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REGULAR AUTOMORPHISMS AND CALOGERO–MOSER FAMILIES

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ABSTRACT. We study the subvariety of fixed points of an automorphism of a Calogero–Moser space induced by a regular element of finite order of the normalizer of the associated complex reflection group W. We determine some of (and conjecturally all) the $\mathbb{C}^\times$-fixed points of its unique irreducible component of maximal dimension in terms of the character table of W. This is inspired by the mysterious relations between the geometry of Calogero–Moser spaces and unipotent representations of finite reductive groups, which is the theme of another paper [Pure Appl. Math. Q. 21 no. 1 (2025), 131–200].

INTRODUCTION

If $\mathbf{G}$ is a split reductive group over a finite field $\mathbb{F}_q$ with q elements with Weyl group W, Deligne and Lusztig [12] defined a particular class of irreducible characters of the finite group $G = \mathbf{G}(\mathbb{F}_q)$, called the *unipotent characters* of G. To W, one can also associate a Calogero–Moser space $\mathcal{Z}$ *at equal parameters*, which is a complex irreducible normal affine Poisson variety endowed with a $\mathbb{C}^\times$-action [13]. The main theme of a previous paper of the author [4] is the observation that many aspects of the combinatorics of unipotent characters of G have a conjectural analogue in the geometry of $\mathcal{Z}$, thanks to the Poisson structure and the $\mathbb{C}^\times$-action. Here are two examples:

- Unipotent characters have been partitioned by Lusztig [20] into *families* and it has been conjectured by Gordon–Martino [17] that these families are in bijection with $\mathbb{C}^\times$-fixed points of $\mathcal{Z}$. Note that this conjecture has been proved in all cases except types E_6, E_7 and E_8 (see [17, 1, 7]).
- For d a natural number, Broué–Malle–Michel [9] defined a partition of unipotent characters into *d-Harish-Chandra series* (generalizing the classical partition into *Harish-Chandra series*, which corresponds to the case $d = 1$). This partition is conjecturally related to the stratification of $\mathcal{Z}^{\boldsymbol{\mu}_d}$ by symplectic leaves (here, $\boldsymbol{\mu}_d$ denotes the group of complex d-th roots of

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unity); the reader may find more details in [4, §12.C]. See [4, Part IV] for a list of cases where this conjecture is proved.

In the second point, whenever d is a *regular number* in the sense of Springer [21], it has been observed by Broué–Malle–Michel [10, Rem. 4.21] that the families of unipotent characters which meet the principal d-Harish-Chandra series are characterized by a property involving character values of W (again, more details may be found in [4, Ex. 12.9]). If one believes in the analogy between unipotent characters and geometry of $\mathcal{Z}$, this suggests [4, Conj. 7.5] a conjectural characterization of $\mathbb{C}^\times$-fixed points meeting the unique irreducible component of $\mathcal{Z}^{\boldsymbol{\mu}_d}$ of maximal dimension in terms of the character table of W. The proof of one direction of this conjecture is the theme of the present paper.

Note that the conjecture [4, Conj. 7.5] involves only the Calogero–Moser space and can be studied without any reference to unipotent characters. Moreover, since Calogero–Moser spaces are defined for any finite complex reflection group (and not only for Weyl groups) and for a bigger family of parameters, this conjecture is somewhat more general than what has been explained above in this introduction, which can be seen as a motivation for the results obtained here. Therefore, from now on, we will work in this more general context of complex reflection groups. Let V be a finite-dimensional complex vector space and let W be a finite subgroup of $\mathbf{GL}_{\mathbb{C}}(V)$ generated by reflections (i.e., automorphisms of V whose fixed points space is an hyperplane). To some parameter k, Etingof and Ginzburg [13] associated a normal irreducible affine complex variety $\mathcal{Z}_k = \mathcal{Z}_k(V, W)$ called a (generalized) *Calogero–Moser space*. If τ is an element of finite order of the normalizer of W in $\mathbf{GL}_{\mathbb{C}}(V)$ stabilizing the parameter k, it induces an automorphism of $\mathcal{Z}_k$.

We denote by V_{reg} the open subset of V on which W acts freely, and we assume that $V_{\mathrm{reg}}^\tau \neq \varnothing$ (then τ is called *regular*). In this case, there exists a unique irreducible component $(\mathcal{Z}_k^\tau)_{\max}$ of $\mathcal{Z}_k^\tau$ of maximal dimension (as it will be explained in Section 2). Recall that $\mathcal{Z}_k$ is endowed with a $\mathbb{C}^\times$-action and that we have a surjective map $\mathrm{Irr}(W) \to \mathcal{Z}_k^{\mathbb{C}^\times}$ defined by Gordon [16] (induced by the action of the center of a rational Cherednik algebra on *baby Verma modules*) whose fibers are called the *Calogero–Moser k-families* of W. Here, $\mathrm{Irr}(W)$ is the set of irreducible characters of W. If $p \in \mathcal{Z}_k^{\mathbb{C}^\times}$, we denote by $\mathfrak{F}_p$ its associated Calogero–Moser k-family. It is a natural question to wonder which $\mathbb{C}^\times$-fixed points of $\mathcal{Z}_k^\tau$ belong to $(\mathcal{Z}_k^\tau)_{\max}$. The aim of this note is to provide a partial answer in terms of the character table of W:

Theorem A. *Assume that $V_{\mathrm{reg}}^\tau \neq \varnothing$. Let $p \in \mathcal{Z}_k^{\mathbb{C}^\times}$ be such that $\tau(p) = p$. If there exists χ in $\mathfrak{F}_p^\tau$ such that $\tilde{\chi}(\tau) \neq 0$, then $p \in (\mathcal{Z}_k^\tau)_{\max}$.*

In this statement, if χ is a τ-stable irreducible character of W, we denote by $\tilde{\chi}$ an extension of χ to the finite group $W\langle\tau\rangle$ (see [18, Cor. 11.22] for the existence of $\tilde{\chi}$): note that $|\tilde{\chi}(\tau)|^2$ does not depend on the choice of $\tilde{\chi}$ (see [18, Cor. 6.17]). Our proof of Theorem A makes an extensive use of the Gaudin operators introduced in [6, §8.3.B]: whenever $\tilde{\chi}(\tau) \neq 0$, the decomposition of a representation affording χ with respect to generalized eigenspaces of the Gaudin operators allows us to construct a τ-fixed point p' in $(\mathcal{Z}_k^\tau)_{\max}$ such that $p = \lim_{\xi \to 0} \xi \cdot p'$.

We conjecture that the converse of Theorem A holds [4, Conj. 7.5]:

Conjecture B. *Assume that $V_{\mathrm{reg}}^\tau \neq \varnothing$. Let $p \in \mathcal{Z}_k^{\mathbb{C}^\times}$ be such that $\tau(p) = p$ and $p \in (\mathcal{Z}_k^\tau)_{\max}$. Then there exists χ in $\mathfrak{F}_p^\tau$ such that $\tilde{\chi}(\tau) \neq 0$.*

General notation. Throughout this paper, we will abbreviate $\otimes_{\mathbb{C}}$ as $\otimes$, and all varieties will be algebraic, complex, quasi-projective and reduced. If $\mathcal{X}$ is an affine variety, we denote by $\mathbb{C}[\mathcal{X}]$ its coordinate ring.

If X is a subset of a vector space V (or of its dual V^*), and if Γ is a subgroup of $\mathbf{GL}_{\mathbb{C}}(V)$, we denote by Γ_X the pointwise stabilizer of X. If moreover Γ is finite, we will identify $(V^\Gamma)^*$ and $(V^*)^\Gamma$.

1. Calogero–Moser spaces and families

Hypothesis and notation. We fix in this paper a finite-dimensional complex vector space V and a finite subgroup W of $\mathbf{GL}_{\mathbb{C}}(V)$. We set

$$\mathrm{Ref}(W) = \{s \in W \mid \mathrm{codim}_{\mathbb{C}} V^s = 1\}$$

and we assume throughout this paper that

$$W = \langle \mathrm{Ref}(W) \rangle,$$

i.e., that W is a complex reflection group.

1.A. **About W.** We set $\varepsilon : W \to \mathbb{C}^\times$, $w \mapsto \det(w)$. We identify $\mathbb{C}[V]$ (resp. $\mathbb{C}[V^*]$) with the symmetric algebra $\mathrm{S}(V^*)$ (resp. $\mathrm{S}(V)$).

We denote by $\mathcal{A}$ the set of *reflecting hyperplanes* of W, namely

$$\mathcal{A} = \{V^s \mid s \in \mathrm{Ref}(W)\}.$$

If $H \in \mathcal{A}$, we denote by α_H an element of V^* such that $H = \mathrm{Ker}(\alpha_H)$, and by $\alpha_H^\vee$ an element of V such that $V = H \oplus \mathbb{C}\alpha_H^\vee$ and the line $\mathbb{C}\alpha_H^\vee$ is W_H-stable. We set $e_H = |W_H|$. Note that W_H is cyclic of order e_H and that $\mathrm{Irr}(W_H) = \{\mathrm{Res}_{W_H}^W \varepsilon^j \mid 0 \leq j \leq e - 1\}$. We denote by $\varepsilon_{H,j}$ the (central) primitive idempotent of $\mathbb{C}W_H$ associated with the character $\mathrm{Res}_{W_H}^W \varepsilon^{-j}$, namely

$$\varepsilon_{H,j} = \frac{1}{e_H} \sum_{w \in W_H} \varepsilon(w)^j w \in \mathbb{C}W_H.$$

If Ω is a W-orbit of reflecting hyperplanes, we write e_Ω for the common value of all the e_H, where $H \in \Omega$. We denote by $\aleph$ the set of pairs (Ω, j) where $\Omega \in \mathcal{A}$ and $0 \leq j \leq e_\Omega - 1$. The vector space of families of complex numbers indexed by $\aleph$ will be denoted by $\mathbb{C}^\aleph$; elements of $\mathbb{C}^\aleph$ will be called *parameters*. If $k = (k_{\Omega,j})_{(\Omega,j)\in\aleph} \in \mathbb{C}^\aleph$, we define $k_{H,j}$ for all $H \in \Omega$ and $j \in \mathbb{Z}$ by $k_{H,j} = k_{\Omega,j_0}$, where Ω is the W-orbit of H and j_0 is the unique element of $\{0, 1, \ldots, e_H - 1\}$ such that $j \equiv j_0 \mod e_H$.

We denote by V_{reg} the set of elements v of V such that $W_v = 1$. It is an open subset of V, and recall from the Steinberg–Serre theorem [8, Thm. 4.7] that

$$V_{\mathrm{reg}} = V \setminus \bigcup_{H \in \mathcal{A}} H.$$

In particular, V_{reg} is a principal open affine subset of V and

$$\mathbb{C}[V_{\mathrm{reg}}] = \mathbb{C}[V][1/\prod_{H \in \mathcal{A}} \alpha_H].$$

1.B. **Rational Cherednik algebra at $t = 0$.** Let $k \in \mathbb{C}^{\aleph}$. We define the *rational Cherednik algebra* $\mathbf{H}_k$ (*at* $t = 0$) to be the quotient of the algebra $\mathrm{T}(V \oplus V^*) \rtimes W$ (the semi-direct product of the tensor algebra $\mathrm{T}(V \oplus V^*)$ with the group W) by the relations

$$\begin{cases} [x, x'] = [y, y'] = 0, \\ [y, x] = \displaystyle\sum_{H \in \mathcal{A}} \sum_{j=0}^{e_H - 1} e_H (k_{H,j} - k_{H,j+1}) \frac{\langle y, \alpha_H \rangle \cdot \langle \alpha_H^\vee, x \rangle}{\langle \alpha_H^\vee, \alpha_H \rangle} \varepsilon_{H,j} \end{cases} \tag{1.1}$$

for all x, $x' \in V^*$, y, $y' \in V$. Here, $\langle\, ,\, \rangle : V \times V^* \to \mathbb{C}$ is the standard pairing. The first commutation relations imply that we have morphisms of algebras $\mathbb{C}[V] \to \mathbf{H}_k$ and $\mathbb{C}[V^*] \to \mathbf{H}_k$. Recall [13, Thm. 1.3] that we have an isomorphism of $\mathbb{C}$-vector spaces

$$\mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*] \xrightarrow{\sim} \mathbf{H}_k$$

induced by multiplication (this is the so-called *PBW-decomposition*).

Remark 1.1. Let $(l_\Omega)_{\Omega \in \mathcal{A}/W}$ be a family of complex numbers and let $k' \in \mathbb{C}^{\aleph}$ be defined by $k'_{\Omega,j} = k_{\Omega,j} + l_\Omega$. Then $\mathbf{H}_k = \mathbf{H}_{k'}$. This means that there is no loss of generality if we consider, for instance, only parameters k such that $k_{\Omega,0} = 0$ for all Ω, or only parameters k such that $k_{\Omega,0} + k_{\Omega,1} + \cdots + k_{\Omega,e_\Omega - 1} = 0$ for all Ω (as in [6]).

1.C. **Calogero–Moser space.** We denote by $\mathbf{Z}_k$ the center of the algebra $\mathbf{H}_k$; it is well-known [13, Theo 3.3 and Lem. 3.5] that $\mathbf{Z}_k$ is an integral domain, which is integrally closed. Moreover, it contains $\mathbb{C}[V]^W$ and $\mathbb{C}[V^*]^W$ as subalgebras [16, Prop. 3.6]. So, by the PBW-decomposition, $\mathbf{Z}_k$ contains $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$, and it is a free $\mathbf{P}$-module of rank $|W|$ (see [13, Prop. 4.15]). We denote by $\mathcal{Z}_k$ the affine algebraic variety whose ring of regular functions $\mathbb{C}[\mathcal{Z}_k]$ is $\mathbf{Z}_k$: this is the *Calogero–Moser space* associated with the datum (V, W, k). It is irreducible and normal.

We set $\mathcal{P} = V/W \times V^*/W$, so that $\mathbb{C}[\mathcal{P}] = \mathbf{P}$ and the inclusion $\mathbf{P} \hookrightarrow \mathbf{Z}_k$ induces a morphism of varieties

$$\Upsilon_k : \mathcal{Z}_k \longrightarrow \mathcal{P}$$

which is finite and flat.

1.D. **Calogero–Moser families.** Using the PBW-decomposition, we define a $\mathbb{C}$-linear map $\mathbf{\Omega}^{\mathbf{H}_k} : \mathbf{H}_k \longrightarrow \mathbb{C}W$ by

$$\mathbf{\Omega}^{\mathbf{H}_k}(fwg) = f(0)g(0)w$$

for all $f \in \mathbb{C}[V]$, $g \in \mathbb{C}[V^*]$ and $w \in \mathbb{C}W$. This map is W-equivariant for the action on both sides by conjugation, so it induces a well-defined $\mathbb{C}$-linear map

$$\mathbf{\Omega}^k : \mathbf{Z}_k \longrightarrow \mathrm{Z}(\mathbb{C}W).$$

Recall from [6, Cor. 4.2.11] that $\mathbf{\Omega}^k$ is a morphism of algebras.

Calogero–Moser families were defined by Gordon using his theory of *baby Verma modules* [16, §4.2 and §5.4]. We explain here an equivalent definition given in [6, §7.2]. If $\chi \in \mathrm{Irr}(W)$, we denote by $\omega_\chi : \mathrm{Z}(\mathbb{C}W) \to \mathbb{C}$ its central character (i.e., $\omega_\chi(z) = \chi(z)/\chi(1)$ is the scalar by which z acts on an irreducible representation affording the character χ). We say that two characters χ and χ' belong to the same *Calogero–Moser k-family* if $\omega_\chi \circ \mathbf{\Omega}^k = \omega_{\chi'} \circ \mathbf{\Omega}^k$.

In other words, the map $\omega_\chi \circ \mathbf{\Omega}^k : \mathbf{Z}_k \to \mathbb{C}$ is a morphism of algebras, so it might be viewed as a point $\varphi_k(\chi)$ of $\mathcal{Z}_k$, which is easily checked to be $\mathbb{C}^\times$-fixed. This defines a surjective map [16, §5.4]

$$\varphi_k : \mathrm{Irr}(W) \longrightarrow \mathcal{Z}_k^{\mathbb{C}^\times}$$

whose fibers are the Calogero–Moser k-families. If $p \in \mathcal{Z}_k^{\mathbb{C}^\times}$, we denote by $\mathfrak{F}_p$ the corresponding Calogero–Moser k-family.

1.E. **Alternative parameters.** Let $\mathcal{C}$ denote the space of maps $\mathrm{Ref}(W) \to \mathbb{C}$ which are constant on conjugacy classes of reflections. The element

$$\sum_{(\Omega,j)\in\aleph} \sum_{H\in\Omega} (k_{H,j} - k_{H,j+1}) e_H \varepsilon_{H,j}$$

of $\mathrm{Z}(\mathbb{C}W)$ is supported only by reflections, so there exists a unique map $c_k \in \mathcal{C}$ such that

$$\sum_{(\Omega,j)\in\aleph} \sum_{H\in\Omega} (k_{H,j} - k_{H,j+1}) e_H \varepsilon_{H,j} = \sum_{s\in\mathrm{Ref}(W)} (\varepsilon(s) - 1) c_k(s) s.$$

Then the map $\mathbb{C}^\aleph \to \mathcal{C}$, $k \mapsto c_k$ is linear and surjective. With this notation, we have

$$[y, x] = \sum_{s\in\mathrm{Ref}(W)} (\varepsilon(s) - 1) c_k(s) \frac{\langle y, \alpha_s\rangle \cdot \langle \alpha_s^\vee, x\rangle}{\langle \alpha_s^\vee, \alpha_s\rangle} s$$

for all $y \in V$ and $x \in V^*$. Here, $\alpha_s = \alpha_{V^s}$ and $\alpha_s^\vee = \alpha_{V^s}^\vee$.

1.F. **Actions on the Calogero–Moser space.** The Calogero–Moser space $\mathcal{Z}_k$ is endowed with a $\mathbb{C}^\times$-action and an action of the stabilizer of k in $\mathrm{N}_{\mathbf{GL}_\mathbb{C}(V)}(W)$, which are described below.

1.F.1. *Grading, $\mathbb{C}^\times$-action.* The algebra $\mathrm{T}(V \oplus V^*) \rtimes W$ can be $\mathbb{Z}$-graded in such a way that the generators have the following degrees:

$$\begin{cases} \deg(y) = -1 & \text{if } y \in V, \\ \deg(x) = 1 & \text{if } x \in V^*, \\ \deg(w) = 0 & \text{if } w \in W. \end{cases}$$

This descends to a $\mathbb{Z}$-grading on $\mathbf{H}_k$, because the defining relations (1.1) are homogeneous. Since the center of a graded algebra is always graded, the subalgebra $\mathbf{Z}_k$ is also $\mathbb{Z}$-graded. So the Calogero–Moser space $\mathcal{Z}_k$ inherits a regular $\mathbb{C}^\times$-action. Note also that, by definition, $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$ is clearly a graded subalgebra of $\mathbf{Z}_k$.

1.F.2. *Action of the normalizer.* The group $\mathrm{N}_{\mathbf{GL}_\mathbb{C}(V)}(W)$ acts on the set $\aleph$ and so on the space of parameters $\mathbb{C}^\aleph$. If $\tau \in \mathrm{N}_{\mathbf{GL}_\mathbb{C}(V)}(W)$, then τ induces an isomorphism of algebras $\mathbf{H}_k \longrightarrow \mathbf{H}_{\tau(k)}$. So, if $\tau(k) = k$, then it induces an action on the algebra $\mathbf{H}_k$ (and so on its center $\mathbf{Z}_k$ and on the Calogero–Moser space $\mathcal{Z}_k$).

Notation. From now on, and until the end of this paper, we fix a parameter $k \in \mathbb{C}^\aleph$ and a *regular* element τ of *finite* order of $\mathrm{N}_{\mathbf{GL}_\mathbb{C}(V)}(W)$ such that $\tau(k) = k$.

We denote by $\mathcal{Z}_k^\tau$ the variety of fixed points of τ in $\mathcal{Z}_k$, endowed with its reduced structure. All the above constructions are τ-equivariant; for instance, the map $\varphi_k : \mathrm{Irr}(W) \longrightarrow \mathcal{Z}_k^{\mathbb{C}^\times}$ is τ-equivariant.

Let us recall the following consequence [21, Prop. 3.5 and Thm. 4.2] of the above hypothesis:

Theorem 1.2 (Springer)**.** *The group W^τ acts as a reflection group on V^τ and the natural map $V^\tau/W^\tau \to (V/W)^\tau$ is an isomorphism of varieties.*

Corollary 1.3. *The natural map $(V_{\mathrm{reg}}^\tau \times V^{*\tau})/W^\tau \to ((V_{\mathrm{reg}} \times V^*)/W)^\tau$ is an isomorphism of varieties.*

Proof. Since W acts freely on the variety $V_{\mathrm{reg}} \times V^*$, the quotient variety $(V_{\mathrm{reg}} \times V^*)/W$ is smooth. Consequently, the variety of fixed points $((V_{\mathrm{reg}} \times V^*)/W)^\tau$ is also smooth. Similarly, $(V_{\mathrm{reg}}^\tau \times V^{*\tau})/W^\tau$ is smooth. Since a bijective morphism between smooth complex varieties is an isomorphism (by Zariski's main theorem), we only need to show that the above natural map is bijective.

First, if (v_1, v_1^*) and (v_2, v_2^*) are two elements of $V_{\mathrm{reg}}^\tau \times V^{*\tau}$ belonging to the same W-orbit, there exists $w \in W$ such that $v_2 = w(v_1)$. Since v_1 and v_2 are τ-stable, we also have $\tau(w)(v_1) = v_2$, and so $v_1 = w^{-1}\tau(w)(v_1)$. Since $v_1 \in V_{\mathrm{reg}}$, this forces $\tau(w) = w$, and the injectivity follows.

Now, if $(v, v^*) \in V_{\mathrm{reg}} \times V^*$ is such that its W-orbit is τ-stable, then the W-orbit of v is τ-stable. So Theorem 1.2 shows that we may assume that $\tau(v) = v$. The hypothesis implies that there exists $w \in W$ such that $\tau(v) = w(v)$ and $\tau(v^*) = w(v^*)$. But $\tau(v) = v \in V_{\mathrm{reg}}$, so $w = 1$. In particular, $\tau(v^*) = v^*$, and the surjectivity follows. ∎

2. Irreducible component of maximal dimension

Let $(\mathcal{Z}_k)_{\mathrm{reg}}$ denote the open subset $\Upsilon_k^{-1}(V_{\mathrm{reg}}/W \times V^*/W)$. By [13, Prop. 4.11], we have a $\mathbb{C}^\times$-equivariant and τ-equivariant isomorphism

$$(\mathcal{Z}_k)_{\mathrm{reg}} \simeq (V_{\mathrm{reg}} \times V^*)/W.$$

This shows that $(\mathcal{Z}_k)_{\mathrm{reg}}$ is smooth and so $(\mathcal{Z}_k)_{\mathrm{reg}}^\tau$ is also smooth. By Corollary 1.3, this implies that

$$(\mathcal{Z}_k)_{\mathrm{reg}}^\tau \simeq (V_{\mathrm{reg}}^\tau \times V^{*\tau})/W^\tau.$$

In particular, it is irreducible. We denote by $(\mathcal{Z}_k^\tau)_{\max}$ its closure; it is an irreducible closed subvariety of $\mathcal{Z}_k^\tau$.

Moreover, $(\mathcal{Z}_k)_{\mathrm{reg}}^\tau$ has dimension $2\dim V^\tau$ by Corollary 1.3. So $\dim \mathcal{Z}_k^\tau \geq 2\dim V^\tau = \dim(\mathcal{Z}_k^\tau)_{\max}$. But, on the other hand, $\Upsilon_k(\mathcal{Z}_k^\tau) \subset (V/W)^\tau \times (V^*/W)^\tau$. Since Υ_k is a finite morphism, we get from Theorem 1.2 that $\dim \mathcal{Z}_k^\tau \leq 2\dim V^\tau$. Hence

$$\dim \mathcal{Z}_k^\tau = \dim(\mathcal{Z}_k^\tau)_{\max} = 2\dim V^\tau.$$

This shows that $(\mathcal{Z}_k^\tau)_{\max}$ is an irreducible component of maximal dimension of $\mathcal{Z}_k^\tau$ and that

$$\Upsilon_k((\mathcal{Z}_k^\tau)_{\max}) = (V/W)^\tau \times (V^*/W)^\tau. \tag{2.1}$$

Proposition 2.1. *The closed subvariety $(\mathcal{Z}_k^\tau)_{\max}$ of $\mathcal{Z}_k^\tau$ is the unique irreducible component of maximal dimension.*

Proof. Let $\mathcal{X}$ be an irreducible component of $\mathcal{Z}_k^\tau$ of dimension $2\dim V^\tau$. Since Υ_k is finite, the image $\Upsilon_k(\mathcal{X})$ is closed in $V/W \times V^*/W$, irreducible of dimension $2\dim(V^\tau)$ and contained in $(V/W)^\tau \times (V^*/W)^\tau$. By Theorem 1.2, we get that $\Upsilon_k(\mathcal{X}) = (V/W)^\tau \times (V^*/W)^\tau$.

Let $\mathcal{U} = \Upsilon_k^{-1}(V_{\mathrm{reg}}/W \times V^*/W) \cap \mathcal{X}$. Then $\mathcal{U}$ is a non-empty open subset of $\mathcal{X}$; since $\mathcal{X}$ is irreducible, this forces $\mathcal{U}$ to have dimension $2\dim(V^\tau)$. But $\mathcal{U}$ is contained in $(\mathcal{Z}_k)_{\mathrm{reg}}^\tau$, which is irreducible of the same dimension, so the closure of $\mathcal{U}$ contains $(\mathcal{Z}_k)_{\mathrm{reg}}^\tau$. This proves that $\mathcal{X} = (\mathcal{Z}_k^\tau)_{\max}$. $\square$

It is natural to ask which $\mathbb{C}^\times$-fixed points of $\mathcal{Z}_k$ belong to $(\mathcal{Z}_k^\tau)_{\max}$. Inspired by the representation theory of finite reductive groups (see [11] and [10, Rem. 4.21]), we propose an answer to this question in terms of the character table of the finite group $W\langle\tau\rangle$ (see [4, Ex. 12.9] for some explanations). We first need some notation.

If $\chi \in \mathrm{Irr}(W)$, we denote by E_χ a $\mathbb{C}W$-module affording the character χ. If moreover χ is τ-stable, we fix a structure of $\mathbb{C}W\langle\tau\rangle$-module on E_χ extending the structure of $\mathbb{C}W$-module, and we denote by $\tilde{\chi}$ its associated irreducible character of $W\langle\tau\rangle$. Note that the real number $|\tilde{\chi}(\tau)|^2$ does not depend on the choice of $\tilde{\chi}$.

Conjecture 2.2. *Recall that τ is regular. Let $p \in \mathcal{Z}_k^{\mathbb{C}^\times}$ be such that $\tau(p) = p$. Then p belongs to $(\mathcal{Z}_k^\tau)_{\max}$ if and only if $\sum_{\chi \in \mathfrak{F}_p^\tau} |\tilde{\chi}(\tau)|^2 \neq 0$.*

Remark 2.3. Let $\mathfrak{F}$ be a τ-stable Calogero–Moser family. Then $\mathfrak{F}$ contains a unique irreducible character $\chi_{\mathfrak{F}}$ with minimal b-invariant [6, Thm. 7.4.1], where the b-invariant of an irreducible character χ is the minimal natural number j such that χ occurs in the j-th symmetric power of the natural representation V of W. From this characterization, we see that $\chi_{\mathfrak{F}}$ is τ-stable. In particular, any τ-stable Calogero–Moser family contains at least one τ-stable character.

In general, we are only able to prove the "if" part of Conjecture 2.2.

Theorem 2.4. *Recall that τ is regular. Let $p \in \mathcal{Z}_k^{\mathbb{C}^\times}$ be such that $\tau(p) = p$. If $\sum_{\chi \in \mathfrak{F}_p^\tau} |\tilde{\chi}(\tau)|^2 \neq 0$, then p belongs to $(\mathcal{Z}_k^\tau)_{\max}$.*

The next two sections are devoted to the proof of Theorem 2.4.

3. Verma modules

3.A. **Definition.** Recall that $\mathbb{C}[V] \rtimes W$ is a subalgebra of $\mathbf{H}_k$ (it is the image of $1 \otimes \mathbb{C}W \otimes \mathbb{C}[V]$ by the PBW-decomposition 1.B). If E is a $\mathbb{C}W$-module, we denote by $E^{\#}$ the $(\mathbb{C}[V^*] \rtimes W)$-module extending E by letting any element $f \in \mathbb{C}[V^*]$ act by multiplication by $f(0)$. If $\chi \in \operatorname{Irr}(W)$, we define an $\mathbf{H}_k$-module $\Delta(\chi)$ as follows:

$$\Delta(\chi) = \mathbf{H}_k \otimes_{\mathbb{C}[V^*] \rtimes W} E_\chi^{\#}.$$

Then $\Delta(\chi)$ is called a *Verma module* of $\mathbf{H}_k$ (see [6, §5.4.A]; in this reference, $\Delta(\chi)$ is denoted by $\Delta(E_\chi^{\#})$). Let $\mathbf{H}_k^{\mathrm{reg}}$ denote the localization of $\mathbf{H}_k$ at $\mathbf{P}_{\mathrm{reg}} = \mathbb{C}[V_{\mathrm{reg}}/W] \otimes \mathbb{C}[V^*/W]$. By [13, Prop. 4.11], we have an isomorphism $\mathbb{C}[V_{\mathrm{reg}} \times V^*] \rtimes W \simeq \mathbf{H}_k^{\mathrm{reg}}$. We denote by $\Delta^{\mathrm{reg}}(\chi)$ the localization of $\Delta(\chi)$ at $\mathbf{H}_k^{\mathrm{reg}}$. So, by restriction to $\mathbb{C}[V_{\mathrm{reg}} \times V^*]$, the localized Verma module $\Delta^{\mathrm{reg}}(\chi)$ might be viewed as a W-equivariant coherent sheaf on $V_{\mathrm{reg}} \times V^*$. We also view $e\Delta(\chi)$ as a coherent sheaf on $\mathcal{Z}_k$, so that $e\Delta^{\mathrm{reg}}(\chi)$ may be viewed as a coherent sheaf on $(V_{\mathrm{reg}} \times V^*)/W$. If $p \in \mathcal{Z}_k$ (or if $(v, v^*) \in V_{\mathrm{reg}} \times V^*$), we denote by $e\Delta(\chi)_p$ (respectively $e\Delta(\chi)_{W\cdot(v,v^*)} = e\Delta^{\mathrm{reg}}(\chi)_{W\cdot(v,v^*)}$, respectively $\Delta^{\mathrm{reg}}(\chi)_{v,v^*}$) the restriction of $e\Delta(\chi)$ (respectively of $e\Delta(\chi)$ or $e\Delta^{\mathrm{reg}}(\chi)$, respectively $\Delta^{\mathrm{reg}}(\chi)$) at the point p (respectively $W \cdot (v, v^*) \in (V_{\mathrm{reg}} \times V^*)/W \simeq (\mathcal{Z}_k)_{\mathrm{reg}}$, respectively (v, v^*)). It follows from the definition that the support of $e\Delta(\chi)$ is contained in $\Upsilon_k^{-1}(V/W \times 0)$, and recall that, through the isomorphism $\mathcal{Z}_k^{\mathrm{reg}} \simeq (V_{\mathrm{reg}} \times V^*)/W$, $\Upsilon_k^{-1}(V_{\mathrm{reg}}/W \times 0)$ is not necessarily contained in $(V_{\mathrm{reg}} \times \{0\})/W$.

Lemma 3.1. *Let $\chi \in \operatorname{Irr}(W)$ and let $p \in \mathcal{Z}_k^{\mathbb{C}^\times}$. Then $e\Delta(\chi)_p \neq 0$ if and only if $\chi \in \mathfrak{F}_p$.*

Proof. Let $\mathfrak{p}_0$ denote the maximal ideal of the algebra $\mathbf{P} = \mathbb{C}[\mathcal{P}]$ consisting of functions which vanish at 0. Then $\Delta(\chi)/\mathfrak{p}_0\Delta(\chi)$ is a representation of the *restricted rational Cherednik algebra* $\mathbf{H}_k/\mathfrak{p}_0\mathbf{H}_k$ which coincides with the *baby Verma module* defined by Gordon [16, §4.2]. As $\mathcal{Z}_k^{\mathbb{C}^\times} = \Upsilon_k^{-1}(0)$, the result follows from the very definition of Calogero–Moser families in terms of baby Verma modules and the fact that it is equivalent to the definition given in §1.D. ☐

3.B. **Bialynicki–Birula decomposition.** We denote by $\mathcal{Z}_k^{\mathrm{att}}$ the *attracting set* of $\mathcal{Z}_k$ for the action of $\mathbb{C}^\times$, namely

$$\mathcal{Z}_k^{\mathrm{att}} = \{p \in \mathcal{Z}_k \mid \lim_{\xi \to 0} {}^\xi p \text{ exists}\}.$$

Recall from [6, Chap. 9] the following facts.

Proposition 3.2. *With the above notation, we have:*

(a) *The map* $\lim : \mathcal{Z}_k^{\mathrm{att}} \longrightarrow \mathcal{Z}_k^{\mathbb{C}^\times}$, $p \mapsto \lim_{\xi \to 0} {}^{\xi}p$ *is a morphism of varieties.*
(b) $\mathcal{Z}_k^{\mathrm{att}} = \Upsilon_k^{-1}(V/W \times \{0\})$.
(c) *If* $\mathcal{I}$ *is an irreducible component of* $\mathcal{Z}_k^{\mathrm{att}}$*, then* $\mathcal{I}$ *is* $\mathbb{C}^\times$*-stable and* $\Upsilon_k(\mathcal{I}) = V/W \times \{0\}$ *and* $\lim(\mathcal{I})$ *is a single point.*
(d) *If* $\chi \in \mathrm{Irr}(W)$*, then the support of* $e\Delta(\chi)$ *is a union of irreducible components of* $\mathcal{Z}_k^{\mathrm{att}}$.
(e) *If* $\mathcal{I}$ *is an irreducible component of* $\mathcal{Z}_k^{\mathrm{att}}$*, then there exists* $\chi \in \mathrm{Irr}(W)$ *such that the support of* $e\Delta(\chi)$ *contains* $\mathcal{I}$.

Proof. (a) is classical (see, for instance, [6, §9.1]). For (b), see [6, Lem. 9.3.2]. (c) is explained at the end of [6, §9.3]. For (d) and (e), see [6, (8.1.3) and Prop. 9.3.3]. □

We characterize points of $\mathcal{Z}_k^{\mathbb{C}^\times}$ belonging to $(\mathcal{Z}_k^\tau)_{\max}$ in terms of Verma modules:

Lemma 3.3. *Let* $p \in \mathcal{Z}_k^{\mathbb{C}^\times}$ *and assume that* $\tau(p) = p$*. Then* $p \in (\mathcal{Z}_k^\tau)_{\max}$ *if and only if there exist* $\chi \in \mathfrak{F}_p^k$ *and* $(v, v^*) \in V_{\mathrm{reg}}^\tau \times V^{*\tau}$ *such that* $e\Delta(\chi)_{W\cdot(v,v^*)} \neq 0$.

Proof. Let $(\mathcal{Z}_k^\tau)_{\max}^{\mathrm{att}}$ denote the attracting set of $(\mathcal{Z}_k^\tau)_{\max}$. Then (2.1) and Proposition 3.2 (b) imply that $\Upsilon_k((\mathcal{Z}_k^\tau)_{\max}^{\mathrm{att}}) = (V/W)^\tau \times \{0\}$. Since Υ_k is a finite morphism, the same arguments used in [6, Chap. 9] to prove Proposition 3.2 above yields the following statements:

(a) The map $\lim : (\mathcal{Z}_k^\tau)_{\max}^{\mathrm{att}} \longrightarrow (\mathcal{Z}_k^\tau)_{\max}^{\mathbb{C}^\times}$, $p \mapsto \lim_{\xi \to 0} {}^{\xi}p$ is a morphism of varieties.
(b) $(\mathcal{Z}_k^\tau)_{\max}^{\mathrm{att}} = (\mathcal{Z}_k^\tau)_{\max} \cap \Upsilon_k^{-1}((V/W)^\tau \times \{0\})$.
(c) If $\mathcal{I}$ is an irreducible component of $(\mathcal{Z}_k^\tau)_{\max}^{\mathrm{att}}$, then $\mathcal{I}$ is $\mathbb{C}^\times$-stable and $\Upsilon_k(\mathcal{I}) = (V/W)^\tau \times \{0\}$ and $\lim(\mathcal{I})$ is a single point.

Assume that $p \in (\mathcal{Z}_k^\tau)_{\max}$. Let $\mathcal{I}$ be an irreducible component of $(\mathcal{Z}_k^\tau)_{\max}^{\mathrm{att}} \cap \lim^{-1}(p)$. Then $\mathcal{I}$ is contained in an irreducible component $\mathcal{I}'$ of $(\mathcal{Z}_k^\tau)_{\max}^{\mathrm{att}}$. Since $\lim(\mathcal{I}')$ is a single point by (c), we have $\lim(\mathcal{I}') = \{p\}$ and so $\mathcal{I} = \mathcal{I}'$. Still by (c), this says that $\Upsilon_k(\mathcal{I}) = (V/W)^\tau \times \{0\}$. So let $q \in \mathcal{I}$ be such that $\Upsilon_k(q) \in (V_{\mathrm{reg}}/W)^\tau \times \{0\}$.

Now, let $\mathcal{J}$ be an irreducible component of $\mathcal{Z}_k^{\mathrm{att}}$ containing $\mathcal{I}$. By Proposition 3.2 (e), there exists $\chi \in \mathrm{Irr}(W)$ such that the support of $e\Delta(\chi)$ contains $\mathcal{J}$. In particular, $e\Delta(\chi)_p \neq 0$ and so $\chi \in \mathfrak{F}_p$ by Lemma 3.1. But also $e\Delta(\chi)_q \neq 0$. Since $q \in (\mathcal{Z}_k^\tau)_{\max}$ and $\Upsilon_k(q) \in V_{\mathrm{reg}}/W$, it follows that there exists $(v, v^*) \in V_{\mathrm{reg}}^\tau \times V^{*\tau}$ such that $e\Delta(\chi)_{W\cdot(v,v^*)} \neq 0$, as desired.

Conversely, assume that there exist both $\chi \in \mathfrak{F}_p^k$ and $(v, v^*) \in V_{\mathrm{reg}}^\tau \times V^{*\tau}$ such that $e\Delta(\chi)_{W\cdot(v,v^*)} \neq 0$. Let $\mathcal{I}$ be an irreducible component of $\mathcal{Z}_k^{\mathrm{att}}$ contained in the support of $e\Delta(\chi)$. Then $p \in \mathcal{I}$ and so $p = \lim W\cdot(v, v^*)$. Since $W\cdot(v, v^*) \in (\mathcal{Z}_k^\tau)_{\max}$ by the definition of $(\mathcal{Z}_k^\tau)_{\max}$, this implies that $p \in (\mathcal{Z}_k^\tau)_{\max}$, as desired. □

4. Gaudin algebra

4.A. **Definition.** We recall here the definition of *Gaudin algebra* [6, §8.3.B]. First, let $\mathbb{C}[V_{\mathrm{reg}}][W]$ denote the group algebra of W over the algebra $\mathbb{C}[V_{\mathrm{reg}}]$ (and not the semi-direct product $\mathbb{C}[V_{\mathrm{reg}}] \rtimes W$). For $y \in V$, let

$$\mathcal{D}_y^k = \sum_{s \in \mathrm{Ref}(W)} \varepsilon(s) c_k(s) \frac{\langle y, \alpha_s \rangle}{\alpha_s} s \in \mathbb{C}[V_{\mathrm{reg}}][W].$$

Now, let $\mathrm{Gau}_k(W)$ be the sub-$\mathbb{C}[V_{\mathrm{reg}}]$-algebra of $\mathbb{C}[V_{\mathrm{reg}}][W]$ generated by the $\mathcal{D}_y^k$'s (where y runs over V); it will be called the *Gaudin algebra* (*with parameter* k) associated with W.

Let $\mathbb{C}(V)$ denote the function field of V (which is the fraction field of $\mathbb{C}[V]$ or of $\mathbb{C}[V_{\mathrm{reg}}]$) and let $\mathbb{C}(V)\mathrm{Gau}_k(W)$ denote the subalgebra $\mathbb{C}(V) \otimes_{\mathbb{C}[V_{\mathrm{reg}}]} \mathrm{Gau}_k(W)$ of the group algebra $\mathbb{C}(V)[W]$. Recall [6, §8.3.B] that

$$\mathrm{Gau}_k(W) \text{ is a commutative algebra,}$$

but that $\mathbb{C}(V)\mathrm{Gau}_k(W)$ is generally non-split, as shown by the examples treated in [2, §4] and [19].

4.B. **Generalized eigenspaces.** If $v \in V_{\mathrm{reg}}$, we denote by $\mathcal{D}_y^{k,v}$ the specialization of $\mathcal{D}_y^k$ at v, namely $\mathcal{D}_y^{k,v}$ is the element of the group algebra $\mathbb{C}W$ equal to

$$\mathcal{D}_y^{k,v} = \sum_{s \in \mathrm{Ref}(W)} \varepsilon(s) c_k(s) \frac{\langle y, \alpha_s \rangle}{\langle v, \alpha_s \rangle} s.$$

Now, if $v^* \in V^*$ and if M is a $\mathbb{C}W$-module, we define M^{k,v,v^*} to be the common generalized eigenspace of the operators $\mathcal{D}_y^{k,v}$ for the eigenvalue $\langle y, v^* \rangle$ for y running over V. Namely,

$$M^{k,v,v^*} = \big\{ m \in M \mid \forall\, y \in V,\ (\mathcal{D}_y^{k,v} - \langle y, v^* \rangle \,\mathrm{Id}_M)^{\dim(M)}(m) = 0 \big\}.$$

Then

$$M = \bigoplus_{v^* \in V^*} M^{k,v,v^*}, \tag{4.1}$$

since $\mathrm{Gau}_k(W)$ is commutative.

Lemma 4.1. *Let $\chi \in \mathrm{Irr}(W)$ and let $(v, v^*) \in V_{\mathrm{reg}} \times V^*$. Then the following are equivalent:*

(1) $e\Delta(\chi)_{W \cdot (v,v^*)} \neq 0$.
(2) $\Delta^{\mathrm{reg}}(\chi)_{v,v^*} \neq 0$.
(3) $E_\chi^{k,v,v^*} \neq 0$.

Proof. The equivalence between (1) and (2) follows from the Morita equivalence between $\mathbb{C}[V_{\mathrm{reg}} \times V^*]^W$ and $\mathbb{C}[V_{\mathrm{reg}} \times V^*] \rtimes W$ proved in [6, Lem. 3.1.8 (b)]. Now, as a $(\mathbb{C}[V_{\mathrm{reg}}] \rtimes W)$-module, $\Delta^{\mathrm{reg}}(\chi) \simeq \mathbb{C}[V_{\mathrm{reg}}] \otimes E_\chi$, and the action of $y \in V \subset \mathbb{C}[V^*]$ is given by the operator $-\mathcal{D}_y^k \in \mathbb{C}[V_{\mathrm{reg}}][W]$ (see [6, §8.3.B]). Now, $\Delta^{\mathrm{reg}}(\chi)_{v,v^*} \simeq E_\chi$ as a $\mathbb{C}$-vector space; on this vector space, the action of an element $f \in \mathbb{C}[V_{\mathrm{reg}}]$ is given by multiplication by $f(v)$, while the action of an element $y \in V$ is given by the operator $\langle y, v^* \rangle \,\mathrm{Id}_{E_\chi} - \mathcal{D}_y^{k,v}$ (see [6, Thm. 4.1.7]). This shows the equivalence between (2) and (3). □

4.C. **Proof of Theorem A (i.e., Theorem 2.4).** Let $\chi \in \mathrm{Irr}(W)$ be τ-stable and such that $\tilde{\chi}(\tau) \neq 0$, and let $v \in V_{\mathrm{reg}}^{\tau}$. By Lemmas 3.1 and 4.1, it is sufficient to show that there exists $v^* \in V^{*\tau}$ such that $E_\chi^{k,v,v^*} \neq 0$.

For this, let $\mathcal{E}$ denote the set of $v^* \in V^*$ such that $E_\chi^{k,v,v^*} \neq 0$. Then it follows from (4.1) that

$$(*) \qquad E_\chi = \bigoplus_{v^* \in \mathcal{E}} E_\chi^{k,v,v^*}.$$

Since $\tau(v) = v$, we have

$$^{\tau}\mathcal{D}_y^{k,v} = \sum_{s \in \mathrm{Ref}(W)} \varepsilon(s) c_k(s) \frac{\langle y, \alpha_s \rangle}{\langle v, \alpha_s \rangle} \tau s \tau^{-1} = \sum_{s \in \mathrm{Ref}(W)} \varepsilon(s) c_k(s) \frac{\langle y, \tau^{-1}(\alpha_s) \rangle}{\langle v, \tau^{-1}(\alpha_s) \rangle} s = \mathcal{D}_{\tau(y)}^{k,v}.$$

Consequently,

$$^{\tau}E_\chi^{k,v,v^*} = E_\chi^{k,v,\tau(v^*)}.$$

But $\tilde{\chi}(\tau) = \mathrm{Tr}(\tau, E_\chi) \neq 0$, so τ must fix at least one of the generalized eigenspaces in the decomposition $(*)$. In other words, this implies that there exists $v^* \in \mathcal{E}$ such that $\tau(v^*) = v^*$, as desired. The proof is complete. $\square$

5. Complements: further conjectures, examples

5.A. **Conjectures.** The variety $\mathcal{Z}_k$ is endowed with a Poisson structure [13, §1] and so the variety of fixed points $\mathcal{Z}_k^\tau$ inherits a Poisson structure too, as well as all its irreducible components. Recall from Springer's Theorem 1.2 that W^τ is a reflection group for its action on V^τ, so we can define a set of pairs $\aleph_\tau$ for the pair (V^τ, W^τ), just as $\aleph$ was defined for the pair (V, W); and, for each parameter $l \in \mathbb{C}^{\aleph_\tau}$, we can define a Calogero–Moser space $\mathcal{Z}_l(V^\tau, W^\tau)$. The following conjecture is a particular case of [3, Conj. B] (see [3] for a discussion about the cases where this conjecture is known to hold).

Conjecture 5.1. *Recall that τ is regular. Then there exist a linear map $\lambda : \mathbb{C}^{\aleph} \to \mathbb{C}^{\aleph_\tau}$ and, for each $k \in \mathbb{C}^{\aleph}$, a $\mathbb{C}^\times$-equivariant isomorphism of Poisson varieties*

$$\iota_k : (\mathcal{Z}_k^\tau)_{\max} \xrightarrow{\sim} \mathcal{Z}_{\lambda(k)}(V^\tau, W^\tau).$$

If the existence of such a $\mathbb{C}^\times$-equivariant isomorphism

$$\iota_k : (\mathcal{Z}_k^\tau)_{\max} \xrightarrow{\sim} \mathcal{Z}_{\lambda(k)}(V^\tau, W^\tau)$$

is known, but it is not known whether it preserves the Poisson structure, then we say that "Conjecture 5.1$^-$ holds".

Assume now that Conjecture 5.1$^-$ holds and keep its notation. Then ι_k restricts to a map $\iota_k : (\mathcal{Z}_k)_{\max}^{\mathbb{C}^\times} \xrightarrow{\sim} \mathcal{Z}_{\lambda(k)}(V^\tau, W^\tau)^{\mathbb{C}^\times}$. If $p \in \mathcal{Z}_{\lambda(k)}(V^\tau, W^\tau)^{\mathbb{C}^\times}$, we denote by $\mathfrak{F}_{\iota_k(p)}^{(\tau)}$ the corresponding Calogero–Moser $\lambda(k)$-family of W^τ. The next conjecture, still inspired by the representation theory of finite reductive groups (see again [4, Ex. 12.9] for some explanations), makes Conjecture B more precise:

Conjecture 5.2. *Recall that τ is regular and assume that Conjecture* 5.1$^-$ *holds. If $p \in (\mathcal{Z}_k^\tau)^{\mathbb{C}^\times}_{\max}$, then*
$$\sum_{\chi \in \mathfrak{F}_p^\tau} |\tilde{\chi}(\tau)|^2 = \sum_{\psi \in \mathfrak{F}^{(\tau)}_{\iota_k(p)}} \psi(1)^2.$$

Note that this last conjecture is compatible with the fact that
$$\sum_{\chi \in \mathrm{Irr}(W)^\tau} |\tilde{\chi}(\tau)|^2 = |W^\tau| = \sum_{\psi \in \mathrm{Irr}(W^\tau)} \psi(1)^2,$$
where the first equality follows from the second orthogonality relation for characters applied to $W\langle\tau\rangle$. Indeed, $\sum_{\theta \in \mathrm{Irr}(W\langle\tau\rangle)} |\theta(\tau)|^2 = |C_{W\langle\tau\rangle}(\tau)|$ and $\theta(\tau) \neq 0$ implies that θ is an extension of a τ-invariant character χ of W by [18, Thm. 6.11]; the equality then follows from the fact that $|\theta(\tau)|$ depends only on χ and that there are $|W\langle\tau\rangle|/|W|$ extensions of χ by [18, Cor. 6.17].

5.B. **Roots of unity.** We consider in this subsection a particular (but very important) case of the general situation studied in this paper. We fix a natural number $d \geq 1$ and a primitive d-th root of unity ζ_d. The group of d-th roots of unity is denoted by $\boldsymbol{\mu}_d$. An element $w \in W$ is called *ζ_d-regular* if the element $\zeta_d^{-1} w$ of $\mathrm{N}_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$ is regular. In other words, w is ζ_d-regular if and only if its ζ_d-eigenspace meets V_{reg}. The existence of a ζ_d-regular element is not guaranteed; we say that d is a *regular number* of W if such an element exists.

Hypothesis. We assume in this subsection that d is a regular number of W. We denote by w_d a ζ_d-regular element and we also set $\tau_d = \zeta_d^{-1} w_d$, so that τ_d is a regular element of $\mathrm{N}_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$.

Recall from [21, Thm. 4.2 (iv)] that w_d is uniquely defined up to conjugacy. Note that
$$V^{\tau_d} = \mathrm{Ker}(w_d - \zeta_d \,\mathrm{Id}_V), \qquad W^{\tau_d} = C_W(w_d) \qquad \text{and} \qquad \mathcal{Z}_k^{\tau_d} = \mathcal{Z}_k^{\boldsymbol{\mu}_d}.$$
Since τ_d induces an inner automorphism of W, all the irreducible characters are τ_d-stable. Moreover, if $\chi \in \mathrm{Irr}(W)$, then $\tilde{\chi}(\tau_d) = \xi \chi(w_d)$ for some root of unity ξ, so $|\tilde{\chi}(\tau_d)|^2 = |\chi(w_d)|^2$. This allows us to reformulate both Theorem A and Conjecture B in this case.

Conjecture 5.3. *Recall that d is a regular number. Let $p \in \mathcal{Z}_k^{\mathbb{C}^\times}$. Then p belongs to $(\mathcal{Z}_k^{\boldsymbol{\mu}_d})_{\max}$ if and only if $\sum_{\chi \in \mathfrak{F}_p} |\chi(w_d)|^2 \neq 0$.*

Theorem 5.4. *Recall that d is regular. Let $p \in \mathcal{Z}_k^{\mathbb{C}^\times}$ be such that $\sum_{\chi \in \mathfrak{F}_p} |\chi(w_d)|^2 \neq 0$. Then p belongs to $(\mathcal{Z}_k^{\boldsymbol{\mu}_d})_{\max}$.*

Example 5.5 (Symmetric group)**.** We assume here that $W = \mathfrak{S}_n$ acting on $V = \mathbb{C}^n$ by permutation of the coordinates for some $n \geq 2$. The canonical basis of $\mathbb{C}^n$ is denoted by $(y_1, \ldots, y_n)$. Then there is a unique orbit of hyperplanes, that we denote by Ω, and $e_\Omega = 2$. To avoid too easy cases, we also assume that $k_{\Omega,0} \neq k_{\Omega,1}$ (so that $\mathcal{Z}_k$ is smooth [13, Cor. 1.14]) and that $d \geq 2$. Saying that d is a regular number is equivalent to saying that d divides n or $n-1$. Therefore, we will denote by j the unique element of $\{0,1\}$ such that d divides $n-j$, and we set $r=(n-j)/d$. Then w_d is the product of r disjoint cycles of length d, so one can choose, for instance,

$$w_d = (1,2,\dots,d)(d+1,d+2,\dots,2d)\cdots((r-1)d+1,(r-1)d+2,\dots,rd).$$

Then V^{τ_d} is r-dimensional, with basis $(v_1,\dots,v_r)$ where $v_a = \sum_{b=1}^{d} \zeta_d^{-b} e_{(a-1)d+b}$ and the group $C_W(w_d) \simeq G(d,1,r)$ acts "naturally" as a reflection group on $V^{\tau_d} = \bigoplus_{a=1}^{r} \mathbb{C}v_a$.

We also need some combinatorics. We denote by $\mathrm{Part}(n)$ (resp. $\mathrm{Part}^d(r)$) the set of partitions of n (resp. of d-partitions of r). If $\lambda \in \mathrm{Part}(n)$, we denote by χ_λ the irreducible character of $\mathfrak{S}_n$ (with the convention of [15]: for instance, $\chi_n = 1$ and $\chi_{1^n} = \varepsilon$), by $\mathrm{cor}_d(\lambda)$ the d-core of λ, and by $\mathrm{quo}_d(\lambda)$ its d-quotient. We let $\mathrm{Part}(n,d)$ denote the set of partitions of n whose d-core is the unique partition of $j \in \{0,1\}$. Then the map

$$\mathrm{quo}_d : \mathrm{Part}(n,d) \longrightarrow \mathrm{Part}^d(r)$$

is bijective. Finally, if $\mu \in \mathrm{Part}^d(r)$, we denote by χ_μ the associated irreducible character of $C_W(w_d) = G(d,1,r)$ (with the convention of [14]). It follows from the Murnaghan–Nakayama rule that

$$\chi_\lambda(w_d) \neq 0 \quad \text{if and only if} \quad \lambda \in \mathrm{Part}(n,d), \tag{5.1}$$

and that

$$\chi_\lambda(w_d) = \pm\chi_{\mathrm{quo}_d(\lambda)}(1) \tag{5.2}$$

for all $\lambda \in \mathrm{Part}(n,d)$ (see, for instance, [9, p. 47]).

Now, the smoothness of $\mathcal{Z}_k$ implies that the map $\varphi_k : \mathrm{Irr}(\mathfrak{S}_n) \longrightarrow \mathcal{Z}_k^{\mathbb{C}^\times}$ is bijective (so that Calogero–Moser k-families of $\mathfrak{S}_n$ are singleton), and it follows from the main theorem of [5] that Conjecture 5.1 holds (except that we do not know if the isomorphism respects the Poisson structure), so that we have a $\mathbb{C}^\times$-equivariant isomorphism of varieties

$$\iota_k : (\mathcal{Z}_k^{\boldsymbol{\mu}_d})_{\max} \xrightarrow{\sim} \mathcal{Z}_{\lambda(k)}(V^{\tau_d}, G(d,1,r))$$

for some explicit $\lambda(k) \in \mathbb{C}^{\aleph_{\tau_d}}$. Moreover, $\mathcal{Z}_{\lambda(k)}(V^{\tau_d}, G(d,1,r))$ is smooth so that the map $\varphi_{\lambda(k)}^{\tau_d} : \mathrm{Irr}(G(d,1,r)) \longrightarrow \mathcal{Z}_{\lambda(k)}(V^{\tau_d}, G(d,1,r))^{\mathbb{C}^\times}$ is bijective (that is, Calogero–Moser $\lambda(k)$-families of $G(d,1,r)$ are singleton). Now, by [5, Thm. 4.21], we have that

$$\varphi_k(\chi_\lambda) \in (\mathcal{Z}_k^{\boldsymbol{\mu}_d})_{\max} \quad \text{if and only if} \quad \lambda \in \mathrm{Part}(n,d), \tag{5.3}$$

and that

$$\iota_k(\varphi_k(\chi_\lambda)) = \varphi_{\lambda(k)}^{\tau_d}(\chi_{\mathrm{quo}_d(\lambda)}) \tag{5.4}$$

for all $\lambda \in \mathrm{Part}(n,d)$. Then (5.1), (5.2), (5.3) and (5.4) show that Conjectures 5.3 and 5.2 hold for the symmetric group.

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