β.)

Broadly speaking, the exponent β explains the behaviour of the h-function at the intersection of the first and second regimes of r. In most of the h-functions that we computed so far, the h-functions have a vertical tangent at this intersecting point from the right. Obviously, all the h-functions have a horizontal tangent at this intersecting point from the left, since the h-function will always be zero in the first regime of r. So far, we have obtained the values of $\beta \in [0, 1)$ for simply connected regions. In this paper, we have obtained the exponent $\beta = 1/2$ for some regions. However, for the computational difficulty due to the complicated formulas of $h(r)$, we did not explain the exponent β for all domains described in this paper.

In the first method to compute the h-function, we use the *Riemann mapping theorem*, which states that any simply connected region is conformally equivalent to the interior of the unit disc. However, in this paper, we mostly use the conformal mapping from the given simply connected domain Ω to a halfplane. In the second method, we use the conformal transformation that shows the map between the unit disc and our target domain. First, we use the *Cayley map*, which transforms the interior of the unit disc to the lower halfplane. Then, we use appropriate conformal map from the given domain Ω to the interior of the unit disc. Then, we use the composition of these maps to obtain the connection between the domain Ω and the lower halfplane. See the initial part of Section 2 for the detailed explanation of the computational process for these two methods.

This paper is organised as follows. In Section 2, we give some necessary definitions and useful information that we use in this paper. In Section 3, we compute the h-function of the exterior of a parabola that is symmetrically positioned about the negative real axis and passing through the origin, when the basepoint z_0 is fixed on the positive real axis. In Section 4, we explain the h-functions for the complement of union of two intersecting discs of equal size with a chosen basepoint z_0, and also describe the h-functions of the union of two intersecting discs of equal size with some locations of the basepoint z_0. In Section 5, we explore the h-function for the complement of a round lollipop that is symmetrically positioned about the real axis, with a purely imaginary basepoint z_0. In Section 6, we compute the h-function of the complement of a round lollipop, but with the replacement of the ray to a slit, again with a purely imaginary basepoint z_0. In Section 7, we explain the h-function of a round lollipop along with a line segment when the basepoint z_0 is fixed on the real axis. In Section 8, we calculate the h-function for the complement of a disc with two line segments where the line segments meet the disc on the real axis, when the basepoint z_0 is fixed on the real axis. In Section 9, we describe the

h-function for the complement of two discs which are joined with a common line segment. Further, in some sections, we check how the behaviour of the h-function changes when part of the boundary is removed from the given domain.

2. Background

In this section, we provide some definitions, and also explain the process by which we compute h-functions. In Section 2.1, we describe the steps that we follow in computing h-functions. In Section 2.2, we explain certain types of asymptotics that we usually use for h-functions.

2.1. **Computation of h-functions.** This section describes two different methods to compute h-functions of simply connected domains. Both methods involve the use of conformal maps.

In the first method, we find the conformal map $f(z)$ which transforms the given domain Ω to a halfplane. Then for $r \in (0, \infty)$, we find the set $E_r := \partial\Omega \cap \overline{B(z_0, r)}$ in Ω and its image $f(E_r)$ in the halfplane. Also, in the halfplane, we identify the image $f(z_0)$ of the basepoint z_0. Finally, we find the *angle of sight*, which is the angle subtended by $f(E_r)$ at $f(z_0)$. Then, the normalised angle of sight produces the h-function formula. We use this method for most of the sections of this paper.

In the second method, we use the conformal map in two directions. One is for the conformal map $F(\zeta)$ from the interior of the unit disc D_ζ to our target domain Ω. The second is the *classical Cayley map*

$$R(\zeta, \tau, \bar{\tau}) = -\frac{1}{\tau}\left(\frac{\zeta - \tau}{\zeta - \bar{\tau}}\right),$$

which transforms the interior of the unit disc D_ζ to the lower halfplane, and maps the boundary of the disc ∂D_ζ to the real axis, where $\zeta \in D_\zeta$ and $\tau, \bar{\tau} \in \partial D_\zeta$. See [2, page 11].

Next, we formulate a harmonic function $W(\zeta, \tau, \bar{\tau})$ whose imaginary part solves our Dirichlet problem in D_ζ. Section 9 describes this approach in detail.

2.2. **Asymptotics.** In this section, we explain two types of asymptotics that we use throughout this paper. These are as follows:

(i) as $r \downarrow d$;
(ii) as $r \downarrow r^*$ $(> d)$.

Walden and Ward introduced the asymptotic behaviour as r decreases to d, namely *purely exponential asymptotics*, in [8]. The second type of asymptotics as $r \downarrow r^*$ $(> d)$ has been studied by Matsumoto [6]. He showed that the h-function can be asymptotically linear as $r \downarrow r^*$ $(> d)$, where the derivative of $h(r)$ is discontinuous at r^*. However, "can the h-function be asymptotically linear as $r \downarrow d$?" is still an open problem.

Definition 2.1 (Purely exponential asymptotics, Walden and Ward [8]). Let Ω be a domain and take a basepoint z_0 in Ω. Let d be the shortest distance between the basepoint z_0 and the boundary $\partial\Omega$. Then the h-function $h(r)$ of the domain Ω has *purely exponential asymptotics with exponent β as r decreases to d*, written as

$$h(r) \sim c(r-d)^{\beta} \quad \text{as } r \downarrow d,$$

if there is a real number β such that the limit

$$c := \lim_{r \to d^+} \frac{h(r)}{(r-d)^{\beta}}$$

exists and is positive.

Theorem 2.2 (Matsumoto [6])**.** *For each $c > 0$, there exists a domain Ω and $z_0 \in \Omega$ such that for some $r^* > d$,*

$$\lim_{r \downarrow r^*} \frac{h(r) - h(r^*)}{(r-r^*)^{\gamma}} = c \quad \textit{with } \gamma = 1.$$

3. Exterior of a parabola

In this section, we explain the h-function of the exterior of a parabola when the basepoint z_0 is fixed on the real axis. The h-function of the interior of a parabola has already been studied by Walden and Ward, and documented in [8]. See also the survey article [7].

Here, we consider the domain $\Omega = \{z = x + iy \in \mathbb{C} : y^2 > -4x\}$, which is the exterior of the parabola $y^2 = -4x$. We fix the basepoint $z_0 = 1$. Then, the conformal map $\sqrt{z+1} - 1$ transplants the domain Ω onto the right halfplane. See Figure 1.

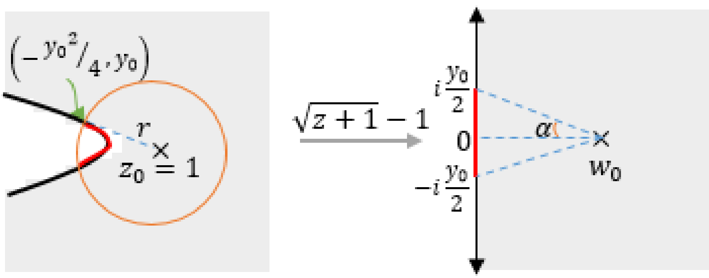


Figure 1. Conformal transformation from the exterior of a parabola to a halfplane. The set E_r (left) and its image $f(E_r)$ (right) are shown in red.

From Figure 1, we have that

$$\left(1 + \frac{y_0^2}{4}\right)^2 + y_0^2 = r^2.$$

Thus,

$$y_0^2 = -12 \pm 4\sqrt{8 + r^2}.$$

Since $y_0^2 > 0$, we have

$$y_0 = \sqrt{-12 + 4\sqrt{8+r^2}}.$$

From Figure 1, the angle of sight in the halfplane is 2α, which can be obtained from the expression

$$\tan\alpha = \frac{y_0/2}{w_0},$$

where $w_0 = \sqrt{2}-1$. Hence, the h-function is

$$h(r) = \frac{2}{\pi}\arctan\left\{\frac{\sqrt{-3+\sqrt{8+r^2}}}{\sqrt{2}-1}\right\},$$

which is the normalised angle of sight in the halfplane. The h-function graph for the above domain Ω with basepoint $z_0 = 1$ is shown in Figure 2.

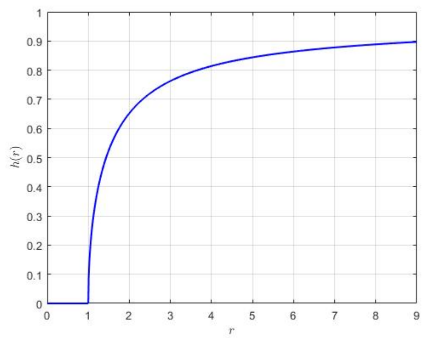


FIGURE 2. Graph of the h-function for the exterior of the parabola $y^2 = -4x$ with basepoint $z_0 = 1$.

Next, we focus on the asymptotic behaviour near $r = d = 1$. As r decreases to d,

$$h(r) \sim \frac{2}{\pi(\sqrt{2}-1)}\sqrt{f(r)}, \tag{3.1}$$

where $f(r) = -3 + \sqrt{8+r^2}$. Now, by the Taylor expansion of $f(r)$ at $r = 1$, we have

$$\begin{aligned} f(r) &= f(1) + \frac{f'(1)}{1!}(r-1) + \frac{f''(1)}{2!}(r-1)^2 + \cdots \\ &\sim \frac{1}{3}(r-1). \end{aligned}$$

Then,

$$\sqrt{f(r)} \sim \frac{1}{\sqrt{3}}(r-1)^{1/2}. \tag{3.2}$$

By substituting (3.2) in (3.1), we obtain

$$h(r) \sim \frac{2}{\pi\sqrt{3}(\sqrt{2}-1)}(r-1)^{1/2} = c(r-d)^{\beta},$$

where $c = \frac{2}{\pi\sqrt{3}(\sqrt{2}-1)}$ and $\beta = 1/2$.

4. Union of two intersecting discs of equal size

In this section, we explain the h-function of a domain Ω that consists of two intersecting circles as its boundary. The domain Ω is either the union of two intersecting discs of equal size or the complement of the union of two intersecting discs of equal size. In Section 4.1, we compute the h-function for the complement of the union of two intersecting equal-sized discs with a specific location of the basepoint z_0 on the real axis. In Section 4.2, we calculate the h-function for the union of two intersecting equal-sized discs with various locations of the basepoint z_0.

4.1. Complement of the union of two intersecting discs of equal size.

The domain Ω is the complement of the union of two intersecting discs of equal size, where the discs are centred at 2 and 3.5, both with unit radius. We fix the basepoint $z_0 = 0$. See Figure 3.

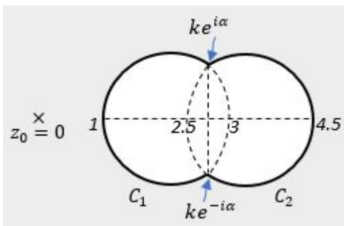


Figure 3. Domain $\Omega = \mathbb{C}\backslash\left(\overline{B(2,1)} \cup \overline{B(3.5,1)}\right)$ with various positions of basepoint z_0.

When the radius of the capture circle is smaller than 1, the h-function is zero. When the radius r increases past 1, the capture circle encloses an increasing portion of the boundary $\partial\Omega$ of the domain Ω. By solving the equations of both boundary circles, we identify that these two circles intersect at $(11/4, \sqrt{7}/4)$ and $(11/4, -\sqrt{7}/4)$. By converting these co-ordinates into the polar system, both circles intersect at $ke^{i\alpha}$ and $ke^{-i\alpha}$, where $k = \sqrt{8}$ and $\alpha = \arccos\left(\frac{11}{4k}\right)$.

Since the domain Ω is simply connected, we transform the domain Ω onto a halfplane or to the interior of the unit disc. We map the domain Ω onto a halfplane. We start with the Möbius map $(z - ke^{i\alpha})/(z - ke^{-i\alpha})$ that sends the points $ke^{i\alpha}$, $ke^{-i\alpha}$ and 0 to the points 0, ∞ and $e^{2i\alpha}$, respectively. This Möbius map transforms the domain Ω to the interior of a wedge domain with the interior angle 2γ, where $\gamma = \arccos(3/4)$. Then, the conformal map $z^{\pi/2\gamma}$ transplants the interior of the wedge domain onto the right halfplane.

For the regime $1 \le r < k$, the capture circle encloses part of the left-hand circular boundary in $\partial\Omega$, but does not enclose any part of the right-hand circular boundary in $\partial\Omega$. In this case, the h-function is

$$h(r) = \frac{1}{\pi} \arctan \left\{ \frac{\left(\rho_2^{\pi/2\gamma} - \rho_1^{\pi/2\gamma}\right) \cos\left(\frac{\pi\alpha}{\gamma}\right)}{1 + (\rho_1\rho_2)^{\pi/2\gamma} - \left(\rho_1^{\pi/2\gamma} + \rho_2^{\pi/2\gamma}\right) \sin\left(\frac{\pi\alpha}{\gamma}\right)} \right\},$$

where

$$\rho_1 = \frac{4(8-r^2)}{31 - 3r^2 + \sqrt{7(9-r^2)(r^2-1)}} \quad \text{and} \quad \rho_2 = \frac{4(8-r^2)}{31 - 3r^2 - \sqrt{7(9-r^2)(r^2-1)}}.$$

For the regime $k \le r < 4.5$, the capture circle encloses the whole left-hand circular boundary in $\partial\Omega$, and also encloses part of the right-hand circular boundary in $\partial\Omega$. In this case, the h-function is

$$h(r) = \frac{1}{\pi} \arctan \left\{ \frac{\left[\rho_3^{\pi/2\gamma} + \rho_4^{\pi/2\gamma} + 2\sin\left(\frac{\pi\alpha}{\gamma}\right)\right] \cos\left(\frac{\pi\alpha}{\gamma}\right)}{(\rho_3\rho_4)^{\pi/2\gamma} + \left(\rho_3^{\pi/2\gamma} + \rho_4^{\pi/2\gamma}\right) \sin\left(\frac{\pi\alpha}{\gamma}\right) - \cos\left(\frac{2\pi\alpha}{\gamma}\right)} \right\},$$

where

$$\rho_3 = \frac{16(r^2-8)}{12r^2 - 47 + \sqrt{7(4r^2-25)(81-4r^2)}}$$

and

$$\rho_4 = \frac{16(r^2-8)}{12r^2 - 47 - \sqrt{7(4r^2-25)(81-4r^2)}}.$$

For the regime $r \ge 4.5$, the capture circle encloses the entire boundary of the domain Ω. Therefore, $h(r) = 1$ for this range of r. Figure 4 expresses the h-function graph of Ω when the basepoint z_0 is fixed at the origin. Moreover, this h-function has a horizontal tangent at $r = k$.

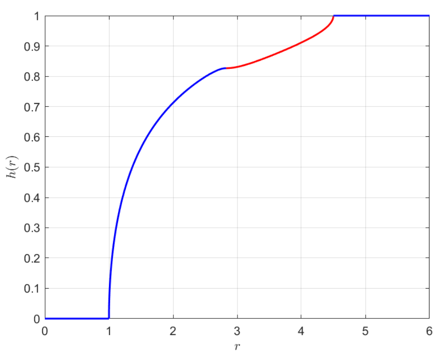


FIGURE 4. Graph of the h-function for the domain $\Omega = \mathbb{C}\backslash\left(\overline{B(2,1)} \cup \overline{B(3.5,1)}\right)$ with basepoint $z_0 = 0$.

As noted in the Introduction, Mahenthiram [4] found several domains with corners or cusps whose h-functions have a horizontal tangent at r^*, where r^* is the distance between the basepoint z_0 to the corners or cusps in the boundary of the domains. Also, Greco [1] found the similar behaviour at r^* in the exterior of a wedge with spike domain when the basepoint z_0 is fixed along the line of the spike.

4.2. **Union of two intersecting discs of equal size.** Consider the domain Ω which is the union of two discs centred at 0 and 1.5, both with unit radius. By solving the equations of both boundary circles, we identify that these two circles intersect at $(3/4, \sqrt{7}/4)$ and $(3/4, -\sqrt{7}/4)$. By converting these co-ordinates into the polar system, both circles intersect at $e^{i\alpha}$ and $e^{-i\alpha}$, where $\alpha = \arccos(3/4)$. See Figure 5.

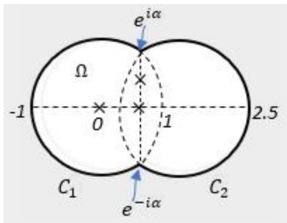


FIGURE 5. Domain $\Omega = B(0,1) \cup B(1.5,1)$ with various positions of the basepoint z_0.

To compute the h-function, we map the domain Ω to a halfplane. We start with the Möbius map $-(z-e^{i\alpha})/(z-e^{-i\alpha})$ that sends the points $e^{i\alpha}$, $e^{-i\alpha}$ and 0 to the points 0, ∞ and $-e^{2i\alpha}$, respectively. This Möbius map transforms the domain Ω to the exterior of a wedge domain with the exterior angle $2(\pi-\alpha)$. Then, the conformal map $z^{\pi/2(\pi-\alpha)}$ transforms the exterior of the wedge domain to the right halfplane.

Now, we discuss the h-function of this domain Ω with various locations of the basepoint z_0. We fix z_0 at three specific positions such as 0.75, $0.75+0.25i$ and 0. See Figure 5.

A. Fix the basepoint $z_0 = 0.75$. That is, the basepoint z_0 is fixed exactly middle in the intersecting region of both discs. Let $y_0 = \sqrt{7}/4$. Now, for the regime $r < y_0$, the h-function is zero. Similarly for the regime $r \geq 1.75$, the h-function is equal to 1. Now, for the regime $y_0 \leq r < 1.75$, the capture circle equally encloses part of the boundary of both discs. In this case, the h-function is

$$h(r) = \frac{2}{\pi}\arctan\left\{\frac{1+(\rho_1\rho_2)^{\pi/2(\pi-\alpha)}}{\rho_2^{\pi/2(\pi-\alpha)} - \rho_1^{\pi/2(\pi-\alpha)}}\right\},$$

where

$$\rho_1 = \frac{4(16r^2-7)}{3(16r^2+7)+\sqrt{7(16r^2-1)(49-16r^2)}}$$

and

$$\rho_2 = \frac{4(16r^2-7)}{3(16r^2+7)-\sqrt{7(16r^2-1)(49-16r^2)}}.$$

Now, we compare the h-function of the interior of the unit disc with the same basepoint $z_0 = 0.75$. To map this domain Ω to the halfplane, we use the Möbius map $(1+z)/(1-z)$ that sends the points -1, 1 and 0 to the points 0, ∞ and 1, respectively. For the regime $0 < r < 0.25$, the h-function is zero. Similarly, the h-function takes the value 1 for the regime $r \geq 1.75$. For the regime $0.25 \leq r < 1.75$, the h-function is

$$h(r) = \frac{2}{\pi}\arctan\left\{7\sqrt{\frac{16r^2-1}{49-16r^2}}\right\}.$$

Figure 6 shows the comparison between the h-function of the interior of the unit disc (red color graph) and the domain $\Omega = B(0,1) \cup B(1.5,1)$ which is the union of two intersecting discs where one is the unit disc (blue color graph), with the same basepoint $z_0 = 0.75$. Moreover, the h-function $h(r)$ of the interior of the unit disc has a point of inflection at $r = 1.432491$, while $h(r)$ for the domain Ω with basepoint $z_0 = 0.75$ has a point of inflection at $r = 1.44799$.

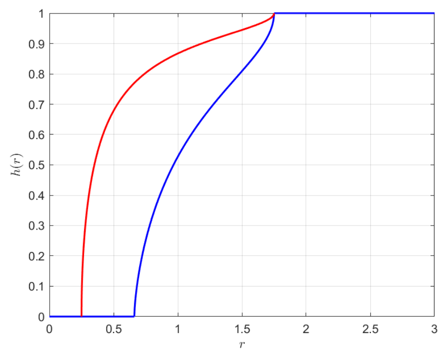


FIGURE 6. Comparing the graphs of the h-function for the interior of the unit disc (red) and the domain $\Omega = B(0,1) \cup B(1.5,1)$ with the same basepoint $z_0 = 0.75$ (blue).

B. Fix the basepoint $z_0 = 0.75 + 0.25i$. That is, the basepoint z_0 is fixed in the intersecting region of both discs, but in an off-center location (complex point). We can write $z_0 = ke^{i\theta}$, where $k = \sqrt{10}/4$ and $\theta = \arccos(3/4k)$. Let $y_0 = \sqrt{7}/4$. Now, for the regime $r < y_0 - 0.25$, the h-function is zero. For the regime $y_0 - 0.25 \leq r < y_0 + 0.25$, the capture circle encloses part of the boundary in the top side of both discs, and also equally encloses the boundary in both discs. In this case, the h-function is

$$h(r) = \frac{2}{\pi}\arctan\left\{\left[\left(\frac{\sqrt{7}+1}{\sqrt{7}-1}\right)\rho_1\right]^{\pi/2(\pi-\alpha)}\right\},$$

where

$$\rho_1 = \frac{4[3(8r^2-3)+\sqrt{160-(13-8r^2)^2}]}{43+72r^2+3\sqrt{160-(13-8r^2)^2}+3\sqrt{7(160-(13-8r^2)^2)}+\sqrt{7}(13-8r^2)}.$$

For the regime $r \geq y_0 + 0.25$, the capture circle encloses part of the boundary in the top and bottom sides of both discs, and also equally encloses the boundary in both discs. In this case, the h-function is

$$h(r) = \frac{2}{\pi}\left\{\arctan\left[\left(\frac{\sqrt{7}+1}{\sqrt{7}-1}\right)\rho_1\right]^{\pi/2(\pi-\alpha)} + \arctan\left[\left(\frac{\sqrt{7}-1}{\sqrt{7}+1}\right)\frac{1}{\rho_2}\right]^{\pi/2(\pi-\alpha)}\right\},$$

where

$$\rho_2 = \frac{4\big[3(8r^2-3)-\sqrt{160-(13-8r^2)^2}\big]}{43+72r^2-3\sqrt{160-(13-8r^2)^2}-3\sqrt{7(160-(13-8r^2)^2)}+\sqrt{7}(13-8r^2)}.$$

Figure 7 shows the h-function graph for the domain Ω with basepoint $z_0 = 0.75 + 0.25i$. Moreover, this h-function has a point of inflection at $r = 1.48941$.

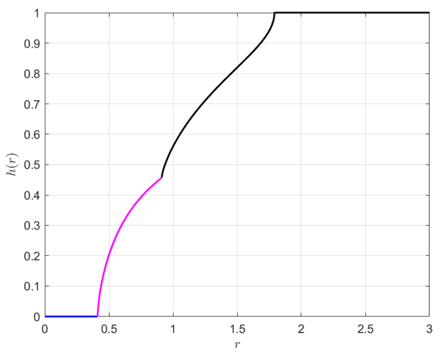


FIGURE 7. Graph of the h-function for the domain $\Omega = B(0,1) \cup B(1.5,1)$ with basepoint $z_0 = 0.75 + 0.25i$.

C. Fix the basepoint $z_0 = 0$. That is, the basepoint z_0 is fixed at the centre of the left-hand disc. In this case, we expect a jump in the graph of the h-function, because there is a sudden increase in the harmonic measure when r takes the value 1. For the regime $0 < r < 1$, we have $h(r) = 0$ and for the regime $r \geq 2.5$, we have $h(r) = 1$. When $r = 1$, the capture circle encloses the whole boundary of the left-hand disc in Ω. In this case, $h(r) = 0.8506$. For the regime $1 \leq r < 2.5$, the capture circle encloses the entire boundary of the left-hand disc, and also encloses part of the right-hand disc in Ω. In this case, the h-function is

$$h(r) = \frac{1}{\pi}\arctan\left\{\frac{\left(\rho_1^{\pi/2(\pi-\alpha)} - \rho_2^{\pi/2(\pi-\alpha)}\right)\cos\left[\frac{(\pi-2\alpha)\pi}{2(\pi-\alpha)}\right]}{1+(\rho_1\rho_2)^{\pi/2(\pi-\alpha)} + \left(\rho_1^{\pi/2(\pi-\alpha)} + \rho_2^{\pi/2(\pi-\alpha)}\right)\sin\left[\frac{(\pi-2\alpha)\pi}{2(\pi-\alpha)}\right]}\right\},$$

where

$$\rho_1 = \frac{16(r^2-1)}{3(4r^2+3)+\sqrt{7(25-4r^2)(4r^2-1)}}$$

and

$$\rho_2 = \frac{16(r^2-1)}{3(4r^2+3)-\sqrt{7(25-4r^2)(4r^2-1)}}.$$

Figure 8 shows the h-function graph of the domain Ω when the basepoint z_0 is fixed at the origin. Moreover, this h-function has a point of inflection at $r = 2.105$.

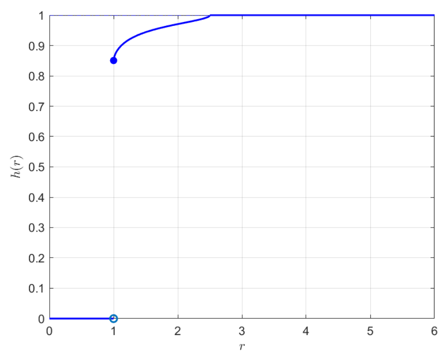


FIGURE 8. Graph of the h-function for the domain $\Omega = B(0,1) \cup B(1.5,1)$ with basepoint $z_0 = 0$.

5. Complement of a round lollipop with complex basepoint

In this section, we describe the h-function of a lollipop domain when the basepoint z_0 is purely imaginary. The h-function for the lollipop domains when the basepoint is fixed on the real axis has been described in [4]. Consider the domain Ω which is the complement of the disc $\{z \in \mathbb{C} : |z| \leq 1\}$ along with the ray $(-\infty, -1]$; see Figure 9.

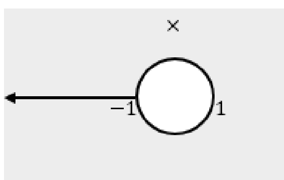


FIGURE 9. Round lollipop domain $\Omega = \mathbb{C}\backslash(\overline{B(0,1)} \cup (-\infty,-1])$ with basepoint $z_0 = 2i$.

For the regime $1 \le r < \sqrt{5}$, the capture circle encloses only part of the upper half boundary of the disc. In this case, the h-function is

$$h(r) = \frac{1}{\pi} \arctan \left\{ \frac{\sqrt{2k(1-\sin\alpha)} \sin\frac{\theta}{2}}{k + \sin\alpha - \sqrt{2k}\cos\frac{\theta}{2}\left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\right)} \right\},$$

where

$$k = \frac{5}{4}, \quad \cos\theta = \frac{4}{5} \quad \text{and} \quad \sin\alpha = \frac{5-r^2}{4}.$$

That is, for this range of r,

$$h(r) = \frac{1}{\pi} \arctan t,$$

where

$$t = \frac{\sqrt{2(r^2-1)}}{\sqrt{2}(10-r^2) - 3\left[\sqrt{4+\sqrt{(9-r^2)(r^2-1)}} + \sqrt{4-\sqrt{(9-r^2)(r^2-1)}}\right]}.$$

For the regime $\sqrt{5} \le r < 3$, the capture circle encloses the entire upper half boundary of the disc, and also encloses part of the ray and the lower half boundary of the disc. In this case, the h-function is

$$\begin{aligned} h(r) = \frac{1}{\pi} \arctan &\left\{ \frac{\sqrt{k}\left(\sqrt{\delta} + \sqrt{2}\sin\frac{\alpha}{2}\right)\sin\frac{\theta}{2}}{k + \sqrt{2k}\cos\frac{\theta}{2}\sin\frac{\alpha}{2} - \sqrt{\delta}\left(\sqrt{k}\cos\frac{\theta}{2} + \sqrt{2}\sin\frac{\alpha}{2}\right)} \right\} \\ + \frac{1}{\pi} \arctan &\left\{ \frac{\sqrt{k}\left(\sqrt{\delta} - \sqrt{2}\cos\frac{\alpha}{2}\right)\sin\frac{\theta}{2}}{k + \sqrt{2k}\cos\frac{\theta}{2}\cos\frac{\alpha}{2} + \sqrt{\delta}\left(\sqrt{k}\cos\frac{\theta}{2} + \sqrt{2}\cos\frac{\alpha}{2}\right)} \right\}, \end{aligned}$$

where $\delta = \frac{r^2-3+2\sqrt{r^2-4}}{2\sqrt{r^2-4}}$.

Now, for the regime $r \ge 3$, the capture circle encloses the entire boundary of the disc and part of the ray. In this case, the h-function is

$$h(r) = \frac{1}{\pi} \operatorname{arcsec} \left\{ \frac{r\sqrt{4r^2-15}}{\sqrt{r^2-4} + 2(3-r^2)} \right\}.$$

We also note that the h-function is exactly the same for this round lollipop domain when the basepoint is $z_0 = -2i$, since the reflection does not affect the h-function.

The h-function plot for the above lollipop domain $\Omega = \mathbb{C}\backslash(\overline{B(0,1)} \cup (-\infty, -1])$ with basepoint $z_0 = 2i$ or $z_0 = -2i$ is shown in Figure 10.

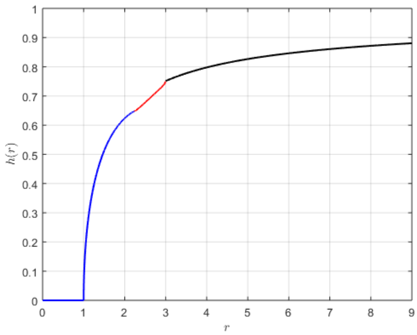


FIGURE 10. Graph of the h-function for the round lollipop domain $\Omega = \mathbb{C}\backslash(\overline{B(0,1)} \cup (-\infty, -1])$ with basepoint $z_0 = 2i$ or $z_0 = -2i$.

6. Finite lollipop with pure imaginary basepoint

In this section, we explain the h-function of a lollipop domain with a complex basepoint, but we change the boundary ray as a line segment. The h-function of this domain when the basepoint is fixed on the real axis has already been described in [4].

Consider the domain $\Omega = \mathbb{C}\backslash(\{z : |z| \leq 1\} \cup [1, 3])$. Fix the basepoint $z_0 = 2i$. See Figure 11.

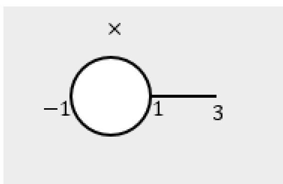


FIGURE 11. Finite lollipop domain Ω with basepoint $z_0 = 2i$.

To transform the domain Ω to a halfplane, we start with the Joukowski map to transplant the domain Ω to the complement of the line segment $[-1, 5/3]$. Next, we use the Möbius map $(3z-5)/3(z+1)$ that sends the points -1, 1 and $5/3$ to the points ∞, $-1/3$ and 0, respectively. This Möbius map transplants the domain $\mathbb{C}\backslash([-1, 5/3])$ to the domain $\mathbb{C}\backslash(-\infty, 0]$. Finally, we use the square-root transformation to map the domain $\mathbb{C}\backslash(-\infty, 0]$ to the right halfplane. Through these conformal transformations, the basepoint z_0 is shifted from $2i$ to $\sqrt{k}e^{(\pi-\theta)/2}$, where $k = \sqrt{481}/15$ and $\theta = \operatorname{arcsec}\big(15\sqrt{481}/159\big)$.

For the regime $1 \leq r < \sqrt{5}$, the capture circle encloses part of the upper half boundary of the disc, but does not enclose any part of the line segment and the lower half boundary of the disc. In this case, the h-function is

$$h(r) = \frac{1}{\pi} \arctan \left\{ \frac{\sqrt{k}(\sqrt{\eta_1} - \sqrt{\eta_0}) \sin \frac{\theta}{2}}{k + \sqrt{\eta_0 \eta_1} - \sqrt{k}(\sqrt{\eta_0} + \sqrt{\eta_1}) \cos \frac{\theta}{2}} \right\},$$

where

$$\eta_0 = \frac{20 - 3\sqrt{(r^2-1)(9-r^2)}}{3(4 + \sqrt{(r^2-1)(9-r^2)})} \quad \text{and} \quad \eta_1 = \frac{20 + 3\sqrt{(r^2-1)(9-r^2)}}{3(4 - \sqrt{(r^2-1)(9-r^2)})}.$$

For the regime $\sqrt{5} \leq r < 3$, the capture circle encloses the whole upper half boundary of the disc, and also encloses part of the lower half boundary of the disc and part of the line segment. In this case, the h-function is

$$\begin{aligned} h(r) = {} & \frac{1}{\pi} \arctan \left\{ \frac{\sqrt{k}(\sqrt{\delta} + \sqrt{\eta_1}) \sin \frac{\theta}{2}}{\sqrt{\delta \eta_1} - k + \sqrt{k}(\sqrt{\delta} - \sqrt{\eta_1}) \cos \frac{\theta}{2}} \right\} \\ & + \frac{1}{\pi} \arctan \left\{ \frac{\sqrt{k}(\sqrt{\eta_0} - \sqrt{\delta}) \sin \frac{\theta}{2}}{\sqrt{\delta \eta_0} + k + \sqrt{k}(\sqrt{\eta_0} + \sqrt{\delta}) \cos \frac{\theta}{2}} \right\}, \end{aligned}$$

where $\delta = \frac{10\sqrt{r^2-4} - 3(r^2-3)}{3(r^2-3+2\sqrt{r^2-4})}$.

For the regime $3 \leq r < \sqrt{13}$, the capture circle encloses the whole boundary of the disc, and also encloses part of the line segment. In this case,

$$h(r) = \frac{1}{\pi} \arctan \left\{ \frac{2\sqrt{k\delta} \sin \frac{\theta}{2}}{\delta - k} \right\}.$$

Figure 12 shows the h-function graph for this lollipop domain Ω with basepoint $z_0 = 2i$.

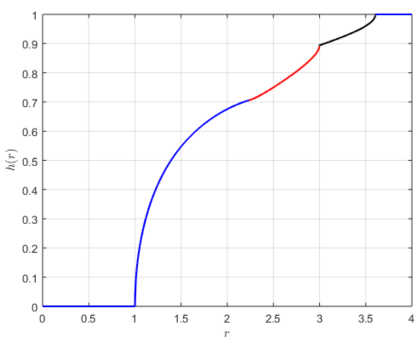


FIGURE 12. Graph of the h-function for the finite lollipop $\Omega = \mathbb{C} \backslash (\overline{B(0,1)} \cup [1,3])$ with basepoint $z_0 = 2i$.

Note that there is no change in the h-function when we change the position of the basepoint to $-2i$, since the reflection does not affect the h-function.

7. Lollipop with a joined line segment

In this section, we explain the h-function of the complement of a lollipop joined with a deleted spike when the basepoint z_0 is fixed on the real axis. The h-function of lollipop domains has already been documented in [4]. When we add a new boundary component to the lollipop, the whole calculation of $h(r)$ will be changed, since we map the whole domain to a halfplane. Also, the number of regimes will be increased in this case.

Consider the domain Ω which is the complement of the disc $\{z \in \mathbb{C} : |z| \leq 1\}$, the ray $(-\infty, -1]$ and the line segment $[1, b]$, where $b > 1$. We fix the basepoint $z_0 = b + d$ in Ω.

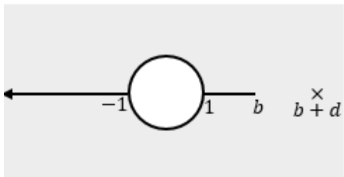


Figure 13. Domain $\Omega = \mathbb{C} \backslash \left(\overline{B(0,1)} \cup [1, b] \cup (-\infty, -1]\right)$ with basepoint $z_0 = b + d$.

For the regime $d \leq r < b + d - 1$, the capture circle only encloses the part of the spike in the boundary $\partial\Omega$. In this case, the h-function is

$$h(r) = \frac{2}{\pi} \arctan \sqrt{\frac{(b+d)}{d(b^2+d^2-1)} \left[\frac{[(b+d-r)(br-bd+1)-b]}{b+d-r}\right]}.$$

For the regime $b + d - 1 \leq r < b + d + 1$, the capture circle encloses the whole boundary of the spike, and part of the boundary of the disc in $\partial\Omega$. In this case, the h-function is

$$h(r) = \frac{2}{\pi} \arctan \sqrt{\frac{b(r^2-1)+(b+d)(1-bd)}{d(b^2+d^2-1)}}.$$

For the regime $r \geq b + d + 1$, the capture circle encloses the whole boundary of the spike and the disc, and part of the boundary of the ray in $\partial\Omega$. In this case, the h-function is

$$h(r) = \frac{2}{\pi} \arctan \sqrt{\frac{(b+d)}{d(b^2+d^2-1)} \left[\frac{bd(2b+d)+(b^2+1)(r-d)+br[r-2(b+d)]}{r-(b+d)}\right]}.$$

Figure 14 shows the h-function graph of the above domain Ω with $b = 2$ when the basepoint is $z_0 = b + d = 3$.

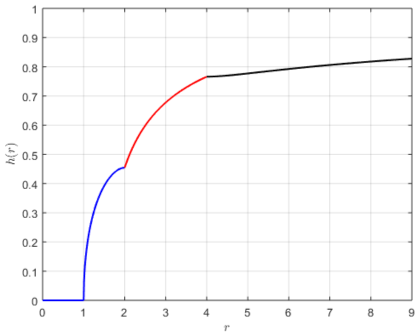


FIGURE 14. Graph of the h-function for the domain $\Omega = \mathbb{C}\backslash\left(\overline{B(0,1)} \cup [1,2] \cup (-\infty,-1]\right)$ with basepoint $z_0 = 3$.

Now, we move our focus to the asymptotics of the h-function $h(r)$ near $r = d$, $r = b + d - 1$ and $r = b + d + 1$ at which two regimes meet. Near $r = d$,

$$
\begin{aligned}
h(r) &\sim \frac{2}{\pi}\sqrt{\frac{(b+d)(b^2-1)}{bd(b^2+d^2-1)}}(r-d)^{1/2} \\
&= c(r-d)^{\beta},
\end{aligned}
$$

where $c = \frac{2}{\pi}\sqrt{\frac{(b+d)(b^2-1)}{bd(b^2+d^2-1)}}$ and $\beta = 1/2$.

Near $r = b + d - 1$,

$$
\begin{aligned}
& h(r) - h(b+d-1) \\
&\quad = \frac{2}{\pi}\arctan\sqrt{\frac{b(r^2-1)+(b+d)(1-bd)}{d(b^2+d^2-1)}} \\
&\qquad - \frac{2}{\pi}\arctan\sqrt{\frac{b((b+d-1)^2-1)+(b+d)(1-bd)}{d(b^2+d^2-1)}} \\
&\quad = \frac{2}{\pi}\arctan\left\{\frac{\sqrt{d(b^2+d^2-1)}\left[\sqrt{br^2+d-bd(b+d)} - \sqrt{b(b+d-1)^2+d-bd(b+d)}\right]}{d(b^2+d^2-1)+\sqrt{[br^2+d-bd(b+d)][b(b+d-1)^2+d-bd(b+d)]}}\right\} \\
&\quad \sim \frac{2}{\pi}\frac{\sqrt{d(b^2+d^2-1)}}{[b^3+d^3+2b^2d+b-2b^2-2bd]}f(r),
\end{aligned}
\tag{7.1}
$$

where $f(r) = \sqrt{br^2 + d - bd(b+d)} - \sqrt{b(b+d-1)^2 + d - bd(b+d)}$. By using the Taylor expansion for the function $f(r)$ at $r = b+d-1$, we write

$$\begin{aligned} f(r) &= f(b+d-1) + \frac{f'(b+d-1)}{1!}[r-(b+d-1)] + \frac{f''(b+d-1)}{2!}[r-(b+d-1)]^2 + \cdots \\ &\sim \frac{b(b+d-1)}{\sqrt{b(b+d-1)+d-bd(b+d)}}[r-(b+d-1)], \end{aligned} \tag{7.2}$$

since $f(b+d-1) = 0$ and $f'(b+d-1) = \frac{b(b+d-1)}{\sqrt{b(b+d-1)+d-bd(b+d)}}[r-(b+d-1)]$.

By substituting (7.2) in (7.1), we obtain

$$\begin{aligned} h(r) - h(b+d-1) &\sim \frac{2}{\pi}\frac{\sqrt{d(b^2+d^2-1)}}{[b^3+d^3+2b^2d+b-2b^2-2bd]} \\ &\qquad \times \frac{b(b+d-1)}{\sqrt{b(b+d-1)+d-bd(b+d)}}[r-(b+d-1)] \\ &= c_0[r-r^*]^\gamma, \end{aligned}$$

where

$$c_0 = \frac{2}{\pi}\frac{\sqrt{d(b^2+d^2-1)}}{[b^3+d^3+2b^2d+b-2b^2-2bd]}\frac{b(b+d-1)}{\sqrt{b(b+d-1)+d-bd(b+d)}},$$

$r^* = b+d-1$ and $\gamma = 1$. We are also interested to know the behaviour of the asymptotics near $r = r^* = b+d-1$ when $b \to 1$, that is, when the length of the spike reduces to the point 1. When $b \to 1$, the constant

$$c_0 \to \frac{2}{\pi}\frac{1}{d\sqrt{1-d}},$$

which does not blow up. However, this does not mean that this h-function has the exponent $\beta = 1$, because the left-hand side term $h(r) - h(b+d-1)$ consists of a substracting term from the h-function $h(r)$.

Near $r = b+d+1$,

$$\begin{aligned} &h(r) - h(b+d+1) \\ &\quad = \frac{2}{\pi}\arctan\sqrt{\frac{(b+d)}{d(b^2+d^2-1)}\left[\frac{bd(2b+d)+(b^2+1)(r-d)+br[r-2(b+d)]}{r-(b+d)}\right]} \\ &\qquad - \frac{2}{\pi}\arctan\left[\sqrt{\frac{(b+d)}{d(b^2+d^2-1)}}(b+1)\right] \\ &\quad = \frac{2}{\pi}\arctan\left\{\frac{\rho\sqrt{m}-\rho(b+1)\sqrt{r-(b+d)}}{\sqrt{r-(b+d)}+\rho^2(b+1)\sqrt{m}}\right\}, \end{aligned}$$

where $\rho = \sqrt{\frac{(b+d)}{d(b^2+d^2-1)}}$ and $m = bd(2b+d) + (b^2+1)(r-d) + br[r-2(b+d)]$. Then,

$$h(r) - h(b+d+1) \sim \frac{2}{\pi}\frac{1}{[1+\rho^2(b+1)^2]}f_0(r),$$

where $f_0(r) = \rho\sqrt{m} - \rho(b+1)\sqrt{r-(b+d)}$. Again, by using the Taylor expansion for the function $f_0(r)$ at $r = b+d+1$, we obtain

$$f_0(r) \sim \frac{\rho\left[(1-3b^2-4bd-2b)\sqrt{(b+d)(b+d+1)} - (b+1)^2\right]}{2(b+1)\sqrt{(b+d)(b+d+1)}}.$$

Hence,

$$\begin{aligned} &h(r) - h(b+d+1) \\ &\quad\sim \frac{\rho}{\pi}\frac{\left[(1-3b^2-4bd-2b)\sqrt{(b+d)(b+d+1)} - (b+1)^2\right]}{[1+\rho^2(b+1)^2](b+1)\sqrt{(b+d)(b+d+1)}}[r-(b+d+1)] \\ &\quad= c_1[r-(b+d+1)]^{\gamma}, \end{aligned}$$

where $c_1 = \frac{\rho}{\pi}\frac{\left[(1-3b^2-4bd-2b)\sqrt{(b+d)(b+d+1)} - (b+1)^2\right]}{[1+\rho^2(b+1)^2](b+1)\sqrt{(b+d)(b+d+1)}}$ and $\gamma = 1$.

8. Complement of a disc joined with two line segments

In this section, we describe the h-function of the complement of the unit disc along with two line segments which meets the real axis at 1 and -1. Again, we add a line segment to a lollipop when it has a line segment in its boundary instead of a ray.

Consider the domain $\Omega = \mathbb{C}\backslash(\{z : |z| \leq 1\} \cup [-3,-1] \cup [1,3])$; see Figure 15.

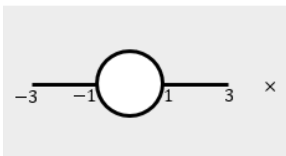


Figure 15. Domain $\Omega = \mathbb{C}\backslash\left(\overline{B(0,1)} \cup [-3,-1] \cup [1,3]\right)$ with basepoint $z_0 = 4$.

To compute the h-function, we transform the domain Ω to a halfplane. First, we use the Joukowski map to transplant the domain Ω to the complement of the line segment $[-5/3, 5/3]$. Then, we use the Möbius map $(3z-5)/(3z+5)$ that sends the points $5/3$, $-5/3$ and 2 to the points 0, ∞ and $1/11$, respectively. This Möbius map transplants the domain $\mathbb{C}\backslash[-5/3, 5/3]$ to the domain $\mathbb{C}\backslash(-\infty, 0]$. Finally, we use the square-root transformation to map this current domain $\mathbb{C}\backslash(-\infty, 0]$ to the halfplane.

For the regime $1 \leq r < 3$, the capture circle only encloses part of the line segment $(1, 3]$. In this case, the h-function is

$$h(r) = \frac{2}{\pi}\arctan\sqrt{\frac{91(3r^2-14r+11)}{11(-3r^2+34r-91)}}.$$

For the regime $3 \leq r < 5$, the capture circle encloses the entire line segment $[1,3]$ and part of the boundary of the disc in $\partial\Omega$. In this case, the h-function is

$$h(r) = \frac{2}{\pi} \arctan \sqrt{\frac{91(3r^2-11)}{11(91-3r^2)}}.$$

For the regime $5 \leq r < 7$, the capture circle encloses the entire line segment $[1,3]$ and the boundary of the disc in $\partial\Omega$. Also, it encloses part of the line segment $(-3,-1]$. In this case, the h-function is

$$h(r) = \frac{2}{\pi} \arctan \sqrt{\frac{91(3r^2-14r+11)}{11(-3r^2+34r-91)}}.$$

Figure 16 shows the h-function graph for the domain Ω with basepoint $z_0 = 4$.

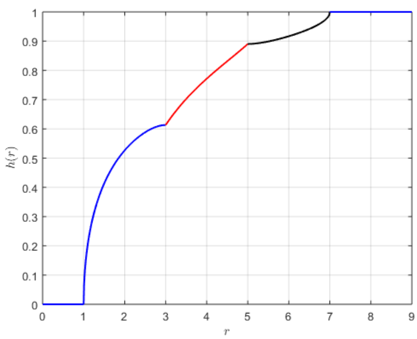


Figure 16. Graph of the h-function for the domain $\Omega = \mathbb{C}\backslash\left(\overline{B(0,1)} \cup [-3,-1] \cup [1,3]\right)$ with basepoint $z_0 = 4$.

In this domain, we have

$$h'(r) = \begin{cases} \dfrac{\sqrt{1001}(3-r)(5-r)}{\pi r(4r-15)\sqrt{(3r-11)(r-1)(-3r^2+34r-91)}} & \text{for } 1 \leq r < 3; \\ \dfrac{\sqrt{1001}}{3\pi r\sqrt{(3r^2-11)(91-3r^2)}} & \text{for } 3 \leq r < 5; \\ \dfrac{\sqrt{1001}(3-r)(5-r)}{\pi r(4r-15)\sqrt{(3r-11)(r-1)(-3r^2+34r-91)}} & \text{for } 5 \leq r < 7. \end{cases}$$

We note that h'_- and h'_+ denote the left-hand and the right-hand derivatives of the function h, respectively. In our case, $h'_-(1) = 0$, $h'_+(1) = \infty$, $h'_-(3) = 0$, $h'_+(3) = \frac{\sqrt{1001}}{32\cdot 15\pi}$, $h'_-(5) = \frac{\sqrt{1001}}{32\cdot 9\pi}$ and $h'_+(5) = 0$. Therefore, at $r = 1$, the h-function has a horizontal tangent from the left and a vertical tangent from the right. Similarly, the h-function has a horizontal tangent at $r = 3$ from the left only, and a horizontal tangent at $r = 5$ from the right only.

Moreover, we have $h''_-(1) = 0$, $h''_+(1) = \infty$. Also, $h''_-(3) \neq h''_+(3)$ with both left-hand and right-hand derivatives being not equal to either 0 or ∞. Similarly, $h''_-(5) \neq h''_+(5)$ with both both left-hand and right-hand derivatives being not equal to either 0 or ∞.

Now, we move our focus to the asymptotics on the h-function of this domain Ω near the values of r where two regimes meet. Near $r = 1$,

$$h(r) \sim \frac{2}{\pi}\sqrt{\frac{91}{11}}\sqrt{\frac{2}{15}}(r-1)^{1/2} = c(r-d)^{\beta},$$

where $c = \frac{2}{\pi}\sqrt{\frac{91}{11}}\sqrt{\frac{2}{15}}$, $d = 1$ and $\beta = 1/2$.

Near $r = r^* = 3$,

$$\begin{aligned} h(r) - h(3) &= \frac{2}{\pi}\arctan\sqrt{\frac{91(3r^2-14r+11)}{11(-3r^2+34r-91)}} - \frac{2}{\pi}\arctan\left\{\sqrt{\frac{91}{11}}\frac{1}{2}\right\} \\ &= \frac{2}{\pi}\arctan\left\{\sqrt{1001}\left[\frac{2\sqrt{3r^2-11}-\sqrt{91-3r^2}}{22\sqrt{91-3r^2}+91\sqrt{3r^2-11}}\right]\right\} \\ &\sim \frac{\sqrt{1001}}{270\pi}f(r), \end{aligned} \tag{8.1}$$

where $f(r) = 2\sqrt{3r^2-11} - \sqrt{91-3r^2}$. By using the Taylor expansion of $f(r)$ at $r = 3$, we obtain

$$f(r) \sim \frac{45}{8}(r-3). \tag{8.2}$$

Applying (8.2) to the expression (8.1), we obtain

$$h(r) - h(3) \sim c_0(r-r^*)^{\gamma},$$

where $c_0 = \frac{\sqrt{1001}}{48\pi}$ and $\gamma = 1$.

Near $r = r^{**} = 5$,

$$\begin{aligned} h(r) - h(5) &= \frac{2}{\pi}\arctan\left\{\sqrt{91}\left[\frac{\sqrt{3r^2-14r+11}-2\sqrt{-3r^2+34r-91}}{11\sqrt{-3r^2+34r-91}+182\sqrt{3r^2-14r+11}}\right]\right\} \\ &\sim \frac{\sqrt{91}}{375\pi}[\sqrt{3r^2-14r+11}-2\sqrt{-3r^2+34r-91}] \\ &\sim \frac{\sqrt{91}}{40\pi}(r-5)^2 \\ &= c_1(r-r^{**})^{\gamma}, \end{aligned}$$

where $c_1 = \frac{\sqrt{91}}{40\pi}$ and $\gamma = 2$.

9. Two discs connected by a common line segment

In this section, we explain the h-function of an unbounded domain whose boundary components are two discs and a line segment which connects both discs. However, for the computation of the h-function, here we use a different approach which is not used in the earlier sections of this paper. The interesting part of this method is the construction of a harmonic function to solve our Dirichlet problem. This method was first mentioned in [2].

Consider the unbounded domain Ω which is the complement of two discs D_1, D_2 and the common line segment $l = [1, 1.700005]$, where the disc D_1 is centred at zero with unit radius, while the disc D_2 is centred at 1.755190 with radius 0.055185. We fix the basepoint $z_0 = -2$.

To compute the h-function of Ω with this given basepoint z_0, we use the method of conformal transformations. In this case, we transform our target domain Ω to the interior of the unit disc. We start with the inverse map $(1/z)$ to transplant the domain Ω to a bounded domain which is the interior of the unit disc, but the complement of the lollipop whose boundary consists of the line segment $[0.588233, 1]$ and the boundary circle that meets the real line at the points $c = 0.552378$ and $d = 0.588233$.

Next, we use the Möbius map $(z - \lambda)/(\lambda z - 1)$ with $\lambda = (c + d)/(1 + cd + \sqrt{(1 - c^2)(1 - d^2)})$ to transform the current region to the concentric annulus excluding the line segment $[-1, -\rho]$, where ρ is the radius of the inner circle of the annulus and the outer circle of the annulus has the unit radius. Next, we use the logarithmic transformation to map this new region to the interior of the rectangle whose vertices are $(0, \pi)$, $(0, -\pi)$, $(\ln \rho, -\pi)$ and $(\ln \rho, \pi)$. Then, we use the map $(1.38403z/\pi)$ to replace the vertices of the rectangle at $(0, 1.38403)$, $(0, -1.38403)$, $(-1.59814, -1.38403)$ and $(-1.59814, -1.38403)$. Now, we can obtain this new rectangle from the unit disc via the Schwarz–Christoffel mapping $f(\zeta)$ by fixing the pre-vertices of this rectangle at the points $e^{5i\pi/12}$, $e^{-5i\pi/12}$, $e^{-7i\pi/12}$ and $e^{7i\pi/12}$, where

$$f(\zeta) = \int_1^\zeta \left[(1 - te^{5i\pi/12})(1 - te^{7i\pi/12})(1 - te^{-7i\pi/12})(1 - te^{-5i\pi/12})\right]^{-1/2} dt.$$

Hence, the composite map

$$F(\zeta) = \frac{\lambda A(\zeta) - 1}{A(\zeta) - \lambda},$$

where

$$A(\zeta) = \exp\left\{\frac{\pi}{1.38403} \times \int_1^\zeta \left[(1 - te^{5i\pi/12})(1 - te^{7i\pi/12})(1 - te^{-7i\pi/12})(1 - te^{-5i\pi/12})\right]^{-1/2} dt\right\},$$

transforms the interior of the unit disc to our target domain Ω. Also, $F(1) = -1$, $F(e^{5i\pi/12}) = 1$, $F(e^{7i\pi/12}) = 1.700005$ and $F(-1) = 1.810375$.

Now, we move our focus to the h-function computation. Let τ and $\bar{\tau}$ be the preimages of the intersecting points in the boundary $\partial\Omega$ by the capture circle, where $\tau = e^{i\phi}$. In the computation of $h(r)$, we use the classical Cayley map

$$R(\zeta, \tau, \bar{\tau}) = -\frac{1}{\tau}\left(\frac{\zeta - \tau}{\zeta - \bar{\tau}}\right)$$

which transforms the interior of the unit disc to the lower halfplane. For $|\zeta| = 1$,

$$\begin{aligned}
\overline{R(\zeta,\tau,\bar{\tau})} &= -\frac{1}{\bar{\tau}}\left(\frac{\bar{\zeta}-\bar{\tau}}{\bar{\zeta}-\tau}\right) \\
&= -\tau\left(\frac{(1/\zeta)-(1/\tau)}{(1/\zeta)-(1/\bar{\tau})}\right) \\
&= -\bar{\tau}\left(\frac{\zeta-\tau}{\zeta-\bar{\tau}}\right) \\
&= -\frac{1}{\tau}\left(\frac{\zeta-\tau}{\zeta-\bar{\tau}}\right) \\
&= R(\zeta,\tau,\bar{\tau}).
\end{aligned}$$

Thus, R is real on $|\zeta| = 1$. We also fix the branch of the argument function $\arg R$ such that $-\pi \leq \arg R < \pi$. Therefore,

$$\arg R(\zeta,\tau,\bar{\tau}) = \begin{cases} 0 & \text{for } 0 < |\arg\zeta| < \phi; \\ -\pi & \text{for } \phi \leq |\arg\zeta| < \pi. \end{cases}$$

Now, we define the harmonic function

$$W(\zeta,\tau,\bar{\tau}) = \frac{1}{\pi}\left[\log R(\zeta,\tau,\bar{\tau}) + i\pi\right].$$

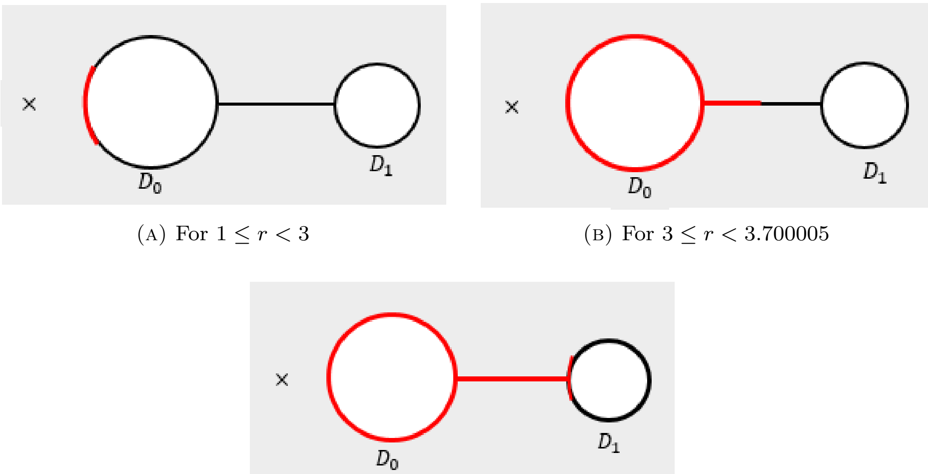
Then,

$$\begin{aligned}
\operatorname{Im}[W(\zeta,\tau,\bar{\tau})] &= \frac{1}{\pi}\left[\arg R(\zeta,\tau,\bar{\tau})\right] + 1 \\
&= \begin{cases} 1 & \text{for } 0 < |\arg\zeta| < \phi; \\ 0 & \text{for } \phi \leq |\arg\zeta| < \pi. \end{cases}
\end{aligned}$$

Therefore, the harmonic function $\operatorname{Im}[W(\zeta,\tau,\bar{\tau})]$ solves our Dirichlet problem in the interior of the unit disc. Thus, the composite function $\operatorname{Im}[W] \circ F^{-1}$ solves our Dirichlet problem in our target domain Ω. Hence, the h-function is given by

$$h(r) = \operatorname{Im}[W(\zeta_0,\tau,\bar{\tau})].$$

For the regime $1 \leq r < 3$, the capture circle encloses part of the boundary of the unit disc in $\partial\Omega$. In this case, the angle $\phi \in [0, 5\pi/12)$. Similarly, for the regime $3 \leq r < 3.700005$, the capture circle encloses the whole boundary of the unit disc in $\partial\Omega$, and also encloses part of the line segment in $\partial\Omega$. In this case, the angle $\phi \in [5\pi/12, 7\pi/12)$. For the regime $r \geq 3.810375$, the capture circle encloses the whole boundary of the unit disc and the line segment in $\partial\Omega$, and also encloses part of the boundary of the other circle in $\partial\Omega$. In this case, the angle $\phi \in [7\pi/12, \pi)$. Figure 17 shows the subset E_r for the three regimes of r.



(A) For $1 \leq r < 3$

(B) For $3 \leq r < 3.700005$

(C) For $r \geq 3.700005$

FIGURE 17. The set E_r (shown in red) for three regimes of r.

The h-function graph for the above domain Ω with basepoint $z_0 = -2$ is shown in Figure 18.

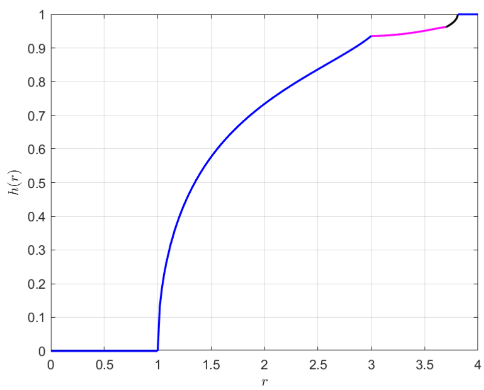


FIGURE 18. Graph of the h-function for the domain $\Omega = \mathbb{C}\backslash(\{|z| \leq 1\} \cup [1, 1.700005] \cup \{|z - 1.755190| \leq 0.055185\})$ with basepoint $z_0 = -2$.

In this section, our h-function expression is in an implicit form. Also, the conformal map $F(\zeta)$ consists of an integral form within it. Therefore, it is hard to investigate the asymptotics at the values of r where two regimes meet.

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