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ON THE MODULI SPACE OF LEFT-INVARIANT METRICS ON THE COTANGENT BUNDLE OF THE HEISENBERG GROUP

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Abstract. The focus of the paper is on the study of the moduli space of left-invariant pseudo-Riemannian metrics on the cotangent bundle of the Heisenberg group. We use algebraic approach to obtain orbits of the automorphism group acting in a natural way on the space of left invariant metrics. However, geometric tools such as the classification of hyperbolic plane conics are often needed. For the metrics obtained by the classification, we study geometric properties: curvature, Ricci tensor, sectional curvature, holonomy, and parallel vector fields. The classification of algebraic Ricci solitons is also presented, as well as the classification of pseudo-Kähler and pp-wave metrics. We obtain description of parallel symmetric tensors for each metric and show that they are derived from parallel vector fields. Finally, we study the totally geodesic subalgebras and show that for each subalgebra of the observed algebra there is a metric which makes it totally geodesic.

1. Introduction

The space of metrics is called *moduli space* and is defined as the orbit space of the action of $\mathbb{R}^\times \operatorname{Aut}(\mathfrak{g})$ on the space $\mathfrak{M}(G)$ of left-invariant metrics on G. Here $\operatorname{Aut}(\mathfrak{g})$ denotes the automorphism group of the corresponding Lie algebra and $\mathbb{R}^\times$ is the scalar group. There are two in some sense dual approaches to the classification problem, both based on the moduli space of left-invariant (pseudo-)Riemannian metrics on the Lie group. The first approach consists of fixing a Lie algebra basis such that the commutator relations are as simple as possible, and then fitting the inner product to it by the action of the automorphism group. This approach was first introduced by Milnor [35], who used it to classify all left-invariant Riemannian metrics on 3-dimensional unimodular Lie groups. The second way is to start from the basis which puts the inner product in the simplest form, where the Lie brackets

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can be arbitrary but satisfy the Jacobi identity, and in this way define a hypersurface with feasible Lie brackets. Note that in both cases the orbits of $\mathrm{Aut}(\mathfrak{g})$ induce the isometry classes, while $\mathbb{R}^\times$ induces the scaling. For a more detailed exposition of the two approaches, we refer to [24, 29].

Interestingly, the Riemannian case is well studied and understood, while the pseudo-Riemannian case seems to be more difficult and still has many open questions. Milnor's classification of 3-dimensional Lie groups with left-invariant positive definite metric [35] has become a classical reference, while the corresponding Lorentz classification [13] followed twenty years later. In dimension four, only partial results are known. The classification of 4-dimensional Riemannian Lie groups is due to Bérard-Bérgery [4]. Jensen [26] has studied homogeneous Einstein spaces with Riemannian (positive definite) metrics, while Karki and Thompson [27] have studied Einstein manifolds arising from right invariant Riemannian metrics on a 4-dimensional Lie group. Calvaruso and Zaeim [8] have classified left-invariant Lorentz metrics on Lie groups which are Einstein or Ricci-parallel, using the second approach mentioned above. Classification in the case of nilpotent Lie groups in small dimensions has been studied in detail in both the Riemannian [31] and the pseudo-Riemannian setting [5, 43, 25]. Recent results include the classification of pseudo-Riemannian metrics for 4-dimensional solvable Lie groups [44] and in the positive definite case, the moduli space for 6-dimensional nilpotent Lie groups admitting a complex structure with the first Betti number equal to 4 was determined [38]. For arbitrary dimensions, the Lorentz classification of left-invariant metrics on the Heisenberg group H_{2n+1} [47] and the classification of Ricci solitons on nilmanifolds [32] are worth mentioning.

The cotangent bundles play an important role in the standard description of physical systems, both for particles and for fields (see, e.g., [2]). In particular, they appear as configuration spaces of some mechanical systems and are often endowed with rich algebraic and geometric structures (see, e.g., [19, 16, 17]). In this paper, we are interested in the cotangent bundle of the Heisenberg group H_3, mainly because this group is a constant topic of research due to its properties and diverse applications. For example, Herman Weyl was led to an explicit account of the Heisenberg group when he attempted to answer the question of the physical equivalence of the Schrödinger and Heisenberg pictures.

The paper is organized as follows.

First, in Section 2 some basic facts about the algebra $\mathrm{T}^*\mathfrak{h}_3$ and its automorphism group are explained.

In Section 3 we classify all non-isometric left-invariant pseudo-Riemannian metrics on $\mathrm{T}^*\mathfrak{h}_3$. For the classification we use the second approach described above: we fix the commutators and act with automorphisms of the algebra $\mathrm{T}^*\mathfrak{h}_3$ to find representatives of the metrics. The restriction of the metric to the derived subalgebra ${\mathrm{T}^*\mathfrak{h}_3}'$ plays a very important role in the analysis. Each induced signature of ${\mathrm{T}^*\mathfrak{h}_3}'$ is treated in a separate subsection, and in each case we have to use different geometric and algebraic methods for classification. For example, if the induced metric is Lorentzian, we must include some classical results from projective geometry, while the degenerate case requires a more subtle analysis that depends heavily on the signature of the degenerate subspace and often involves the use of Euclidean and hyperbolic rotations. The results are summarised in Theorem 3.4.

Section 4 is devoted to the study of the geometric properties of the obtained metrics. First, we study the curvature properties (Proposition 4.2) and the scalar curvature (Proposition 4.4). In Proposition 4.7 we describe parallel left invariant vector fields and show that all such fields are null. The holonomy of the metrics is quite diverse and is described in Proposition 4.8. However, we leave a deeper understanding of holonomy to further research.

In Subsection 4.2 we classify metrics that are algebraic Ricci solitons. In the Riemannian case (see [32]) such a metric would be unique up to homotety, but since we are working in pseudo-Riemannian settings, we have several non-isometric metrics which are shrinking, expanding or steady solitons.

In Subsection 4.3 we consider the invariant complex structure obtained by Salamon [40] in his classification of complex structures on nilpotent Lie algebras. It is known that the space of the corresponding complex structures is 5-dimensional and that the nonflat, Ricci-flat, pseudo-Kähler metrics are admissible (see [12]). In this section we classify pseudo-Kähler metrics and show that they all belong to the same family of metrics (Proposition 4.14).

It is known that two left-invariant metrics with the same geodesics are affinely equivalent (see [6]) and that the difference of two such metrics is an invariant parallel symmetric tensor. In Proposition 4.17 we show that all such tensors can be obtained using parallel vectors, and therefore it follows from [30] that metrics admitting such tensors are Riemannian extensions of Euclidean space.

There are many well-known facts about totally geodesic subalgebras of a nilpotent Lie algebra (see, e.g., [7]). Therefore, the Subsection 4.5 is devoted to their study. Interestingly, for every subalgebra $\mathfrak{h}$ of $\mathrm{T}^*\mathfrak{h}_3$ there exists a metric which makes it totally geodesic, as shown in Proposition 4.20.

2. Preliminaries

Let us briefly recall the construction of the cotangent Lie algebra.

The *cotangent algebra* $T^*\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ is the semidirect product of $\mathfrak{g}$ and its cotangent space $\mathfrak{g}^*$,

$$T^*\mathfrak{g} := \mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*,$$

i.e., the commutators are defined by

$$[(x,\phi),(y,\psi)] := ([x,y], ad^*(x)(\psi) - ad^*(y)(\phi)), \quad x,y \in \mathfrak{g}, \quad \phi,\psi \in \mathfrak{g}^*.$$

By $ad^* : \mathfrak{g} \to \mathrm{gl}(\mathfrak{g}^*)$ we denote the coadjoint representation

$$(ad^*(x)(\phi))(y) := -\phi(ad(x)(y)) = -\phi([x,y]).$$

The Heisenberg Lie algebra $\mathfrak{h}_3$ is a 3-dimensional nilpotent Lie algebra defined by a nonzero commutator

$$[x_1, x_2] = x_3.$$

The cotangent algebra $\mathrm{T}^*\mathfrak{h}_3$ of $\mathfrak{h}_3$ is a 6-dimensional irreducible 2-step nilpotent algebra with maximal abelian ideal of rank 4 and 3-dimensional centre (see [36, Type 3] or [46, Type III3]).

For simplicity, we will fix the basis $e = (e_1, e_2, e_3, e_4, e_5, e_6)$ such that the Lie algebra $\mathrm{T}^*\mathfrak{h}_3$ is defined by nonzero commutators:

$$[e_1, e_2] = e_6, \quad [e_1, e_3] = -e_5, \quad [e_2, e_3] = e_4. \tag{2.1}$$

Note that these relations can be written in the form

$$[e_i, e_j] = \varepsilon_{ijk} e_{3+k}, \tag{2.2}$$

where ε_{ijk} is the totally antisymmetric Levi-Civita symbol and $i, j, k \in \{1, 2, 3\}$. The commutator subalgebra $\mathrm{T}^*\mathfrak{h}_3{}' = [\mathrm{T}^*\mathfrak{h}_3, \mathrm{T}^*\mathfrak{h}_3]$ and the central subalgebra $\mathcal{Z}(\mathrm{T}^*\mathfrak{h}_3)$ coincide:

$$\mathrm{T}^*\mathfrak{h}_3{}' = \mathbb{R}\langle e_4, e_5, e_6\rangle = \mathcal{Z}(\mathrm{T}^*\mathfrak{h}_3).$$

Lemma 2.1. *The group of automorphisms of the Lie algebra* $\mathrm{T}^*\mathfrak{h}_3$ *in the basis* e *with commutators* (2.1) *is given in block-matrix form*

$$\mathrm{Aut}(\mathrm{T}^*\mathfrak{h}_3) = \left\{ \begin{pmatrix} A & 0 \\ B & A^* \end{pmatrix} \mid \det A \neq 0 \right\}, \tag{2.3}$$

where $A^* := (\det A) A^{-T}$ *and* A, B *are* 3×3 *matrices, or equivalently, as*

$$\mathrm{Aut}(\mathrm{T}^*\mathfrak{h}_3) = \left\{ \begin{pmatrix} \pm(\sqrt{\det C})\, C^{-T} & 0 \\ B & C \end{pmatrix} \mid \det C > 0 \right\}. \tag{2.4}$$

Proof. By definition, the automorphism $F : \mathrm{T}^*\mathfrak{h}_3 \to \mathrm{T}^*\mathfrak{h}_3$ is a linear bijective mapping satisfying

$$F([u, v]) = [F(u), F(v)], \quad u, v \in \mathrm{T}^*\mathfrak{h}_3.$$

The automorphism F maps vectors e_1, e_2, e_3 to arbitrary vectors

$$F(e_j) = \sum_{i=1}^{3} a_{ij} e_i + \sum_{i=1}^{3} b_{ij} e_{3+i} = a_{ij} e_i + b_{ij} e_{3+i}. \tag{2.5}$$

where the 3×3 matrix $B = (b_{ij})$ is arbitrary and the 3×3 matrix $A = (a_{ij})$ must be regular. In the last relation, we have omitted the summation sign because we assume summation over repeated indices, as we will do in what follows. The automorphism F must preserve the commutator subalgebra. This can be written as

$$F(e_{3+j}) = c_{ij} e_{3+i}, \quad j = 1, 2, 3, \tag{2.6}$$

where $C = (c_{ij})$ is a 3×3 matrix. This explains the zero block in the matrix (2.3). Now we find the relation between the matrices A and C.

Using (2.2) and (2.6) we get

$$F([e_i, e_j]) = \varepsilon_{ijk} c_{pk} e_{3+p}, \tag{2.7}$$

$$\begin{aligned} [F(e_i), F(e_j)] &= [a_{ki} e_k + b_{ki} e_{3+k}, a_{mj} e_m + b_{mj} e_{3+m}] = [a_{ki} e_k, a_{mj} e_m] \\ &= a_{ki} a_{mj} \varepsilon_{kmp} e_{3+p}. \end{aligned} \tag{2.8}$$

Comparing the relations (2.7) and (2.8), we get

$$\varepsilon_{ijk}c_{pk} = \varepsilon_{kmp}a_{ki}a_{mj},$$

or, equivalently,

$$c_{pk} = \varepsilon_{kij}\varepsilon_{kmp}a_{ki}a_{mj} = A_{pk},$$

where A_{pk} is the cofactor of the element a_{pk} of the matrix A. Therefore, $A^* = (\det A)(A^{-1})^T = C$ as claimed.

To obtain the second representation, we take the determinant of the relation $C = A^* = (\det A)A^{-T}$ and we obtain $\det C = (\det A)^2 > 0$. □

3. Classification of metrics

In this section, we classify non-isometric left-invariant metrics of arbitrary signature on $\mathrm{T}^*\mathfrak{h}_3$.

If $\mathfrak{g}$ is a Lie algebra and $\langle\cdot,\cdot\rangle$ is an inner product on $\mathfrak{g}$, the pair $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is called a *metric Lie algebra*. The structure of a metric Lie algebra uniquely defines a left-invariant pseudo-Riemannian metric on the corresponding simple connected Lie group G, and vice versa.

Metric algebras are said to be *isometric* if there exists an isomorphism of Euclidean spaces preserving the curvature tensor and its covariant derivatives. This translates to the condition that metric algebras are isometric if and only if they are isometric as pseudo-Riemannian spaces (see [1, Proposition 2.2]). Although two isomorphic metric algebras are also isometric, the converse is not true. In general, two metric algebras may be isometric even if the corresponding Lie algebras are not isomorphic. The test to determine whether any two given solvable metric algebras (i.e., solvmanifolds) are isometric was developed by Gordon and Wilson in [23]. However, according to the results of Alekseevskiĭ [1, Proposition 2.3], in the completely solvable case, isometric means isomorphic.

Since the Lie algebra $\mathrm{T}^*\mathfrak{h}_3$ is nilpotent and therefore completely solvable, the non-isometric metrics on $\mathrm{T}^*\mathfrak{h}_3$ are the non-isomorphic ones.

The isomorphic classes of various left-invariant metrics on $\mathrm{T}^*\mathfrak{h}_3$ can be viewed as orbits of the automorphism group $\mathrm{Aut}(\mathrm{T}^*\mathfrak{h}_3)$ which naturally act on a space of left-invariant metrics. This allows us to use the algebraic approach, although more geometrical tools are often required.

In the basis e of $\mathrm{T}^*\mathfrak{h}_3$, the metric $\langle\cdot,\cdot\rangle$ is represented by a symmetric 6×6 matrix $S_e = (\langle e_i, e_j\rangle)$, which we refer to as the *metric matrix*. The problem of classifying metrics on $\mathrm{T}^*\mathfrak{h}_3$ reduces to finding conjugacy classes of symmetric matrices under the action of the group $\mathrm{Aut}(\mathrm{T}^*\mathfrak{h}_3)$:

$$S_f = F^T S_e F, \quad F \in \mathrm{Aut}(\mathrm{T}^*\mathfrak{h}_3). \tag{3.1}$$

In simple terms, we want to find a new basis $f = (f_1, f_2, f_3, f_4, f_5, f_6)$ of $\mathrm{T}^*\mathfrak{h}_3$ with brackets of the form (2.1) such that the metric matrix S_f in that basis is as simple as possible. Since the commutator algebra $\mathrm{T}^*\mathfrak{h}_3{}' = \mathbb{R}\langle e_4, e_5, e_6\rangle$ is invariant under $\mathrm{Aut}(\mathrm{T}^*\mathfrak{h}_3)$, we cannot change its metrical character, i.e., its signature.

Therefore, given a symmetric metric matrix S_e in the basis e, we find its canonical form depending on the restriction of the metric $\langle \cdot, \cdot \rangle$ on $\mathrm{T}^*\mathfrak{h}_3{}'$.

Let S'_e be the symmetric 3×3 matrix representing the restriction. The restriction of action (3.1) on S' by the automorphism $F \in \mathrm{Aut}(\mathrm{T}^*\mathfrak{h}_3)$ of the form (2.4) is $C^T S'_e C$. Since C is an arbitrary matrix with positive determinant, this action puts S'_e into the canonical form given by the matrix $\mathrm{diag}(\mu_1, \mu_2, \mu_3)$, $\mu_i \in \{1, -1, 0\}$. To introduce the notation, let

$$\begin{aligned}
&E_{30} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I, \quad && E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad && E_{20} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
&E_{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, && E_{10} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, && E_{00} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
&E_{03} = -E_{30}, && E_{12} = -E_{21}, && E_{02} = -E_{20}, \qquad E_{01} = -E_{10}.
\end{aligned} \tag{3.2}$$

The indexes of E_{pq} denote the signature (p, q), i.e., the number of positive and negative vectors, respectively, in the canonical form of S'_e.

Thus, choosing the matrix C in the automorphism F such that the restriction of the metric on $\mathrm{T}^*\mathfrak{h}_3{}'$ has the canonical form E_{pq}, the matrix of the metric $\langle \cdot, \cdot \rangle$ in the new basis becomes

$$S_{pq} = F^T\, S_e\, F = \begin{pmatrix} S & M \\ M^T & E_{pq} \end{pmatrix}, \tag{3.3}$$

where $M = (m_{ij})$ is an arbitrary and $S^T = S = (s_{ij})$ is a symmetric 3×3 matrix.

To further simplify S_{pq}, we choose an automorphism from the subgroup that preserves the E_{pq}-part of the matrix S_{pq}

$$\mathrm{Aut}(E_{pq}) = \{F \in \mathrm{Aut}(\mathrm{T}^*\mathfrak{h}_3) \mid C^T E_{pq} C = E_{pq}\}.$$

The groups $\mathrm{Aut}(E_{pq})$ and $\mathrm{Aut}(E_{qp})$ are isomorphic. In other cases these groups are fundamentally different and therefore we need to discuss separately each case of S_{pq} given by (3.3).

3.1. $\mathrm{T}^*\mathfrak{h}_3{}'$ is definite (case S_{30} and S_{03}).

In this case,

$$\mathrm{Aut}(E_{30}) = \left\{ \begin{pmatrix} \pm A & 0 \\ B & A \end{pmatrix} \mid A^T A = I,\, \det A > 0 \right\}, \tag{3.4}$$

i.e., $A \in SO(3)$ is orthogonal and B is any 3×3 matrix.

Suppose the metric $\langle \cdot, \cdot \rangle$ is represented in the basis e by the matrix S_{30} or S_{03} given by (3.3). Find a new basis f corresponding to $F \in \mathrm{Aut}(E_{30})$ of the form (2.5) for $a_{ij} = \delta_{ij}$, i.e., the matrix is the identity matrix $A = I$. From the form of F given by (3.4), we also have $F(e_{3+i}) = e_{3+i}$. Then

$$\begin{aligned}
\langle F(e_j), F(e_{3+k}) \rangle &= \langle e_j + b_{ij} e_{3+i}, e_{3+k} \rangle \\
&= \langle e_j, e_{3+k} \rangle + b_{ij} \langle e_{3+i}, e_{3+k} \rangle = m_{jk} + b_{ij}\delta_{ik}.
\end{aligned} \tag{3.5}$$

Thus, for $b_{jk} = -m_{kj}$, i.e., for $B = -M^T$, we obtain

$$\langle F(e_j), F(e_{3+k})\rangle = 0, \quad j, k \in \{1,2,3\}.$$

Therefore, F puts the matrix S_{30} into the form

$$\begin{pmatrix} S & 0 \\ 0 & E_{30} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} S & 0 \\ 0 & E_{03} \end{pmatrix},$$

where $S = S^T$ has changed but is denoted by the same letter to simplify notation. Finally, since the symmetric matrix S can be diagonalized by the orthogonal matrix A, by using the automorphism F of the form (3.4) we obtain the canonical form for definite $\mathrm{T}^*\mathfrak{h}_3{}'$,

$$S_{30} = \begin{pmatrix} \Lambda & 0 \\ 0 & E_{30} \end{pmatrix} \quad \text{or} \quad S_{03} = \begin{pmatrix} \Lambda & 0 \\ 0 & E_{03} \end{pmatrix}, \tag{3.6}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$ and $\lambda_1 \geq \lambda_2 \geq \lambda_3$ are nonzero and of arbitrary sign.

3.2. $\mathrm{T}^*\mathfrak{h}_3{}'$ **is Lorentzian (case S_{21} and S_{12}).** The admissible automorphisms are

$$\operatorname{Aut}(E_{21}) = \operatorname{Aut}(E_{12}) = \left\{ \begin{pmatrix} \pm A & 0 \\ B & A \end{pmatrix} \mid A^T E_{21} A = E_{21}, \det A > 0 \right\} \tag{3.7}$$

i.e., $A \in SO(2,1)$ and $\pm A \in O(2,1)$ and B any 3×3 matrix.

Suppose that in the basis e the metric $\langle \cdot,\cdot\rangle$ is represented by the matrix S_{21} or S_{12} given by (3.3).

By similar calculations as in (3.5), one can choose the matrices $A = I$ and $B = -E_{21}M^T$ of the automorphism $F \in \operatorname{Aut}(E_{21})$ such that in the new basis $\langle \cdot,\cdot\rangle$ has the form

$$\begin{pmatrix} S & 0 \\ 0 & E_{21} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} S & 0 \\ 0 & E_{12} \end{pmatrix}. \tag{3.8}$$

Now the 3×3 symmetric matrix S can have definite or Lorenz signature. To obtain the form (3.8), we can act by automorphism $F \in \operatorname{Aut}(E_{21})$ with $B = 0$. This reduces to finding equivalence classes of the action of the group $SO(2,1)$ on the Riemannian and on the Lorentzian symmetric matrix S.

3.2.1. $\mathrm{T}^*\mathfrak{h}_3{}'$ *is Lorentzian,* $\mathrm{T}^*\mathfrak{h}_3{}'^{\perp}$ *is Riemannian.* The case where $\mathrm{T}^*\mathfrak{h}_3{}'^{\perp}$ is Riemannian is the simpler of the two cases.

Lemma 3.1. *Let S be a symmetric matrix with positive eigenvalues. Then there exists a matrix $A \in SO(2,1)$ such that $A^T S A$ is diagonal.*

Proof. There exists an orthogonal matrix $T \in SO(3)$ such that

$$T^{-1} S T = D = \operatorname{diag}(d_1, d_2, d_3), \quad d_i > 0.$$

Then $S = TDT^{-1}$ and we denote the symmetric matrix $\sqrt{S} = T\sqrt{D}\,T^{-1}$, where $\sqrt{D} = \operatorname{diag}(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3})$. The matrix $\sqrt{S}^{-1} E_{21} \sqrt{S}^{-1} = (\sqrt{S}^{-1})^T E_{21} \sqrt{S}^{-1}$ is also symmetric (and has the same signature as E_{21}). Therefore, it can be diagonalized by the orthogonal matrix $R \in SO(3)$:

$$R^T\big((\sqrt{S}^{-1})^T E_{21}\sqrt{S}^{-1}\big)R = \operatorname{diag}\left(\frac{1}{\delta_1^2}, \frac{1}{\delta_2^2}, -\frac{1}{\delta_3^2}\right) = \Delta^{-1}E_{21}\Delta^{-1},$$

where $\Delta = \operatorname{diag}(\delta_1, \delta_2, \delta_3)$. If we set $A = \sqrt{S}^{-1}R\Delta$, then $\det A > 0$ and

$$A^T E_{21} A = E_{21}, \quad A^T S A = \Delta^2 = \operatorname{diag}(\delta_1^2, \delta_2^2, \delta_3^2),$$

which completes the proof. □

Suppose the metric $\langle\cdot,\cdot\rangle$ is represented by the matrix S_{21} or S_{12} given by (3.8), and the matrix S is positive definite (or negative definite). It follows from Lemma 3.1 that there exists a matrix $A \in SO(2,1)$ that diagonalizes S. The corresponding automorphism $F \in \operatorname{Aut}(E_{21})$, given by (3.7) with $B = 0$, brings the metric into the canonical form

$$S_{21} = \begin{pmatrix} \pm\Delta & 0 \\ 0 & E_{21} \end{pmatrix} \quad \text{or} \quad S_{12} = \begin{pmatrix} \pm\Delta & 0 \\ 0 & E_{12} \end{pmatrix}, \tag{3.9}$$

where $\Delta = \operatorname{diag}(\delta_1^2, \delta_2^2, \delta_3^2)$.

3.2.2. $\mathrm{T}^*\mathfrak{h}_3{}'$ *is Lorentzian,* $\mathrm{T}^*\mathfrak{h}_3{}'^{\perp}$ *is Lorentzian.* Suppose the metric $\langle\cdot,\cdot\rangle$ is represented by the matrix S_{21} (or S_{12}) given by (3.8), and the matrix S is Lorentzian, i.e., of signature $(2,1)$ (or signature $(1,2)$).

Finding the canonical form of S_{21} using the automorphism $F \in \operatorname{Aut}(E_{21})$, given by (3.7), reduces to:

Problem 1. Find equivalence classes of symmetric matrices S of Lorentz signature under the action of the group $O(2,1)$.

It is useful to consider the group $O(2,1)$ as a group of isometries of the hyperbolic plane. This is best seen in the Klein projective model of the hyperbolic plane [14].

Any symmetric nondegenerate matrix H can be viewed as a projective conic section $\Gamma(H)$ which satisfies the equation

$$\Gamma(H): \quad 0 = x^T H x,$$

where $x = (x_1\ x_2\ x_3)^T$ denotes the column vector of homogeneous coordinates $(x_1 : x_2 : x_3)$. For example, the Absolute of the Klein model $0 = x_1^2 + x_2^2 - x_3^2$ is a conic $\Gamma(E_{21})$. We restrict our attention only to conics represented by a symmetric matrix of signature $(2,1)$, since we have treated the case of signature $(3,0)$ (and $(0,3)$) in the previous subsection. Moreover, the matrix S with signature $(3,0)$ represents the "empty set" of the conics in real projective geometry.

The projective mapping $x \to Cx$, represented by the non-degenerate 3×3 matrix C, maps the conic $\Gamma(H)$ to the conic $\Gamma(C^T H C)$. Therefore, the condition $C \in O(2,1)$ for the matrix of the projective mapping is equivalent to preserving the Absolute $\Gamma(E_{21})$, i.e., C is a hyperbolic isometry.

Moreover, if $H = S$, the matrix of the metric we want to simplify, then we can consider the metric S as "conic" $\Gamma(S)$. Therefore, Problem 1 of classifying metrics is equivalent to the problem of classifying hyperbolic conics:

Problem 1*. Find the canonical forms of projective conics under the group of hyperbolic isometries.

Note that the conic $\Gamma(S)$ must not belong to the interior of the Absolute (i.e., the hyperbolic plane or the de Sitter space), since the group $O(2,1)$ also acts on its exterior (the anti-de Sitter space).

The classification of hyperbolic conics is a classical and well-known result [42, 33]. In the original paper [42] there are nine types of conics in the classification, but in later literature [33, 39, 28] 12 types appear. However, all of these classifications are mostly given by images only. In the paper [21] there are equations, but the classification is too complicated and in our case we do not need to distinguish between all 12 types. We get only 4 types, because we consider conics in the projective plane as a whole and not conics in the hyperbolic plane, which is the intersection of the projective plane and the interior of the Absolute. Our classification below uses the concept explained in [39].

We recall some basic facts about hyperbolic isometries in the projective Klein model (see, e.g., [14]). Let $\Gamma = \Gamma(H)$, $H^T = H$ be a non-degenerate conic. Let the point $P(\xi_1 : \xi_2 : \xi_3)$ be a *pole* and the line

$$p : p_1x_1 + p_2x_2 + p_3x_3 = 0, \quad \text{i.e.,} \quad p(p_1 : p_2 : p_3)$$

its *polar* with respect to Γ if $\lambda p = HP$, where $\lambda \neq 0$ is used to emphasize the homogenous nature of the coordinates. Note that $P \in p$ holds if and only if $P \in \Gamma(H)$. It is well known that projective mappings (or changes of coordinates) $x \to Cx$ preserve the pole-polar relation.

The group of hyperbolic isometries is generated by homologies ϕ_P (Klein reflection) with centre $P \notin \Gamma(E_{21})$ and its polar p with respect to the Absolute. The Klein reflection $\phi_P(M)$ of a point M is defined as the point M' such that points M, M', P, P_M are harmonic, where P_M is the intersection of PM and p.

In what follows, we are interested in two conics: for $H = E_{21}$, the conic $\Gamma(E_{21})$ which represents the Absolute and defines the group of admissible transformations $O(2,1)$; and for $H = S$, the conic $\Gamma(S)$ which represents the metric to be simplified.

The conic $\Gamma(S)$ is invariant with respect to the Klein reflection ϕ_P if P and p are also common pole and polar for both conics $\Gamma(E_{21})$ and $\Gamma(S)$. In this case, the point P is called the *centre of symmetry* and p the *line of symmetry* of $\Gamma(S)$. The basic idea is that the equation of a conic simplifies if the coordinates of its centre of symmetry are "nice".

The condition that P and p are common pole and polar for both $\Gamma(E_{21})$ and $\Gamma(S)$ is $\lambda_1 p = E_{21}P$, $\lambda_2 p = SP$, or equivalently

$$SP = \lambda E_{21}P \iff (E_{21}S)P = \lambda P, \quad \lambda \neq 0. \tag{3.10}$$

The nontrivial solution $P \neq (0:0:0)$ of this equation exists if and only if

$$\chi_S(\lambda) := \det(S - \lambda E_{21}) = 0. \tag{3.11}$$

Note that $\chi_S(\lambda)$ is not a characteristic polynomial of the matrix S. Moreover, it is clear from (3.10) that the solution of (3.11) is an eigenvalue, and P is an eigenvector of the *nonsymmetric* matrix $E_{21}S$.

Multiplying (3.10) by P^T from the left, we obtain, for the common pole P,

$$|P|_S^2 = P^T S P = \lambda P^T E_{21} P = \lambda |P|^2, \tag{3.12}$$

where we have denoted by $|P|_S^2$ the norm of P with respect to the metric S (i.e., $\langle\cdot,\cdot\rangle$) and by $|P|$ the norm with respect to the "hyperbolic" metric defined by E_{21}.

Since $\chi_S(\lambda)$ is of degree 3, there is at least one real eigenvalue $\lambda_1 \neq 0$ corresponding to the common pole P_1.

Case 1 $|P_1| > 0$ *(equivalently, P_1 is in the exterior of the Absolute).*

We can choose a new pseudo-orhonormal basis $f = (f_1, f_2, f_3) = C \in O(2,1)$ of $\mathrm{T}^*{\mathfrak{h}_3{}'}^{\perp}$ such that $f_1 = \frac{P_1}{|P_1|}$, and f_2, f_3 are arbitrary. In the new coordinates we have $P_1(1:0:0)$, the matrix of the Absolute is unchanged and

$$\lambda p_1 = E_{21} P_1 = (1\ 0\ 0)^T.$$

The matrix S of the metric $\langle\cdot,\cdot\rangle$ has changed to $\bar{S} = C^T S C = (\bar{s}_{ij})$, which we want to determine. But regardless of the change of coordinates, P_1 and p_1 are pole and polar with respect to the same conic $\Gamma(\bar{S})$:

$$(\lambda\ 0\ 0)^T = \lambda p_1 = \bar{S} P_1 = (\bar{s}_{11}\ \bar{s}_{12}\ \bar{s}_{13})^T,$$

and we have $\bar{s}_{12} = \bar{s}_{13} = 0$. Also,

$$\bar{s}_{11} = \langle f_1, f_1\rangle = |f_1|_S^2 = \lambda_1 |f_1|^2 = \lambda_1.$$

Therefore, for the case $|P_1|^2 > 0$, we can assume that the matrix S of the metric has the form

$$S = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & s_{22} & s_{23} \\ 0 & s_{23} & s_{33} \end{pmatrix}. \tag{3.13}$$

Now we discuss the possible Jordan forms of the matrix $E_{21}S$.

Case 1a) $E_{21}S$ is diagonalizable: $E_{21}S \sim \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$.

If P_1, P_2, P_3 are corresponding eigenvectors, then the triangle $P_1P_2P_3$ is autopolar with respect to both conics $\Gamma(E_{21})$ and $\Gamma(S)$. This means that P_i is pole of the line P_jP_k for all distinct i, j, k. Since $|P_1| > 0$, i.e., P_1 lies in the exterior of the Absolute, it is easy to prove that exactly one of P_2 and P_3 must lie in the interior – let it be P_3. Thus: $|P_1|^2$, $|P_2|^2 > 0$, $|P_3|^2 < 0$. As in *Case 1*, and more, we choose

$$f_1 = \frac{P_1}{|P_1|}, \quad f_2 = \frac{P_2}{|P_2|}, \quad f_3 = \frac{P_3}{|P_3|}.$$

The fact that $P_1P_2P_3$ is autopolar ensures that $f = (f_1, f_2, f_3) = C \in O(2,1)$. We already know that in the new basis (because of the choice of f_1) the matrix of the metric conic has the form (3.13). The new coordinates of the points are $P_1(1:0:0)$, $P_2(0:1:0)$, $P_3(0:0:1)$. Using the fact that $p_2(0:1:0)$ is polar to the pole P_2 with respect to the metric conic $\Gamma(S)$, we get $s_{23} = 0$. It is easy to verify that the canonical form is

$$S = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \tag{3.14}$$

where two of the λ_i are positive and one is negative.

Case 1b) $E_{21}S$ has the Jordan form $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix}$.

One calculates that $E_{21}S$ has double eigenvalues if and only if

$$(s_{22}+s_{33})^2 - 4s_{23}^2 = 0 \;\Leftrightarrow\; s_{23} = \pm\frac{s_{22}+s_{33}}{2}.$$

It is easy to verify that the automorphism $C = \operatorname{diag}(1,1,-1) \in O(2,1)$ changes s_{23} to $-s_{23}$, so we can assume that $s_{23} = \frac{s_{22}+s_{33}}{2}$. We obtain the canonical form

$$S = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & s_{22} & \frac{s_{22}+s_{33}}{2} \\ 0 & \frac{s_{22}+s_{33}}{2} & s_{33} \end{pmatrix}, \quad \lambda_1 > 0,\; s_{22} \neq s_{33}. \tag{3.15}$$

The condition on the coefficients ensures that the signature of S is $(2,1)$.

Case 1c) $E_{21}S$ has the Jordan form $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & \bar{z} \end{pmatrix}$, $z \in \mathbb{C}$.

It is obtained that $E_{21}S$ has complex conjugate eigenvalues if and only if $(s_{22}+s_{33})^2 - 4s_{23}^2 < 0$. Suppose that $\lambda_1 < 0$. Then both eigenvalues of the matrix $S' = \left(\begin{smallmatrix} s_{22} & s_{23} \\ s_{23} & s_{33} \end{smallmatrix}\right)$ must be positive, i.e., $s_{22}s_{33} - s_{23}^2 > 0$. From the previous two inequalities, we get $(s_{22}-s_{33})^2 < 0$, a contradiction. Therefore, the case $\lambda_1 < 0$ is impossible. For $\lambda_1 > 0$ we must have $s_{22}s_{33} - s_{23}^2 < 0$. The matrix S' represents the restriction of the metric S to the plane spanned by e_2 and e_3, which has the signature $(1,1)$. Null vectors in this plane are

$$v_\pm = \pm s_{33}e_2 + (\mp s_{23} + \sqrt{-\det S'})e_3.$$

The product of their squared norms (with respect to the inner product E_{21}),

$$|v_-|^2|v_+|^2 = s_{33}^2((s_{22}+s_{33})^2 - 4s_{23}^2),$$

is negative and therefore we can choose v_+ to be positive and v_- negative. By hyperbolic rotation,

$$f_2 = \cosh\phi\, e_2 + \sinh\phi\, e_3, \quad f_3 = \sinh\phi\, e_2 + \cosh\phi\, e_3$$

for some ϕ, we get that $f_3 = v_-$ (that would not be possible if $v_\pm$ were null or positive). In the new basis $f_1 = e_1$, f_2, $f_3 = v_-$ we have $s_{33} = \langle f_3, f_3\rangle = |v_-|_S^2 = 0$ (since f_3 is chosen as a null vector).

This results in the canonical form

$$S = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & s_{22} & s_{23} \\ 0 & s_{23} & 0 \end{pmatrix}, \quad \lambda_1 > 0,\; s_{23} \neq 0. \tag{3.16}$$

Case 2 $|P_1| = 0$ (equivalently, P_1 is on the Absolute).

In this case, the Jordan form of $E_{12}S$ is $\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}$. From the relation (3.12) we obtain $|P_1|_S^2 = 0$, and thus

$$P_1 \in \Gamma(E_{21}) \cap \Gamma(S),$$

i.e., the P_1 belong to the intersection of the conics. After the rotation, we can assume that P_1 is any point on the Absolute, for example $P_1(0 : -1 : 1)$. The polar p_1 with respect to the Absolute is $p_1(0 : 1 : 1)$. But p_1 is also the polar of P_1 with respect to $\Gamma(S)$:

$$\lambda p_1 = SP_1 \;\Leftrightarrow\; s_{12} = s_{13}. \tag{3.17}$$

From the condition that P_1 belongs to $\Gamma(S)$, we get

$$s_{33} = -s_{22} + 2s_{23}. \tag{3.18}$$

The condition that λ_1 is triple root of χ_S is equivalent to

$$s_{22} = s_{11} + s_{23}. \tag{3.19}$$

Considering the relations (3.17), (3.18) and (3.19) we obtain that another intersection point of $\Gamma(S)$ and the Absolute is

$$M(-4s_{13}s_{23} : 4s_{13}^2 - s_{23}^2 : 4s_{13}^2 + s_{23}^2).$$

We wish to map the point M to $M_0(1 : 0 : 1)$ by the transformation $C \in O(2,1)$, fixing the point P_1. The required transformation is a homology with centre $\{P\} = MM_0 \cap p_1$ and an axis whose polar $p = E_{21}P$. It can be shown that the matrix of this homology is

$$C = \begin{pmatrix} 8s_{13}^2 & 4s_{13}(s_{23} - 2s_{13}) & 4s_{13}(s_{23} - 2s_{13}) \\ 4s_{13}(s_{23} - 2s_{13}) & -4s_{13}^2 - 4s_{23}s_{13} + s_{23}^2 & (s_{23} - 2s_{13})^2 \\ 4s_{13}(2s_{13} - s_{23}) & -(s_{23} - 2s_{13})^2 & -12s_{13}^2 + 4s_{23}s_{13} - s_{23}^2 \end{pmatrix}.$$

Therefore, we assume that $M_0(1 : 0 : 1) \in \Gamma(S)$ or equivalently $s_{23} = 2s_{13}$ to get the canonical form

$$S = \begin{pmatrix} s_{11} & s_{13} & s_{13} \\ s_{13} & s_{11} - 2s_{13} & -2s_{13} \\ s_{13} & -2s_{13} & -s_{11} - 2s_{13} \end{pmatrix}, \quad s_{11} \neq 0. \tag{3.20}$$

The signature of this matrix is always Lorentzian. Note that when $s_{13} = 0$, we obtain the diagonal form (3.14) considered earlier.

Case 3 $|P_1| < 0$ *(equivalently, P_1 is inside the interior of the Absolute).*

Since $|P_1| < 0$, we can choose the basis $(f_1, f_2, f_3) \in O(2,1)$ such that $f_1 = \frac{P_1}{|P_1|}$. Similar to *Case 1*, we obtain that the metric in this basis is the matrix

$$S = \begin{pmatrix} s_{11} & s_{12} & 0 \\ s_{12} & s_{22} & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}.$$

The zeros of the characteristic polynomial (3.11) are

$$-\lambda_1,\ \lambda_{2/3} = \frac{s_{11} + s_{22} \pm \sqrt{4s_{12}^2 + (s_{22} - s_{22})^2}}{2}.$$

We see that the polynomial cannot have multiple roots or complex conjugate roots, and we obtain only the *Case 1a)*, i.e., the canonical form of the metric is (3.14).

3.3. $\mathrm{T}^*\mathfrak{h}_3{}'$ is degenerate of rank 2 (metrics S_{20}, S_{02}, S_{11}).

3.3.1. *Case S_{20}, S_{02}.* Suppose that in the basis e the metric $\langle\cdot,\cdot\rangle$ is represented by the matrix S_{20} or S_{02} given by (3.3). Therefore, we look for the canonical form

$$\begin{pmatrix} S & M \\ M^T & \pm E_{20} \end{pmatrix} \tag{3.21}$$

with S and M as simple as possible. We first describe the group of isometries of the degenerate inner product E_{20}.

Lemma 3.2. *The subgroup of $Gl_3(\mathbb{R})$ which preserves the degenerate quadratic form represented by the matrix E_{20} is*

$$O_3(2,0) = \left\{ \begin{pmatrix} \lambda & a & b \\ 0 & \cos\phi & \mp\sin\phi \\ 0 & \sin\phi & \pm\cos\phi \end{pmatrix} \mid a, b, \phi, \lambda \in \mathbb{R},\ \lambda \neq 0 \right\}.$$

From this lemma and Lemma 2.1 we derive the subgroup of $\mathrm{Aut}(\mathrm{T}^*\mathfrak{h}_3)$ preserving the form of the matrix S_{20} or S_{02}:

$$\mathrm{Aut}(E_{20}) = \mathrm{Aut}(E_{02}) = \left\{ \begin{pmatrix} \pm A & 0 \\ B & A^* \end{pmatrix} \right\}, \tag{3.22}$$

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ a & \frac{\cos\phi}{\lambda} & \frac{\sin\phi}{\lambda} \\ b & -\frac{\sin\phi}{\lambda} & \frac{\cos\phi}{\lambda} \end{pmatrix}, \quad A^* = \begin{pmatrix} \frac{1}{\lambda^2} & \frac{-a\cos\phi+b\sin\phi}{\lambda} & \frac{-b\cos\phi-a\sin\phi}{\lambda} \\ 0 & \cos\phi & \mp\sin\phi \\ 0 & \sin\phi & \pm\cos\phi \end{pmatrix}.$$

We denote the automorphism $F \in \mathrm{Aut}(E_{20})$ of the form (3.22) by $F(\lambda, a, b, \phi, B)$, $B = (b_{ij})$.

The subalgebra $\mathrm{T}^*\mathfrak{h}_3{}' = \mathbb{R}\langle e_4, e_5, e_6\rangle$ is degenerate, and from (3.2) and (3.3) we see that $e_4 \in \mathrm{T}^*\mathfrak{h}_3{}'^{\perp}$. Moreover, $\mathrm{T}^*\mathfrak{h}_3{}' \cap \mathrm{T}^*\mathfrak{h}_3{}'^{\perp} = \mathbb{R}\langle e_4\rangle$ and therefore $\mathrm{T}^*\mathfrak{h}_3{}' + \mathrm{T}^*\mathfrak{h}_3{}'^{\perp}$ has codimension one in $\mathrm{T}^*\mathfrak{h}_3$.

The search for the canonical form of the metric $S = S_{20}$ consists of several steps in which we apply automorphisms in a very specific order. To simplify the notation, we will always denote the resulting matrix by S and keep the same notation for its entries, even if the entries change.

The automorphisms are quite restrictive in the plane $\mathbb{R}\langle e_2, e_3\rangle$, since we can basically choose a new basis only by rotation. We have three cases corresponding to the following geometric situations:

Case 1. $\mathbb{R}\langle e_2, e_3\rangle \cap \mathrm{T}^*\mathfrak{h}_3{}'^{\perp}$ is non-null.

Case 2. $\mathbb{R}\langle e_2, e_3\rangle \subset \mathrm{T}^*\mathfrak{h}_3{}'^{\perp}$.

Case 3. $\mathbb{R}\langle e_2, e_3\rangle \cap \mathrm{T}^*\mathfrak{h}_3{}'^{\perp}$ is null vector.

The first step is the same for all three cases.

Step 1. On the matrix $S = S_{20}$ we first apply the automorphism $F(1, 0, 0, \phi, B)$, where

$$\cos\phi = \frac{m_{31}}{\sqrt{m_{21}^2 + m_{31}^2}}, \quad \sin\phi = \frac{m_{21}}{\sqrt{m_{21}^2 + m_{31}^2}},$$
$$b_{22} = -m_{22}m_{31} + m_{21}m_{32}, \quad b_{32} = -m_{23}m_{31} + m_{21}m_{33}.$$

This results with the matrix $F^T SF$ with $m_{12} = m_{22} = m_{32} = 0$.

Case 1. $s_{22} \neq 0, m_{31} \neq 0$.

Step 2. We apply the automorphism $F(1, a, b, 0, B)$, where

$$a = \frac{m_{11}s_{23} - m_{31}s_{12}}{m_{31}s_{22}}, \quad b = \frac{-m_{11} + \sqrt[3]{m_{31}}}{m_{31}}, \quad b_{12} = -\frac{s_{23}}{m_{31}},$$

b_{13} is complicated, so it is omitted, and the remaining b_{ij} are zero. After this action we get $s_{12} = 0 = s_{13}$, $m_{11} = \sqrt[3]{m_{13}}$.

Step 3. We apply the automorphism $F(1, 0, 0, 0, B)$, where

$$b_{21} = -m_{12}, \quad b_{23} = -m_{32}, \quad b_{31} = -m_{13}, \quad b_{33} = -m_{33},$$
$$b_{11} = \frac{m_{12}^2 + m_{13}^2 - s_{11}}{2\sqrt[3]{m_{13}}}.$$

Here we have to use the complicated parameter b_{13} again, and the remaining b_{ij} are zero. The resulting matrix of the metric has $s_{11} = m_{12} = m_{13} = m_{32} = m_{33} = 0$.

Step 4. The automorphism $F(\lambda, 0, 0, 0, B)$, with $\lambda = \sqrt[3]{m_{13}}$, $B = 0$ simultaneously sets $m_{11} = m_{13} = 1$, and we obtain the canonical form (3.21) of the metric, with:

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & s_{22} & 0 \\ 0 & 0 & s_{33} \end{pmatrix}, \quad s_{22} \neq 0, s_{33} \neq 0, \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3.23}$$

Case 2. $m_{31} = 0$.

Step 2. By the automorphism $F(1, 0, 0, 0, B)$, $b_{23} = -m_{32}$, $b_{33} = -m_{33}$ we obtain the matrix $F^T SF$ with $m_{32} = 0 = m_{33}$.

Step 3. Now we obtain $s_{11} = s_{12} = s_{13} = 0 = m_{12} = m_{13}$ with appropriate choice of parameters $a, b, b_{12}, b_{13}, b_{11}$.

Step 4. The automorphism $F(\lambda, 0, 0, \phi, B)$, $\lambda = m_{11}$, $B = 0$, where ϕ is chosen such that the corresponding rotation diagonalizes the metric in $\mathbb{R}\langle e_2, e_3\rangle$, gives the canonical form (3.21), with:

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & s_{22} & 0 \\ 0 & 0 & s_{33} \end{pmatrix}, \quad s_{22}, s_{33} \neq 0, \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.24}$$

Case 3. $s_{22} = 0$.

In a similar way, but without using rotation, we obtain the canonical form (3.21):

$$S = \begin{pmatrix} 0 & s_{12} & 0 \\ s_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad s_{12} \neq 0, \quad M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3.25}$$

3.3.2. *Case* S_{11}. Suppose the metric $\langle\cdot,\cdot\rangle$ is represented by the matrix S_{11} given by (3.3).

If $\mathrm{T}^*\mathfrak{h}_3{}'$ has signature $(0,+,-)$, we have the following automorphisms:

$$\mathrm{Aut}(E_{11}) = \left\{ \begin{pmatrix} \pm A & 0 \\ B & A^* \end{pmatrix} \right\},$$

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ a & \frac{\cosh\phi}{\lambda} & \frac{\sinh\phi}{\lambda} \\ b & \frac{\sinh\phi}{\lambda} & \frac{\cosh\phi}{\lambda} \end{pmatrix}, \quad A^* = \begin{pmatrix} \frac{1}{\lambda^2} & \frac{-a\cosh\phi+b\sinh\phi}{\lambda} & \frac{-b\cosh\phi+a\sinh\phi}{\lambda} \\ 0 & \cosh\phi & -\sinh\phi \\ 0 & -\sinh\phi & \cosh\phi \end{pmatrix}.$$

In the plane $\mathbb{R}\langle e_2, e_3\rangle$ the automorphisms act as hyperbolic rotations which do not necessarily diagonalize the metric in this plane. To describe this effect precisely, we need the following lemma.

Lemma 3.3. *The equivalence classes of the symmetric matrix* $S = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ *under the action* F^TSF*, where* $F \in SO(1,1)$*, are:*

$$\begin{pmatrix} a' & 0 \\ 0 & c' \end{pmatrix} \text{ if } 4b^2 \neq (a+c)^2, \tag{3.26}$$

$$\begin{pmatrix} 0 & \frac{c-a}{2} \\ \frac{c-a}{2} & c-a \end{pmatrix} \text{ if } 4b^2 = (a+c)^2,\ |c| > |a|, \tag{3.27}$$

$$\begin{pmatrix} a-c & \frac{a-c}{2} \\ \frac{a-c}{2} & 0 \end{pmatrix} \text{ if } 4b^2 = (a+c)^2,\ |c| < |a|. \tag{3.28}$$

Under the F *which is an anti-isometry, i.e.,* $F^T \operatorname{diag}(1,-1)F = \operatorname{diag}(-1,1)$*, the canonical forms* (3.27) *and* (3.28) *are equivalent.*

Proof. Set $E_{11} = \operatorname{diag}(1,-1)$. The group $SO(1,1)$ consists of hyperbolic rotations and their negatives:

$$\begin{aligned} SO(1,1) &= \{F \in Gl_2(\mathbb{R}) \mid F^TE_{11}F = E_{11}, \det F = 1\} \\ &= \left\{ \pm \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \mid \phi \in \mathbb{R} \right\}. \end{aligned}$$

Case 1. $(a+c)^2 - (2b)^2 > 0$. It is straightforward to verify that a hyperbolic rotation by the "angle" ϕ is such that

$$\cosh 2\phi = \lambda|a+c|, \quad \sinh 2\phi = -2\lambda \operatorname{sgn}(a+c)b,$$

where $\lambda = ((a+c)^2 - (2b)^2)^{-\frac{1}{2}}$ is determined from the condition $\cosh^2 2\phi - \sinh^2 2\phi = 1$, diagonalizes the matrix S and we obtain the form (3.26).

Case 2. $(a+c)^2 - (2b)^2 < 0$. In this case, we diagonalize S with hyperbolic rotation so that

$$\cosh 2\phi = \lambda|2b|, \quad \sinh 2\phi = -2\lambda \operatorname{sgn}(b)(a+c)$$

and $\lambda = ((2b)^2 - (a+c)^2)^{-\frac{1}{2}}$.

Case 3. $(a+c)^2 - (2b)^2 = 0$. Suppose that $b = \frac{a+c}{2}$. In this case, the null directions of the metric S are $(1,-1)$ and $(c,-a)$, $a > 0$. We will apply hyperbolic rotation such that one of the basis vectors is null. The case $|a| = |c|$ is either diagonal or impossible. If $|a| > |c|$, we will take the hyperbolic rotation such that

$$\cosh\phi = \frac{a}{\sqrt{a^2-c^2}}, \quad \sinh\phi = \frac{-c}{\sqrt{a^2-c^2}},$$

to obtain the canonical form (3.28). If $|a| < |c|$, we take the hyperbolic rotation

$$\cosh\phi = \frac{c}{\sqrt{c^2-a^2}}, \quad \sinh\phi = \frac{-a}{\sqrt{c^2-a^2}},$$

to obtain the canonical form (3.27). Note that these two cases are not equivalent under the action of $O(1,1)$, since the null direction of the metric S belongs either to the set of time-like or to the set of space-like vectors of the metric $\mathrm{diag}(1,-1)$, which are preserved under the action of $SO(1,1)$ (and also $O(1,1)$). However, if we allow anti-isometries, we can interchange time-like and space-like vectors, and these two cases are equivalent.

The case $b = -\frac{a+c}{2}$ is similar. $\square$

The classification of metrics of type S_{11} is similar to that of type S_{20}, with a possible difference only when using a rotation. Therefore, we follow the steps from Subsection 3.3.1 in what follows. We look for the canonical form

$$\begin{pmatrix} S & M \\ M^T & \pm E_{11} \end{pmatrix} \tag{3.29}$$

with S and M as simple as possible.

The hyperbolic rotation is not transitive on the vectors of the plane. In "regular" cases, hyperbolic rotation can be used instead of Euclidean rotation and we obtain metrics (3.29) with S and M given by (3.23), (3.24) or (3.25).

Now we discuss the "singular" cases. Already in *Step 1.* the hyperbolic rotation is not possible if $m_{21} = \pm m_{31} \neq 0$. In fact, these two cases are equivalent by an anti-isometric automorphism, so we consider the case $m_{21} = m_{31}$. After long and detailed analysis, we obtain two non-equivalent metrics:

$$\begin{pmatrix} S & M \\ M^T & E_{11} \end{pmatrix}, \ S = \begin{pmatrix} s_{11} & 0 & 0 \\ 0 & s_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ s_{11}, s_{22} \neq 0, \ M = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{3.30}$$

$$\begin{pmatrix} S & M \\ M^T & E_{11} \end{pmatrix}, \ S = \begin{pmatrix} 0 & s_{12} & 0 \\ s_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ s_{12} > 0, \ M = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3.31}$$

If $m_{21} \neq \pm m_{31}$ then we can reach $m_{21} = 0$ by hyperbolic rotation in *Step 1.* and proceed with the remaining steps.

In *Case 1.* rotation is not used and no additional canonical forms are obtained.

In *Case 2.* rotation is used in *Step 4.* to diagonalize the metric in the plane $\mathbb{R}\langle e_2, e_3\rangle$. According to Lemma 3.3, this is not always possible for hyperbolic rotation, so we obtain additional metrics:

$$\begin{pmatrix} S & M \\ M^T & E_{11} \end{pmatrix},\ S=\begin{pmatrix} 0 & 0 & 0 \\ 0 & s_{22} & \frac{1}{2}|s_{22}| \\ 0 & \frac{1}{2}|s_{22}| & 0 \end{pmatrix},\ s_{22}\neq 0,\ M=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.32}$$

$$\begin{pmatrix} S & M \\ M^T & E_{11} \end{pmatrix},\ S=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}|s_{33}| \\ 0 & \frac{1}{2}|s_{33}| & s_{33} \end{pmatrix},\ s_{33}\neq 0,\ M=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.33}$$

Finally, in *Case 3.* rotation is not used, so the classification of metrics with centre of signature $(0,+,-)$ is complete.

3.4. $\mathrm{T}^*\mathfrak{h}_3{}'$ **is degenerate of rank 1 (case S_{10}, S_{01}).** Suppose that in the basis e the metric $\langle\cdot,\cdot\rangle$ is represented by the matrix S_{10} or S_{01} given by (3.3). So, we look for the canonical form

$$\begin{pmatrix} S & M \\ M^T & \pm E_{10} \end{pmatrix} \tag{3.34}$$

with S and M as simple as possible.

When $\mathrm{T}^*\mathfrak{h}_3{}'$ has the signature $(0,0,+)$ or $(0,0,-)$, we have the following group of automorphisms preserving their canonical form:

$$\mathrm{Aut}(E_{10})=\mathrm{Aut}(E_{01})=\left\{\begin{pmatrix} \pm A & 0 \\ B & A^* \end{pmatrix}\right\}, \tag{3.35}$$

$$A=\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix},\quad A^*=\begin{pmatrix} a_{22}a_{33} & -a_{21}a_{33} & a_{21}a_{32}-a_{22}a_{31} \\ -a_{12}a_{33} & a_{11}a_{33} & a_{12}a_{31}-a_{11}a_{32} \\ 0 & 0 & a_{11}a_{22}-a_{12}a_{21} \end{pmatrix},$$

with condition $(a_{11}a_{22}-a_{12}a_{21})^2=1$. Note that the automorphism of the form

$$F=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

interchanges the places of the elements m_{31} and m_{32} in the matrix M. Thus, two cases are distinguished: when $m_{31}\neq 0$ and when $m_{31}=m_{32}=0$. It is worth noting that this is not a simple algebraic distinction. These two cases yield completely different geometric properties (see Proposition 4.8 (iii) below).

Case 1. $m_{31}\neq 0$. In this case, we can obtain the following form of M:

$$\begin{pmatrix} 0 & m_{12} & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{3.36}$$

by performing the next steps:

Step 1. The appropriate choice of elements a_{21} and a_{31} in (3.35) gives us $m_{11} = m_{32} = 0$, while we can set a_{22} and a_{32} such that $m_{21} = t$, $m_{31} = t^2$, where $t \neq 0$ is an arbitrary parameter that is normalized later. Finally, by setting a_{21}, $m_{22} = 0$ is obtained.

Step 2. If we now choose the last row of the matrix B, we obtain $m_{13} = m_{23} = m_{33} = 0$.

Step 3. Choosing the remaining elements of the matrix B, the matrix S reduces in (3.34) to $\pm\lambda E_{01}$, $\lambda \neq 0$.

Step 4. In the last step we normalize both λ and t and obtain the metric (3.34) with $S = \pm E_{01}$ and M in the form (3.36).

Case 2. $m_{31} = m_{32} = 0$. Since the elements a_{31} and a_{32} do not act on the matrix M, the problem reduces to the action $A^T M A$ of the matrix

$$A = \begin{pmatrix} \bar{A} & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{A} \in SL(2).$$

In the first step, depending on the nature of the eigenvalues of the matrix M, we can choose the matrix $\bar{A}$ such that the upper-left 2×2-submatrix of M takes one of the following three forms:

$$\begin{pmatrix} m_{11} & 0 \\ 0 & m_{22} \end{pmatrix}, \quad \begin{pmatrix} m_{11} & m_{12} \\ -m_{12} & m_{11} \end{pmatrix}, \quad \begin{pmatrix} m_{11} & 0 \\ 1 & m_{11} \end{pmatrix}.$$

Next, we can repeat *Step 2.* and *Step 3.* from above. Finally, in the last step, we make the basis vector e_3 to be unit again. Therefore, our metric is $S_{10} = (\pm E_{01}, M, E_{01})$, where M takes one of the following three forms:

$$\begin{pmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} m_{11} & m_{12} & 0 \\ -m_{12} & m_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} m_{11} & 0 & 0 \\ 1 & m_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.37}$$

Note that the case of the metric S_{01} can be considered completely analogously.

3.5. $\mathrm{T}^*\mathfrak{h}_3{}'$ **is degenerate of rank 0 (case S_{00}).** The last case of a completely degenerate centre is the only case that can be considered with a purely algebraic approach. Suppose that in the basis e the metric $\langle \cdot, \cdot \rangle$ is represented by the matrix S_{00} given by (3.3). The group of admissible automorphisms is

$$\mathrm{Aut}(E_{00}) = \left\{ \begin{pmatrix} \pm A & 0 \\ B & A^* \end{pmatrix} \right\}, \tag{3.38}$$

where A, $\det A \neq 0$ and B are arbitrary 3×3 matrices.

If we take the automorphism F of the form (3.38) and act on the matrix S_{00}, we obtain

$$\begin{pmatrix} A^T & B^T \\ 0 & (A^*)^T \end{pmatrix} \begin{pmatrix} S & M \\ M^T & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ B & A^* \end{pmatrix} = \begin{pmatrix} A^T S A + A^T M B + (A^T M B)^T & A^T M A^* \\ (A^T M A^*)^T & 0 \end{pmatrix}. \tag{3.39}$$

From the non-degeneracy of the metric matrix S_{00}, it follows that the matrix M must also be regular. Thus, setting $B = -\frac{1}{2}M^{-1}SA$, the matrix (3.39) takes the form

$$\begin{pmatrix} 0 & A^T M A^* \\ (A^T M A^*)^T & 0 \end{pmatrix}$$

and all that remains is to choose a regular matrix A so that $A^T M A^*$ has the simplest form. However, one must remember that the matrix M is not symmetric, so it is not necessarily diagonalizable. At least one eigenvalue of M must be real, and the other two can be either real (with some multiplicity) or complex conjugate. Therefore, the possible canonical Jordan forms of M are

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & -\lambda_3 \\ 0 & \lambda_3 & \lambda_2 \end{pmatrix}. \quad (3.40)$$

We can take another step to further simplify these forms: setting the automorphism matrix as diagonal, in (3.40) we obtain $\lambda_1 = 1$.

Note that all these metrics have a neutral signature.

3.6. **Main result.** The preceding extensive analysis proves the following theorem.

Theorem 3.4. *The non-isometric left invariant metrics on* $\mathrm{T}^*\mathfrak{h}_3$ *in the basis* e *with commutators* (2.1) *are represented by matrices* $S_{pq} = (S, M, E_{pq})$ *of the form* (3.3)*:*

(i) *if* $\mathrm{T}^*\mathfrak{h}_3{}'$ *is non-degenerate:*
  $S_{30} = (S, 0, \pm E_{30})$*, where* S *is of the form* (3.6)*;*
  $S_{21} = (S, 0, \pm E_{21})$*, where* S *is of the form* (3.9)*,* (3.14)*,* (3.15)*,* (3.16) *or* (3.20)*;*

(ii) *if* $\mathrm{T}^*\mathfrak{h}_3{}'$ *is degenerate of rank* 2*:*
  $S_{20} = (S, M, \pm E_{20})$*, where* S *and* M *take one of the forms* (3.23)*,* (3.24) *or* (3.25)*;*
  $S_{11} = (S, M, \pm E_{11})$*, where* S *and* M *take one of the forms* (3.23) *or* (3.25)*;*
  $S_{11} = (S, M, E_{11})$*, where* S *and* M *take one of the forms* (3.24)*,* (3.30)*,* (3.31)*,* (3.32) *or* (3.33)*;*

(iii) *if* $\mathrm{T}^*\mathfrak{h}_3{}'$ *is degenerate of rank* 1*:*
  $S_{10} = (\pm E_{10}, M, \pm E_{10})$*, where* M *takes one of the forms* (3.36) *or* (3.37) *(all four combinations of* $\pm$ *can occur here);*

(iv) *if* $\mathrm{T}^*\mathfrak{h}_3{}'$ *is degenerate of rank* 0*:*
  $S_{00} = (0, M, 0)$*, where* M *takes one of the forms* (3.40) *with* $\lambda_1 = 1$.

4. Geometrical properties of left-invariant metrics

In this section, we further investigate the metrics obtained in Theorem 3.4. First, their curvature properties are of interest and then we briefly consider the holonomy algebras for each metric. We also obtain the description of the parallel symmetric tensors for each metric and show that they are derived from parallel vector fields. Special types of metrics, such as pp-waves or Ricci solitons, are also studied. Since $\mathrm{T}^*\mathfrak{h}_3$ is even-dimensional, it is natural to study the invariant complex and symplectic structures. This leads to the classification of pseudo-Kähler metrics. Finally, the known facts about the totally geodesic subalgebras of a nilpotent Lie algebra are summarised and it is shown that for every subalgebra of $\mathrm{T}^*\mathfrak{h}_3$ there exists at least one metric which makes it totally geodesic.

4.1. **Curvature and holonomy of the metrics.** If S is the matrix corresponding to the metric $\langle\cdot,\cdot\rangle$ in the basis $(e_1,\ldots,e_6)$, the algebra of its isometries $so(S)\cong so(p,q)$, with $p+q=6$, is spanned by the endomorphisms $e_i\wedge e_j$, $1\le i<j\le 6$, defined by

$$(e_i\wedge e_j)(x):=\langle e_j,x\rangle e_i-\langle e_i,x\rangle e_j,\quad x\in\mathrm{T}^*\mathfrak{h}_3.$$

For the left-invariant vector fields $x,y,z\in\mathrm{T}^*\mathfrak{h}_3$, Koszul's formula reduces to

$$2\langle\nabla_x y,z\rangle=\langle[x,y],z\rangle-\langle[y,z],x\rangle+\langle[z,x],y\rangle, \tag{4.1}$$

which allows us to compute the Levi-Civita connection ∇ of the metric $\langle\cdot,\cdot\rangle$. The curvature R and the Ricci tensor ρ are given by

$$R(x,y)z=\nabla_x(\nabla_y z)-\nabla_y(\nabla_x z)-\nabla_{[x,y]}z,\quad \rho(x,y)=\mathrm{Tr}(z\mapsto R(z,x)y).$$

The scalar curvature is defined as a trace of the Ricci operator.

The metric $\langle\cdot,\cdot\rangle$ is called *flat* if the corresponding curvature tensor is zero everywhere, i.e., $R=0$, and it is *locally symmetric* if $\nabla R=0$. Similarly, the metric is *Ricci-flat* when $\rho=0$ and *Ricci-parallel* when $\nabla\rho=0$.

We can further simplify the above definitions by considering that we are studying curvature operators on the nilpotent Lie group. We define the operators ad_x^*, j_x and φ_x:

$$\langle\mathrm{ad}_x y,z\rangle=\langle y,\mathrm{ad}_x^* z\rangle,\quad j_x y:=\mathrm{ad}_y^* x,\quad \varphi_x:=\mathrm{ad}_x+\mathrm{ad}_x^*.$$

Then the following lemma holds.

Lemma 4.1 ([1]). *In the case of a nilpotent Lie algebra $\mathfrak{g}$, the curvature and Ricci tensors are given by*

$$R(x,y)=\frac{1}{2}(j_{[x,y]}+[j_x,\mathrm{ad}_y^*]+[\mathrm{ad}_x^*,j_y])-\frac{1}{4}([\varphi_x,\varphi_y]+[j_x,\varphi_y]+[\varphi_x,j_y]-[j_x,j_y]),$$
$$\rho(x,y)=-\frac{1}{4}\,\mathrm{tr}(j_x\circ j_y)-\frac{1}{2}\,\mathrm{tr}(\mathrm{ad}_x\circ\mathrm{ad}_y^*),$$

for all left-invariant vector fields $x,y\in\mathfrak{g}$.

In the following statement we describe curvature and Ricci curvature of metrics on $\mathrm{T}^*\mathfrak{h}_3$ depending on the signature of the induced metric of $\mathrm{T}^*\mathfrak{h}_3{}'$.

Proposition 4.2. *The following statements hold:*

(i) *If $\mathrm{T}^*\mathfrak{h}_3{}'$ is nondegenerate the metric cannot be flat or Ricci-flat.*
(ii) *If $\mathrm{T}^*\mathfrak{h}_3{}'$ is degenerate of rank 2, the metrics are $S_{20}=(S,M,\pm E_{20})$, where S and M take the form* (3.24)*, and $S_{11}=(S,M,E_{11})$, where S and M take*

one of the forms (3.24), (3.32), *or* (3.33). *These metrics are Ricci-parallel. Among them, the Ricci-flat metrics are:*

$$S_{20} = (S, M, \pm E_{20}), \qquad S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & s_{22} & 0 \\ 0 & 0 & \pm 1 - s_{22} \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$S_{11} = (S, M, E_{11}), \qquad S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & s_{22} & 0 \\ 0 & 0 & -1 + s_{22} \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$S_{11} = (S, M, E_{11}), \qquad S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$S_{11} = (S, M, E_{11}), \qquad S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -1 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(iii) *If* $\mathrm{T}^*\mathfrak{h}_3{}'$ *is degenerate of rank* 1, *the corresponding metrics* $S_{10} = (\pm E_{10}, M, E_{10})$, *where* M *takes one of the forms* (3.37), *are locally symmetric and Ricci-flat. Specifically, the metrics*

$$S_{10} = (E_{10}, M, E_{10}), \qquad M = \begin{pmatrix} \lambda \pm \sqrt{3} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$S_{10} = (-E_{10}, M, E_{10}), \qquad M = \begin{pmatrix} \lambda_1 & \mp\frac{\sqrt{3}}{2} & 0 \\ \pm\frac{\sqrt{3}}{2} & \lambda_1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

are flat. The Ricci-parallel metric also occurs and has the form

$$S_{10} = (\pm E_{10}, M, \pm E_{10}), \qquad M = \begin{pmatrix} 0 & \lambda & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(iv) *If* $\mathrm{T}^*\mathfrak{h}_3{}'$ *is degenerate of rank* 0, *the corresponding metrics are flat.*

(v) *The only examples of Einstein metrics (i.e., metrics with proportional Ricci curvature and metric tensors) are the trivial, Ricci-flat ones.*

Proof. The proof is simple but tedious, since it must be computed case by case for each canonical form. We give some details for the metric $S_{10} = (E_{10}, M, E_{10})$ with $\mathrm{T}^*\mathfrak{h}_3{}'$ of rank 1, where M is given by (3.36).

By (4.1) we obtain the Levi-Civita connection of the metric in terms of nonzero derivations (taking into account the relation $[x, y] = \nabla_x y - \nabla_y x$)

$$\begin{aligned} &\nabla_{e_1} e_1 = m_{12}(-e_2 + e_3), \quad \nabla_{e_1} e_2 = \frac{1}{2} e_6, \quad \nabla_{e_1} e_3 = -e_5, \\ &2\nabla_{e_1} e_6 = -e_2 + e_3 - e_4 = \nabla_{e_2} e_3 = \nabla_{e_3} e_3, \\ &\nabla_{e_2} e_2 = e_2 - e_3, \quad \nabla_{e_2} e_6 = \frac{1}{2m_{12}} e_5. \end{aligned} \tag{4.2}$$

Note that the vectors e_4, e_5 are parallel. In Proposition 4.7 it was proved that all parallel vectors are given by their linear combination.

Using Lemma 4.1 or directly from the definition of curvature, we obtain that nonzero curvature operators are given by

$$\begin{aligned}
R(e_1,e_2) &= \frac{3}{4m_{12}}(-e_2\wedge e_5 + e_3\wedge e_5) + \frac{4m_{12}-3}{4m_{12}}\, e_4\wedge e_5 + \frac{1}{2}\, e_4\wedge e_6,\\
R(e_1,e_3) &= e_4\wedge e_5 + \frac{1}{2}\, e_4\wedge e_6, \quad R(e_1,e_6) = -\frac{1}{4m_{12}}\, e_5\wedge e_6,\\
R(e_2,e_6) &= R(e_3,e_6) = -\frac{1}{2m_{12}}\, e_4\wedge e_5.
\end{aligned}$$

By reapplying Lemma 4.1 again we obtain that the only nonzero component of the Ricci tensor is

$$\rho(e_1,e_1) = -\frac{1}{2}.$$

One can easily check that this metric is Ricci-parallel, $\nabla\rho \equiv 0$. We also check (see the proof of Proposition 4.8) that $\nabla R \not\equiv 0$, and therefore the metric is not locally symmetric. □

Remark 4.3. Although here we have directly confirmed that there are no Einstein metrics that are not Ricci-flat, this follows from a more general statement (see [45, Proposition 3.1]).

In [35] Milnor proved that in the Riemannian case, if the Lie group G is solvable, every left-invariant metric on G is either flat, or has strictly negative scalar curvature. Note that this is not true in the pseudo-Riemannian setting. In the case of non-degenerate $\mathrm{T}^*\mathfrak{h}_3{}'$, the scalar curvatures can be positive, negative or zero depending on the signature, while in the case of degenerate $\mathrm{T}^*\mathfrak{h}_3{}'$ all but two metrics have zero scalar curvature. More precisely, we have the following statement, which can be obtained from Proposition 4.2 by direct computation.

Proposition 4.4. *The following statements hold:*

(i) *The scalar curvature of the metrics* $(S, 0, E_{pq})$, $p+q=3$ *on* $\mathrm{T}^*\mathfrak{h}_3$ *with non-degenerate centre* $\mathrm{T}^*\mathfrak{h}_3{}'$ *is given by*

$$\tau = -\frac{\operatorname{trace}(SE_{pq})}{2\det S}.$$

(ii) *Metrics* (S, M, E_{pq}), $p+q=2$ *with* S *and* M *given by* (3.23) *and* (3.25) *have nonzero scalar curvature given, respectively, by* $\tau = \mp\frac{\epsilon}{2s_{22}s_{33}}$ *and* $\tau = \pm\frac{\epsilon}{2s_{12}^2}$, *where* ϵ *is the element at position* $(3,3)$ *of* E_{pq}.

(iii) *All other metrics on* $\mathrm{T}^*\mathfrak{h}_3$ *with degenerate centre* $\mathrm{T}^*\mathfrak{h}_3{}'$ *have scalar curvature* $\tau = 0$.

Example 4.5. In [9, Example 5.1] the authors considered a canonical metric defined by

$$\langle (x,\alpha),(x',\alpha')\rangle = \alpha'(x) + \alpha(x') \qquad \forall x, x' \in \mathfrak{h}_3,\ \alpha,\alpha' \in \mathfrak{h}_3^*.$$

This metric is neutral signature and ad-invariant, meaning that, for all $x, y, z \in \mathrm{T}^*\mathfrak{h}_3$, $\langle [x,y], z\rangle = -\langle y, [x,z]\rangle$. Note that this is a special case of our metric $S_{00} = (0, M, 0)$ when M is the identity matrix. This is the only ad-invariant metric on $\mathrm{T}^*\mathfrak{h}_3$ that confirms the result recently obtained in [10].

In what follows, we find all parallel vector fields of metrics on $\mathrm{T}^*\mathfrak{h}_3$. Their existence has important consequences for the holonomy group of metrics as well as for the existence of parallel symmetric tensors (see Section 4.4). They are characterized by the following lemma.

Lemma 4.6. *The left-invariant vector field $x \in \mathfrak{g}$ on the metric Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is parallel (that is, $\nabla_y x = 0$ for all $y \in \mathfrak{g}$) if and only if $x \perp \mathfrak{g}'$ and $\mathrm{ad}_x^* = -\mathrm{ad}_x$.*

Proof. From Koszul's formula (4.1), x is a parallel vector field if and only if, for all $y, z \in \mathfrak{g}$,

$$0 = 2\langle \nabla_y x, z\rangle = \langle \mathrm{ad}_y x - \mathrm{ad}_y^* x - \mathrm{ad}_x^* y, z\rangle = \langle x, \mathrm{ad}_z y\rangle - \langle (\mathrm{ad}_x + \mathrm{ad}_x^*)z, y\rangle.$$

Since $\langle x, \mathrm{ad}_z y\rangle$ is antisymmetric and $\langle (\mathrm{ad}_x + \mathrm{ad}_x^*)z, y\rangle$ is symmetric with respect to y and z, we obtain $\langle (\mathrm{ad}_x + \mathrm{ad}_x^*)z, y\rangle = 0 = \langle x, \mathrm{ad}_z y\rangle$, which is equivalent to $x \perp \mathfrak{g}'$ and $\mathrm{ad}_x^* = -\mathrm{ad}_x$. $\square$

Proposition 4.7. *Let the metric on $\mathrm{T}^*\mathfrak{h}_3$ be given by the matrix S from Theorem 3.4. In all cases, the parallel vector fields are null. Moreover,*

(i) *If ${\mathrm{T}^*\mathfrak{h}_3}'$ is non-degenerate, then there are no parallel vector fields.*
(ii) *If ${\mathrm{T}^*\mathfrak{h}_3}'$ is degenerate of rank 2, then the only parallel vector fields are $x \in \mathbb{R}\langle e_4\rangle$.*
(iii) *If ${\mathrm{T}^*\mathfrak{h}_3}'$ is degenerate of rank 1, then the parallel vector fields are $x \in \mathbb{R}\langle e_4, e_5\rangle$.*
(iv) *If ${\mathrm{T}^*\mathfrak{h}_3}'$ is totally degenerate, all vectors of ${\mathrm{T}^*\mathfrak{h}_3}' = \mathbb{R}\langle e_4, e_5, e_6\rangle$ are parallel.*

Since a nilpotent group is simply connected, the restricted holonomy group coincides with the full holonomy group. By the Ambrose–Singer theorem, the holonomy algebra is generated by curvature operators $R(x,y)$ and their covariant derivatives of arbitrary order. We know that the holonomy algebra is a subalgebra of the isometry algebra, i.e., $so(p,q)$, where (p,q) denotes the signature of the metric.

The results are summarised in the following proposition.

Proposition 4.8. *Let the non-flat metric on* $\mathrm{T}^*\mathfrak{h}_3$ *be given by the matrix* S *from Theorem* 3.4.

- (i) *If* $\mathrm{T}^*\mathfrak{h}_3{}'$ *is non-degenerate, the corresponding metrics have a full holonomy algebra,* $\mathrm{hol}(S) = so(p,q)$, $p+q=6$.
- (ii) *If* $\mathrm{T}^*\mathfrak{h}_3{}'$ *is degenerate of rank 2, then the following cases can occur:*
  - (ii_1) *the holonomy algebra is 10-dimensional* $so(p,q)$, $p+q=5$, *if the corresponding metric is* $S_{20} = (S, M, \pm E_{20})$ *or* $S_{11} = (S, M, \pm E_{11})$, *where the matrices* S *and* M *are one of the forms* (3.23) *or* (3.25), *and* $S_{11} = (S, M, E_{11})$, *where* S *and* M *are of the form* (3.30)*;*
  - (ii_2) *the holonomy algebra is 9-dimensional isomorphic to* $sl_2(\mathbb{R}) \ltimes \mathfrak{g}_{6,54}$, *where* $\mathfrak{g}_{6,54}$ *is a 6-dimensional solvable algebra with 5-dimensional nilradical (see* [37]*) in the case of the metric* $S_{11} = (S, M, E_{11})$, *where* S *and* M *take the form* (3.31)*;*
  - (ii_3) *the holonomy algebra is 4-dimensional and isomorphic to* $\mathbb{R}^4$ *if the corresponding metric is* $S_{20} = (S, M, \pm E_{20})$ *or* $S_{11} = (S, M, \pm E_{11})$, *where the matrices* S *and* M *take the form* (3.24), $S_{11} = (S, M, E_{11})$, *where* S *and* M *take one of the forms* (3.32) *or* (3.33).
- (iii) *If* $\mathrm{T}^*\mathfrak{h}_3{}'$ *is degenerate of rank 1, the non-flat metrics* $S_{10} = (\pm E_{10}, M, \pm E_{10})$, *where* M *takes one of the forms* (3.37), *have a holonomy algebra isomorphic to* $\mathbb{R}$, *while if* M *takes the form* (3.36), *then the holonomy algebra is 5-dimensional and isomorphic to the 2-step nilpotent algebra given by the commutators* $[h_1, h_3] = [h_2, h_4] = h_5$.

Proof. The proof is case-by-case for all types of metrics. We illustrate it for the case ((iii)), i.e., for the metric $S_{10} = (E_{10}, M, E_{10})$, where M is given by (3.36), the same thing we discussed in the proof of Proposition 4.2. From there we know that the curvature operators

$$r_1 := R(e_1, e_2), \quad r_2 := R(e_1, e_3), \quad r_3 := R(e_1, e_6), \quad r_4 := R(e_2, e_3)$$

are linearly independent and generate the space $\mathbb{R}\langle \{R(e_i, e_j) \mid i, j = 1, \dots, 6\}\rangle$. Using the connection formulas (4.2) we calculate their derivatives and see that

$$r_5 := \nabla_{e_1} R(e_1, e_3) = \frac{1}{4}(e_2 \wedge e_4 - e_3 \wedge e_4)$$

is the only operator that does not belong to $\mathbb{R}\langle r_1, r_2, r_3, r_4\rangle$. Now we compute the covariant derivatives of $r_1, \dots, r_5$ and see that they all belong to $\mathbb{R}\langle r_1, r_2, r_3, r_4, r_5\rangle$. Therefore, the holonomy algebra is spanned by curvature operators and their first covariant derivatives, and

$$\mathrm{hol}(S) = \mathbb{R}\langle r_1, r_2, r_3, r_4, r_5\rangle \subset o(4,2),$$

since the signature of S is $(4,2)$ for all $m_{12} \neq 0$. Now we obtain nonzero commutators

$$[r_1, r_3] = \frac{1}{3} r_4, \quad [r_1, r_5] = -\frac{3}{8} r_4, \quad [r_2, r_3] = \frac{1}{4} r_4,$$

which, after setting

$$h_1 = r_2, \quad h_2 = -3r_1 + 4r_2, \quad h_3 = r_3, \quad h_4 = \frac{2}{9} r_5, \quad h_5 = \frac{1}{4} r_4,$$

gives the form formulated in the statement.

Let us now discuss the case ((ii$_2$)) in more detail. Similar to the previous consideration, we obtain that the holonomy algebra is given by the following nonzero commutators:

$$\begin{aligned}
&[h_1,h_2]=2h_2, && [h_2,h_5]=h_3, && [h_4,h_8]=-h_3, && [h_6,h_8]=-h_5,\\
&[h_1,h_3]=h_3, && [h_2,h_6]=h_4, && [h_4,h_9]=h_4, && [h_6,h_9]=h_6,\\
&[h_1,h_4]=h_4, && [h_3,h_7]=h_4, && [h_5,h_7]=h_6, && [h_7,h_8]=h_9,\\
&[h_1,h_5]=-h_5, && [h_3,h_9]=-h_3, && [h_5,h_9]=-h_5, && [h_7,h_9]=2h_7,\\
&[h_1,h_6]=-h_6, && && && [h_8,h_9]=-2h_8.
\end{aligned} \tag{4.3}$$

By the Levi decomposition, we know that the algebra $\mathrm{hol}(S)$ is a semi-direct product of its maximal solvable ideal and a semisimple Lie algebra. Note that $\mathbb{R}\langle h_7,h_8,h_9\rangle \cong sl_2(\mathbb{R})$ and that $\mathbb{R}\langle h_1,\dots,h_6\rangle$ is isomorphic to the 6-dimensional solvable Lie algebra denoted by $\mathfrak{g}_{6,54}$ (with $\lambda=1$, $\gamma=2$) in the classification of Mubarakzyanov [37, Table 4]. It follows that $\mathrm{hol}(S)\cong sl_2(\mathbb{R})\ltimes_\pi \mathfrak{g}_{6,54}$, where the form of $\pi: sl_2(\mathbb{R})\to\mathfrak{g}_{6,54}$ is obtained from the relations (4.3):

$$\pi(x)=\rho(x_7h_7+x_8h_8+x_9h_9)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&x_9&-x_7&0&0\\0&0&x_8&-x_9&0&0\\0&0&0&0&x_9&-x_7\\0&0&0&0&x_8&-x_9\end{pmatrix}.$$

□

We recall that a metric g is called a *pp-wave metric* if there exists a parallel null vector field v such that $R(u,w)=0$ for all $u,w\in v^\perp$.

Proposition 4.9. *The left-invariant pp-wave metrics are* $S_{20}=(S^1,M,\pm E_{20})$, $S_{11}=(S^k,M,\pm E_{11})$, $k=1,2$, *where*

$$M=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad S^1=\begin{pmatrix}0&0&0\\0&s_{22}&0\\0&0&s_{33}\end{pmatrix},\quad S^2=\begin{pmatrix}0&0&0\\0&s_{22}&\frac12|s_{22}|\\0&\frac12|s_{22}|&0\end{pmatrix},$$

with $s_{22},s_{33}\neq 0$, *and* $S_{10}=(\pm E_{10},M,\pm E_{10})$, *where* M *takes one of the forms in* (3.37).

Proof. We have already established that the basis vector e_4 is a parallel null vector field for all metrics from the proposition. The space orthogonal to e_4 is spanned by vectors $e_2,\dots,e_6$ in every case except for the metric S_{10} with

$$M=\begin{pmatrix}m_{11}&m_{12}&0\\-m_{12}&m_{11}&0\\0&0&0\end{pmatrix},$$

where it is spanned by vectors $e_3,\dots,e_6$. However, it is a straightforward calculation to show that $R(e_i,e_j)=0$, $i,j=2,\dots 6$, in all cases. Therefore, the metrics are pp-waves. □

Corollary 4.10. *All left-invariant metrics on* $\mathrm{T}^*\mathfrak{h}_3$ *with abelian holonomy algebra* $\mathbb{R}^k$, $k = 1, 4$, *are homogeneous pp-wave metrics.*

4.2. **Algebraic Ricci solitons on** $\mathrm{T}^*\mathfrak{h}_3$**.** Since the only Einstein metrics are the trivial ones, i.e., Ricci-flat, the next step is to consider a weaker condition – Ricci soliton metrics, i.e., nilsolitons. It was proven in [11] that in the pseudo-Riemannian setting there are four different kinds of nilsolitons. In this paper we focus on a special class of algebraic Ricci solitons. The non-flat left-invariant metric on a Lie group is called an *algebraic Ricci soliton* if it satisfies $\mathrm{Ric} = \gamma I + D$, where γ is an arbitrary constant, Ric is the Ricci operator and D denotes a derivation of a Lie algebra. A Ricci soliton is said to be *shrinking*, *steady* or *expanding* depending on whether $\gamma > 0$, $\gamma = 0$ or $\gamma < 0$, respectively.

If $D = 0$ and $\gamma \neq 0$, the solutions are Einstein metrics that do not exist on $\mathrm{T}^*\mathfrak{h}_3$. If $D = 0$ and $\gamma = 0$, the solutions are the Ricci-flat metrics described in Proposition 4.2. Hence, in the next proposition we describe the solitons for $D \neq 0$.

Proposition 4.11. *Algebraic nilsolitons on* $\mathrm{T}^*\mathfrak{h}_3$ *satisfying* $\mathrm{Ric} \neq 0$ *are:*

(i) *expanding* ($\gamma = -\frac{5}{2\lambda^2}$), *in case of the metric* $S_{30} = (S, 0, E_{30})$, *with* $S = \mathrm{diag}(\lambda, \lambda, \lambda)$*;*

(ii) *shrinking* ($\gamma = \frac{5}{2\lambda^2}$), *in case of the metric* $S_{21} = (S, 0, E_{21})$, *with* $S = \mathrm{diag}(\lambda, \lambda, -\lambda)$*;*

(iii) *steady* ($\gamma = 0$), *in case of the metrics* $S_{20} = (S^1, M, \pm E_{20})$ *or* $S_{11} = (S^k, M, E_{11})$, *where the matrices* S^k ($k = 1, 2, 3$) *and* M *take the forms:*

$$S^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & s_{22} & 0 \\ 0 & 0 & s_{33} \end{pmatrix}, \; s_{22} + s_{33} \neq \pm 1, \quad S^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & s_{22} & \frac{1}{2}|s_{22}| \\ 0 & \frac{1}{2}|s_{22}| & 0 \end{pmatrix}, \; s_{22} \neq 1,$$

$$S^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}|s_{33}| \\ 0 & \frac{1}{2}|s_{33}| & s_{33} \end{pmatrix}, \; s_{33} \neq -1, \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. The proof requires an analysis for each metric from the classification. Let us prove the positive result for case (i); the other cases can be analyzed in a similar way.

For the metric $S_{30} = (S, 0, E_{30})$, with $S = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ the Ricci operator is diagonal, hence $D = \mathrm{Ric} - \gamma I$ also has the diagonal form

$$D = \mathrm{diag}\Big(-\frac{\lambda_2 + \lambda_3}{2\lambda_1\lambda_2\lambda_3} - \gamma, -\frac{\lambda_1 + \lambda_3}{2\lambda_1\lambda_2\lambda_3} - \gamma, -\frac{\lambda_1 + \lambda_2}{2\lambda_1\lambda_2\lambda_3} - \gamma, \\ \frac{1}{2\lambda_2\lambda_3} - \gamma, \frac{1}{2\lambda_1\lambda_3} - \gamma, \frac{1}{2\lambda_1\lambda_2} - \gamma \Big).$$

Since D is derivation, it must satisfy the condition $D[x, y] = [x, Dy] + [Dx, y]$ for all $x, y \in \mathrm{T}^*\mathfrak{h}_3$. By solving this system of equations, we obtain $\lambda_1 = \lambda_2 = \lambda_3$ and $\gamma = -\frac{5}{2\lambda_1^2}$.

We have shown that not all metrics $S_{30} = (S, 0, E_{30})$, where S takes the form (3.6), admit nilsolitons. They exist only in the positive definite and neutral signature case, i.e., only when $S = \lambda E_{30}$, $\lambda \neq 0$. ∎

Remark 4.12. The classification in Proposition 4.11 is up to a sign. Namely, if the metric g is a (shrinking/expanding) algebraic Ricci soliton then the metric $-g$ is an (expanding/shrinking) algebraic Ricci soliton.

It was proved in [32] that the Riemannian left homogenous Ricci soliton metric (equivalently, the algebraic Ricci soliton) on a nilpotent Lie group is unique up to isometry and scaling. Proposition 4.11 confirms that result for the metric Lie algebra $\mathrm{T}^*\mathfrak{h}_3$. However, it also shows that the result does not hold in the pseudo-Riemannian setting, since some of the Ricci soliton metrics ((i))−−((iii)) have the same signature but are not homothetic.

4.3. **Pseudo-Kähler metrics on** $\mathrm{T}^*\mathfrak{h}_3$**.** Let us now classify the pseudo-Kähler metrics on $\mathrm{T}^*\mathfrak{h}_3$.

An *almost complex structure* on a Lie algebra $\mathfrak{g}$ is an endomorphism $J:\mathfrak{g}\to\mathfrak{g}$ satisfying $J^2=-id$. If J is integrable, in the sense that the Nijenhuis tensor

$$N_J(x,y)=[x,y]-[Jx,Jy]+J[Jx,y]+J[x,Jy]$$

of J vanishes, i.e., if it satisfies the condition $N_J(x,y)=0$ for all $x,y\in\mathfrak{g}$, then it is called a *complex structure* on $\mathfrak{g}$.

The centre of $\mathrm{T}^*\mathfrak{h}_3$ is 3-dimensional, so it cannot admit an abelian complex structure, i.e., a complex structure satisfying $[x,y]=[Jx,Jy]$, which means that the centre of the algebra must be J-invariant (consequently, even-dimensional). However, every complex structure on $\mathrm{T}^*\mathfrak{h}_3$ is 3-step nilpotent (see [9, Proposition 4.11 i))] or [12]) and they are all equivalent to the following structure (see [12, 40, 34]):

$$Je_1=e_2,\quad Je_3=-e_6,\quad Je_4=e_5. \tag{4.4}$$

A complex structure J is called *Hermitian* if it preserves the metric: $\langle Jx,Jy\rangle=\langle x,y\rangle$ for all $x,y\in\mathfrak{g}$.

Example 4.13. Let us fix the basis in which the complex structure J has the form (4.4). One can check that J is Hermitian if the corresponding metric is the positive definite metric $S_{30}=(S,0,E_{30})$, with $S=\mathrm{diag}(\lambda,\lambda,1)$, $\lambda>0$.

A *symplectic structure* on a Lie algebra $\mathfrak{g}$ is a closed 2-form $\Omega\in\bigwedge^2\mathfrak{g}^*$ of maximal rank. A pair (J,Ω), where J is complex and Ω is symplectic, is called a *pseudo-Kähler structure* if $\Omega(Jx,Jy)=\Omega(x,y)$ holds for all $x,y\in\mathfrak{g}$.

We already know from [12, Proposition 3.9 i)] that the algebra $\mathrm{T}^*\mathfrak{h}_3$ has a complex structure admitting a 5-dimensional set of compatible symplectic forms. Denote by $\{e^1,\dots,e^6\}$ the dual basis of $\{e_1,\dots,e_6\}$. The Maurer–Cartan equations on $\mathrm{T}^*\mathfrak{h}_3$ are given by

$$de^1=de^2=de^3=0,\quad de^4=e^2\wedge e^3,\quad de^5=-e^1\wedge e^3,\quad de^6=e^1\wedge e^2.$$

The symplectic structure $\Omega=\sum_{i<j}a_{ij}e^i\wedge e^j$, $a_{ij}\in\mathbb{R}$, has to be closed ($d\Omega=0$) and compatible with the complex structure J given by (4.4). Hence, it takes the form

$$\begin{aligned}\Omega=a_{12}e^1\wedge e^2+a_{13}(e^1\wedge e^3-e^2\wedge e^6)+a_{14}(e^1\wedge e^4+e^2\wedge e^5-2e^3\wedge e^6)\\ +a_{15}(e^1\wedge e^5-e^2\wedge e^4)+a_{16}(e^1\wedge e^6+e^2\wedge e^3).\end{aligned}\tag{4.5}$$

The pseudo-Kähler pair (J,Ω) generates a Hermitian structure on a Lie algebra $\mathfrak{g}$ by defining a metric $\langle\cdot,\cdot\rangle$ as

$$\langle x,y\rangle=\Omega(Jx,y) \tag{4.6}$$

for all $x,y\in\mathfrak{g}$. For this Hermitian structure, the condition of parallelism of J with respect to the Levi-Civita connection is satisfied for $\langle\cdot,\cdot\rangle$. In this case, a pair $(J,\langle\cdot,\cdot\rangle)$ is called a *pseudo-Kähler metric* on $\mathfrak{g}$.

From [12, Corollary 3.2] we know that the algebra $\mathrm{T}^*\mathfrak{h}_3$ has compatible pairs (J,Ω) since it admits both symplectic and nilpotent complex structures. In [3, Theorem A] it was proved that the metric associated to any compatible pair (J,Ω) cannot be positive definite since $\mathrm{T}^*\mathfrak{h}_3$ is not abelian. Therefore, the metric from Example 4.13 is not pseudo-Kähler. However, it follows from [20] that any pseudo-Kähler metric on $\mathrm{T}^*\mathfrak{h}_3$ is Ricci-flat. Here we give its classification and explicit form.

Proposition 4.14. *The Lie algebra* $\mathrm{T}^*\mathfrak{h}_3$ *admits Ricci-flat pseudo-Kähler metrics which are not flat. Any pseudo-Kähler metric on* $\mathrm{T}^*\mathfrak{h}_3$ *is equivalent to* $S_{10}=(E_{10},M,E_{10})$*, where* M *has the form of the second matrix in* (3.37)*.*

Proof. We fix the basis, where the complex structure J is given by (4.4) and the symplectic form Ω is given by (4.5). The compatibility condition (4.6) for (J,Ω) gives us that the restriction of the metric on ${\mathrm{T}^*\mathfrak{h}_3}'$ must be degenerate of rank 1. One computes that the metric itself is represented by a symmetric 5-parameter matrix:

$$S=\begin{pmatrix}-a_{12}&0&a_{16}&-a_{15}&a_{14}&-a_{13}\\0&-a_{12}&-a_{13}&-a_{14}&-a_{15}&-a_{16}\\a_{16}&-a_{13}&-2a_{14}&0&0&0\\-a_{15}&-a_{14}&0&0&0&0\\a_{14}&-a_{15}&0&0&0&0\\-a_{13}&-a_{16}&0&0&0&-2a_{14}\end{pmatrix},\quad a_{14}\neq 0.$$

By examining the curvature properties, we conclude that this metric is locally symmetric and Ricci-flat, but not flat. From Proposition 4.2 (iii) we know that this metric must be equivalent to a metric from one of the families $S_{10}=(\pm E_{10},M,\pm E_{10})$, where M takes one of the forms in (3.37). We can do even more. We can find a particular form of the automorphism matrix F such that the matrix F^TSF has the following form:

$$S_{10}=(E_{10},M,E_{10}),\quad M=\begin{pmatrix}\lambda&\frac12&0\\-\frac12&\lambda&0\\0&0&0\end{pmatrix},\quad\text{where }\lambda=-\frac{a_{15}}{2a_{14}}.$$

In this case, the complex structure is $J'=FJF^{-1}$, while the explicit formula for the corresponding symplectic forms can be retrieved from (4.6). □

Remark 4.15. In [41] the author considered three symplectic structures which are special cases of the symplectic structure given by (4.5). For each of these structures, a corresponding metric was determined. However, the author has not noticed that all of these metrics are equivalent.

Remark 4.16. The previous proposition also shows that the differences between the metrics in the classification (Theorem 3.4) are very geometrical, and not just algebraic.

4.4. **Geodesically equivalent metrics.** We say that a metric $\overline{\langle\cdot,\cdot\rangle}$ on a connected manifold M^n is *geodesically equivalent* to $\langle\cdot,\cdot\rangle$ if every geodesic of $\langle\cdot,\cdot\rangle$ is a reparameterized geodesic of $\overline{\langle\cdot,\cdot\rangle}$. We say that they are *affinely equivalent* if their Levi-Civita connections coincide. We call a metric $\overline{\langle\cdot,\cdot\rangle}$ *geodesically rigid* if every metric $\overline{\langle\cdot,\cdot\rangle}$ that is geodesically equivalent to $\langle\cdot,\cdot\rangle$ is proportional to $\langle\cdot,\cdot\rangle$ (by the result of H. Weyl, the proportionality coefficient is a constant). In the Riemannian case, if the metric is not decomposable (not a product of two metrics), it is geodesically rigid. Therefore, it makes sense to look for geodesically equivalent metrics only in the pseudo-Riemannian case.

As proved in [6], two geodesically equivalent invariant metrics on a homogeneous space are affinely equivalent. In particular, this is true for left-invariant metrics on Lie groups. If an invariant metric does not admit a nonproportional affinely equivalent invariant metric, we call it *invariantly rigid.*

The nonproportional, affinely equivalent metrics $\overline{\langle\cdot,\cdot\rangle}$ and $\langle\cdot,\cdot\rangle$ are both parallel with respect to the mutual Levi-Civita connection and hence their difference is a parallel symmetric tensor. Such tensors are closely related to the description of holonomy groups [22]. Metrics admitting such tensors are fully described on general pseudo-Riemannian manifolds in [30] as either Riemannian extensions or using certain complex metrics. In Proposition 4.17 below we show that such (not invariantly rigid) left-invariant metrics on $\mathrm{T}^*\mathfrak{h}_3$ are Riemannian extensions. Moreover, all such parallel tensors on $\mathrm{T}^*\mathfrak{h}_3$ are "made" of parallel vector fields in the following way.

Suppose $v_1,\dots,v_r$ are parallel vector fields with respect to the metric $\langle\cdot,\cdot\rangle$, and $v_1^*,\dots,v_r^*$ are 1-forms metrically dual to these vectors. It is easy to verify that for arbitrary constants $C_{mn}=C_{nm}$, $n,m=1,\dots r$, the metric

$$\overline{\langle\cdot,\cdot\rangle}=\langle\cdot,\cdot\rangle+C_{nm}v_n^*\otimes v_m^* \tag{4.7}$$

is affinely equivalent to $\langle\cdot,\cdot\rangle$, or equivalently, the symmetric tensor $C_{nm}v_n^*\otimes v_m^*$ is parallel.

In [30] it was shown that such a metric $\langle\cdot,\cdot\rangle$ is a Euclidean extension of Riemannian space.

To classify non-invariantly rigid metrics on $\mathrm{T}^*\mathfrak{h}_3$, we follow the algorithm proposed in [6]. To simplify the notation, the matrix S is used to denote the metric $\langle\cdot,\cdot\rangle$.

Proposition 4.17. *If* $\mathrm{T}^*\mathfrak{h}_3{}'$ *is non-degenerate, the corresponding left-invariant metrics are geodesically rigid. If* $\mathrm{T}^*\mathfrak{h}_3{}'$ *is degenerate, then non-trivial affinely equivalent metrics exist and these are exactly the metrics obtained with parallel null vector fields by* (4.7).

Proof. It is clear that if the original metric $\langle\cdot,\cdot\rangle$ has parallel vector fields, the metric (4.7) is affinely equivalent to it.

To prove the converse we perform a case-by-case analysis for each metric from our classification.

Let S be a symmetric matrix representing a left-invariant metric $\langle \cdot, \cdot \rangle$ in the basis $\{e_1, \dots, e_6\}$ and ω its Levi-Civita connection matrix of 1-forms. As proved in [6, Proposition 3.1], the left-invariant metric $\bar{S}$ is geodesically equivalent to S if and only if its matrix $\bar{S}$ in the basis $\{e_1, \dots, e_6\}$ belongs to the subspace

$$\mathrm{aff}(S) := \{\bar{S} \mid \bar{S}\omega + \omega^{\mathrm{T}}\bar{S} = 0\}.$$

Since ω is a matrix of 1-forms, the given relations are six matrix equations.

If S is such that $\mathrm{T}^*\mathfrak{h}_3{}'$ is non-degenerate, we can directly check that $\mathrm{aff}(S)$ is 1-dimensional, that is, S is geodesically rigid. This also follows (without computation) from the fact that such metrics have a full holonomy algebra (Proposition 4.8). Indeed, if a metric is not geometrically rigid, it cannot have full holonomy (see [6]).

We illustrate the proof for the metric $S = S_{10} = (E_{10}, M, E_{10})$ with $\mathrm{T}^*\mathfrak{h}_3{}'$ of rank 1, where M is given by (3.36). The connection matrix ω can be computed from the relations (4.2) and we obtain that $\mathrm{aff}(S)$ is a space of matrices

$$\lambda S + \begin{pmatrix} c^{11} & c^{12} & c^{12} & 0 & 0 & 0 \\ c^{12} & c^{22} & c^{22} & 0 & 0 & 0 \\ c^{12} & c^{22} & c^{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda, c^{11}, c^{12}, c^{22} \in \mathbb{R}. \tag{4.8}$$

Indeed, $\mathrm{aff}(S)$ is the set of all parallel symmetric (left-invariant) tensors for the metric S and we see that it is 4-dimensional. Now we will prove that it consists of parallel vectors using the formula (4.7). Parallel vectors for the metric S are $v_1 = e_4$ and $v_2 = \frac{1}{m_{12}} e_5$ (Proposition 4.7). Their metric dual forms are

$$v_1^* = e^2 + e^3, \quad v_2^* = e^1,$$

where $(e^1, \dots, e^6)$ is the basis of one forms dual to the vectors $(e_1, \dots, e_6)$ in the sense that $e^i(e_j) = \delta^i_j$. Now we see that

$$c^{11}(v_1^* \otimes v_1^*) + c^{12}(v_1^* \otimes v_2^* + v_2^* \otimes v_1^*) + c^{22}(v_2^* \otimes v_2^*)$$

are exactly parallel symmetric tensors in (4.8) which are not proportional to S. They are obtained from parallel vector fields with (4.7). $\square$

Remark 4.18. One can check that non-proportional affinely equivalent metrics are connected by an automorphism of the group. This means that the corresponding Lie groups endowed with these metrics have a family of automorphisms which are not isometries but preserve geodesics.

Remark 4.19. Note that if the metric is Ricci-parallel, i.e., $\nabla\rho = 0$, then $\rho \in \mathrm{aff}(S)$. Obviously, the converse is not true: not all non-invariantly rigid metrics are Ricci-parallel.

4.5. **Totally geodesic subalgebras of** $\mathrm{T}^*\mathfrak{h}_3$**.** A subalgebra $\mathfrak{h}$ of a metric algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is called *totally geodesic* if $\nabla_y z \in \mathfrak{h}$ for all $y, z \in \mathfrak{h}$. If $\mathfrak{h}^\perp$ denotes the orthogonal complement of $\mathfrak{h}$ in $\mathrm{T}^*\mathfrak{h}_3$, then as a direct consequence of Koszul's formula we obtain that $\mathfrak{h}$ is a totally geodesic subalgebra of $\mathrm{T}^*\mathfrak{h}_3$ if and only if

$$\langle [x,y], z\rangle + \langle [x,z], y\rangle = 0 \quad \text{for all } x \in \mathfrak{h}^\perp,\ y, z \in \mathfrak{h}.$$

We say that $\mathfrak{h}^\perp$ is $\mathfrak{h}$-*invariant* if $[x,y] \in \mathfrak{h}^\perp$ for all $x \in \mathfrak{h}^\perp$, $y \in \mathfrak{h}$. A nonzero element $y \in \mathrm{T}^*\mathfrak{h}_3$ is called *geodesic* if it spans a totally geodesic subalgebra $\mathfrak{h}$ and can be characterized by the condition that $\mathfrak{h}^\perp$ is $\mathfrak{h}$-invariant. For nilpotent Lie groups there is an inner product for which a nonzero element y is geodesic (see, e.g., [7]).

Eberlein [18] considered totally geodesic subalgebras of nonsingular 2-step nilpotent Lie algebras, implying that for any noncentral element $x \in \mathfrak{g}$ the adjoint map $\mathrm{ad}(x)$ is surjective on $\mathcal{Z}(\mathfrak{g})$. Later, in [7] the non-singularity condition was replaced by a weaker version: the adjoint map had to be surjective on the derived algebra $[\mathfrak{g},\mathfrak{g}]$. Finally, in [15] the authors gave criteria for a subalgebra to be totally geodesic without the non-singularity condition.

Proposition 4.20. *For every subalgebra* $\mathfrak{h}$ *of* $\mathrm{T}^*\mathfrak{h}_3$ *there is a metric which makes it totally geodesic.*

Proof. Let $\mathfrak{h}$ be an n-dimensional subalgebra of the metric algebra $(\mathrm{T}^*\mathfrak{h}_3, \langle\cdot,\cdot\rangle)$, $\mathrm{T}^*\mathfrak{h}_3 = \nu \oplus \xi$, where ξ denotes the centre of $\mathrm{T}^*\mathfrak{h}_3$ and ν its complement. Then $\mathfrak{h}$ is either abelian ($\mathbb{R}$, $\mathbb{R}^2 \cong \mathbb{R}\langle x, y\rangle$ or $\mathbb{R}^3 \cong \mathbb{R}\langle x, y, z\rangle$, with $x \in \mathrm{T}^*\mathfrak{h}_3$, $y, z \in \xi$) or 2-step nilpotent (isomorphic to one of the algebras $\mathfrak{h}_3$, $\mathfrak{h}_3 \oplus \mathbb{R}$ or $\mathfrak{h}_3 \oplus \mathbb{R}^2$).

Trivially, every subspace of ξ and every abelian subspace of ν are totally geodesic subalgebras of a 2-step nilpotent metric algebra. For other abelian algebras, it suffices to find a metric that makes $\mathfrak{h}$ flat, which implies that $\nabla_y z = 0$ for all $y, z \in \mathfrak{h}$. It is not hard to verify that the metric $S_{00} = (0, M, 0)$, with $M = \mathrm{diag}(1, \lambda_2, \lambda_3)$, is exactly the one required.

The nilpotent case is very similar. First, note that we can always change the basis (by the action of automorphisms (2.3)) so that the corresponding subalgebra $\mathfrak{h}$ is isometric to one of the following: $\mathfrak{h}_3 \cong \mathbb{R}\langle e_1, e_2, e_6\rangle$, $\mathfrak{h}_3 \oplus \mathbb{R} \cong \mathbb{R}\langle e_1, e_2, e_4, e_6\rangle$, $\mathfrak{h}_3 \oplus \mathbb{R}^2 \cong \mathbb{R}\langle e_1, e_2, e_4, e_5, e_6\rangle$. Consider the metric $S_{20} = (S, M, E_{20})$, where S and M are given by (3.25). By a simple calculation we obtain that all three subalgebras are totally geodesic with respect to this metric. □

A subspace $\mathfrak{h} \subseteq (\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is called *isotropic* if $\langle x, y\rangle = 0$ for all $x, y \in \mathfrak{h}$, i.e., $\mathfrak{h} \subset \mathfrak{h}^\perp$. Moreover, $\mathfrak{h}$ is called *totally isotropic* if $\mathfrak{h} = \mathfrak{h}^\perp$.

Example 4.21. In [9, Example 5.2] it was mentioned that on $\mathrm{T}^*\mathfrak{h}_3$ with the canonical metric from Example 4.5 both spaces $\mathfrak{h}_3$ and $\mathfrak{h}_3^*$ are totally isotropic. Here we can see that both spaces are totally isotropic if the metric corresponds to the degenerate centre $\mathrm{T}^*\mathfrak{h}_3{}'$ of rank 0. For the same four families of metrics, the totally geodesic subalgebra $\mathfrak{h}_3 \cong \mathbb{R}\langle e_2, e_3, e_4\rangle$ is also totally isotropic.

Finally, let us consider the decomposition of the totally geodesic subalgebra $\mathfrak{h}$. If the nilpotent metric algebra is non-singular and the corresponding metric is

Riemannian, Eberlein showed that $\mathfrak{h} = (\mathfrak{h} \cap \nu) \oplus (\mathfrak{h} \cap \xi)$ (see [18, Lemma 2.2]). If the non-singularity condition is dropped, according to [15, Theorem 4.10], $\mathfrak{h}$ is either abelian and flat, or it is the direct sum of nonzero subspaces $\mathfrak{h}_+^z = \{x \in \mathfrak{h} \mid R_z(x) = \lambda x,\ \lambda > 0\}$ and $\mathfrak{h}_0^z = \{x \in \mathfrak{h} \mid R_z(x) = 0\}$ for every $z \in \mathfrak{h} \cap \xi$. Here R_z denotes the Jacobi operator $R_z(x) = R(x, z)z$. The following example shows what happens if we consider the pseudo-Riemannian case.

Example 4.22. First, observe the algebra $\mathfrak{h} = \mathfrak{h}_3 \oplus \mathbb{R}^2 \cong \mathbb{R}\langle e_1, e_2, e_4, e_5, e_6\rangle$ and the metric $S_{11} = (E_{10}, M, E_{10})$, where $M = \operatorname{diag}(\mu, \mu, 0)$. It is easy to check that $\mathfrak{h}$ is a totally geodesic subalgebra of a singular metric algebra $(\mathrm{T}^*\mathfrak{h}_3, S_{11})$ and for every element $z = \alpha e_4 + \beta e_5 + \gamma e_6 \in \xi$, $\mathfrak{h}_+^z = \{0\}$ and $\mathfrak{h}_0^z = \mathfrak{h}$.

Now, consider the metric $S_{20} = (S, M, E_{20})$ from the proof of Proposition 4.20. Fix z from the centre ξ of $\mathrm{T}^*\mathfrak{h}_3$. If $z \in \mathbb{R}\langle e_4, e_5\rangle$, then $\mathfrak{h}_+^z = \mathbb{R}\langle e_1, e_2\rangle = \mathfrak{h} \cap \nu$ and $\mathfrak{h}_0^z = \mathbb{R}\langle e_4, e_5, e_6\rangle = \mathfrak{h} \cap \xi$. On the other hand, if $z = e_6$, then $\mathfrak{h}_+^z = \{0\}$, $\mathfrak{h}_0^z = \xi$ and we have the new subspace $\mathfrak{h}_-^z = \{x \in \mathfrak{h} \mid R_z(x) = \lambda x,\ \lambda < 0\} = \mathfrak{h} \cap \nu$. Hence, in both cases we obtained Eberlein's decomposition without the non-singularity condition. However, this is not true for every $z \in \mathfrak{h} \cap \xi$. For example, $z = e_5 + e_6$ yields subspaces $\mathfrak{h}_+^z = \{0\}$, $\mathfrak{h}_0^z = \xi$ and $\mathfrak{h}_-^z = \mathbb{R}\langle e_2, e_4 - \frac{1}{s_{12}} e_1\rangle \neq \mathfrak{h} \cap \nu$.

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