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GRADED ALMOST VALUATION RINGS

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Abstract. Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a commutative ring graded by an arbitrary torsionless monoid Γ. We say that R is a graded almost valuation ring (gr AV-ring) if for every two homogeneous elements a, b of R, there exists a positive integer n such that either a^n divides b^n (in R) or b^n divides a^n. In this paper, we introduce and study the graded version of the almost valuation ring which is a generalization of gr-AVD to the context of arbitrary Γ-graded rings (with zero-divisors). Next, we study the possible transfer of this property to the graded trivial ring extension $A \ltimes E$. Our aim is to provide examples of new classes of Γ-graded rings satisfying the above mentioned property.

1. Introduction

Throughout this paper, all rings are assumed to be commutative with identity and all modules are nonzero unitary, and Γ will denote a torsionless grading monoid (that is, a commutative, cancellative monoid and the quotient group of Γ, $\langle\Gamma\rangle = \{a - b \mid a, b \in \Gamma\}$ is a torsion-free abelian group).

In [4], Anderson and Zafrullah introduced and studied the notion of an almost valuation domain (in short, an AVD) and an almost Bezout domain (in short, an AB-domain) which are generalizations of valuation domain and Bezout domain, respectively. An integral domain R with quotient field K is called an *almost valuation domain* if for every nonzero $x \in K$, there exists an integer $n \geq 1$ such that either $x^n \in R$ or $x^{-n} \in R$. Among other things, they proved that the integral closure of an almost valuation domain is a valuation domain. The notion of gr-AVDs was recentely introduced by Bakkari, Mahdou and Riffi in [8] as follows. Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain and H be the set of nonzero homogeneous elements of R. We say that R is a *graded almost valuation domain* (in short, a gr-AVDs) if for every nonzero homogeneous element $x \in R_H$, there exists an integer $n = n(x) \geq 1$ with x^n or $x^{-n} \in R$; equivalently, for all nonzero homogeneous elements $a, b \in R$, there exists an integer $n = n(a, b) \geq 1$ with $a^n \mid b^n$ or $b^n \mid a^n$ in R. It is clear that any gr-valuation domain is a gr-AVD; the proof of [8, Theorem 5.6] showed that if R is a gr-AVD, then $\bar{R}$ (the integral closure of R)

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is a gr-valuation domain. In [10] and [15], a generalization of AVDs to the context of arbitrary rings was considered as follows. A ring R is called an *almost valuation ring* (in short, an AV-ring) if, for any two elements a and b in R, there exists a positive integer n such that either a^n divides b^n (in R) or b^n divides a^n. Clearly, any valuation ring is an AV-ring; the converse fails [15, Examples 2.3]. However, Proposition 2.2 in [10] showed that any AV-ring is quasi-local with linearly ordered prime ideals.

An integral domain R is an *AB-domain* if for $a, b \in R \setminus \{0\}$ there is n such that (a^n, b^n) is principal. The notion of almost Bezout domains runs along lines somewhat similar to those of Bezout domains (i.e., every two generated, equivalently, every finitely generated, ideal is principal). In [5], Anderson, Knopp, and Lewin continued the study of almost Bezout domains, and after observing that each almost Bezout domain is nearly Bezout, they used the construction $K + XL[X]$ to disprove the converse. In [15], the generalization of the almost Bezout domains to arbitrary commutative rings (with zero-divisors) is considered as follows: R is called an *almost Bezout ring* (AB-ring for short) if, for any two elements a and b in R, there exists a positive integer n such that the ideal (a^n, b^n) is principal.

Our aims is to generalize the concepts of AV-rings and AB-rings to the context of arbitrary Γ-graded rings (with zero-divisors); and then completely transfer these notions to the graded trivial ring extension $A \ltimes E$.

Note that the valuation in the context of rings (with zero-divisors) is defined in two different ways which are not equivalent. Following [14], we say that a ring R is a *valuation ring* if, for any nonzero elements $a, b \in R$, either $Ra \subseteq Rb$ or $Rb \subseteq aR$; In contrast, Huckaba's definition of a valuation ring [9] adds the requirement that at least one of these elements be regular. Any "valuation ring" (in our sense) is, without a doubt, a valuation ring in the sense of [9], but not conversely. This serves as a primary justification for beginning our paper, after recalling some basic background in Section 2 that will be needed in the present work, with a brief section (Section 3) in which we attempt to explain the distinction between the two definitions and their relation with the almost valuation ring.

In Section 4, we introduce the notion of graded almost-valuation rings (gr AV-rings). Among other things, we show that a nontrivially graded ring is never an AV-ring (Proposition 4.3). It is clear that any gr-valuation ring is a gr AV-ring (Proposition 4.4); Examples 4.4 and 4.17 show that the converse fails; but a gr AV-ring $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ must have a unique maximal homogeneous ideal (Corollary 4.7). Also, we characterize the trivial ring extension to be gr-AV ring when it is given a trivial graduation by $\mathbb{Z}_2$ (Proposition 4.16). As an immediate application of this, Example 4.26 shows the failure of Bakkari, Mahdou and Riffi's theorem on the integral closure of a gr-AVD beyond the context of graded integral domains. Then we study the possible transfer of this generalized property for the graded trivial ring extension ($A \ltimes E$). A generalization of the notion of an almost Bezout ring (AB ring) to the context of arbitrary Γ-graded rings brings this section to close, in which we study the possible transfer of this generalized property to the graded trivial ring extensions. For the main transfer result in this paper, see Theorems 4.21 and 4.29.

As we proceed to study the above-mentioned classes of graded rings, the reader may find it helpful to keep in mind the implications shown in the following figure, which are obtained from the results of this paper and are not reversible.

$$\begin{tikzcd}
& \text{Valuation ring} \arrow[dl] \arrow[dr] & \\
\text{Almost valuation ring} \arrow[dr] & & \text{gr-valuation ring} \arrow[dl] \\
& \text{gr-almost valuation ring} \arrow[d] & \\
& \text{gr-quasi-local ring with linearly ordered prime homogeneous ideals} &
\end{tikzcd}$$

Let A be a ring and E be an A-module. Then the ring $A \ltimes E$ with coordinate-wise addition and multiplication given by $(a_1, e_1)\,(a_2, e_2) = (a_1a_2,\, a_1e_2 + a_2e_1)$ is a ring with unity $(1, 0)$ (even R-algebra) called *idealization* of E or the *trivial ring extension* of A by E. Note that A naturally embeds into $A \ltimes E$ by $a \mapsto (a, 0)$. If N is a submodule of E, then $0 \ltimes N$ is an ideal of $A \ltimes E$ and $0 \ltimes E$ is a nilpotent ideal of $A \ltimes E$ of index 2. It is well known that $I \ltimes N$ is an ideal of $A \ltimes E$ if and only if I is an ideal of R and N is a submodule of E such that $IE \subseteq N$, cf. [3, Theorem 3.1].

Let Γ be a commutative monoid. Suppose that $A = \bigoplus_{\alpha\in\Gamma} A_\alpha$ is a Γ-graded ring and $E = \bigoplus_{\alpha\in\Gamma} E_\alpha$ a Γ-graded A-module. Then $A \ltimes E$ is a Γ-graded ring with $(A \ltimes E)_\alpha = A_\alpha \bigoplus E_\alpha$ for every $\alpha \in \Gamma$ (cf. [6, Proposition 2]). Consequently, $h(A \ltimes E) = \cup_{\alpha\in\Gamma}(A \ltimes E)_\alpha$.

2. Preliminaries

This section presents some basic properties of graded rings and modules used in what follows. Let Γ be a torsionless grading monoid (written additively), with an identity element denoted by 0, and the quotient group of Γ, $\langle\Gamma\rangle = \{a - b \mid a, b \in \Gamma\}$ is a torsion-free abelian group. It is well known that a cancellative monoid is torsionless if and only if can be given a total order compatible with the monoid operation [17, p. 123].

Recall that a (not necessarily unital) ring R is called a Γ-graded ring, or simply a graded ring, if $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$, each R_γ is an additive subgroup of R and $R_\gamma R_\delta \subseteq R_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. The set $h(R) = \bigcup_{\gamma\in\Gamma} R_\gamma$ is called the set of homogeneous elements of R. The nonzero elements of R_γ are called homogeneous of degree γ and we write $\deg(r) = \gamma$ if $r \in R_\gamma\backslash\{0\}$. We call the set

$$\Gamma_R = \{\gamma \in \Gamma \mid R_\gamma \neq 0\}$$

the support of R. We say R has a trivial grading, or R is concentrated in degree zero if the support of R is the trivial group, i.e., $R_0 = R$ and $R_\gamma = 0$ for $\gamma \in \Gamma \backslash \{0\}$. Clearly R_0 is a subring of R (intuitively $1 \in R_0$) and every R_α is an R_0-module. Note that every unit of R is homogeneous.

By a graded R-module E, we mean an R-module graded by Γ, that is, a direct sum of subgroups E_α of E such that $R_\alpha E_\beta \subseteq E_{\alpha+\beta}$ for every $\alpha, \beta \in \Gamma$. Let R and R' be two graded rings. Then a ring homomorphism $f : R \to R'$ is called graded if $f(R_\alpha) \subseteq R'_\alpha$ for all $\alpha \in \Gamma$. Let I be an ideal of R. Then I is called a homogeneous ideal of R if one of the following equivalent conditions hold: (i) $I = \bigoplus_{\alpha\in\Gamma} I_\alpha$, where $I_\alpha = I \cap R_\alpha$ for all $\alpha \in \Gamma$, and (ii) $a = a_{\alpha_1} + a_{\alpha_2} + \cdots + a_{\alpha_n} \in I$ implies that $a_{\alpha_i} \in I$, where $a_{\alpha_i} \in R_{\alpha_i}$. Similarly, a submodule N of M is called a homogeneous submodule if and only if $N = \bigoplus_{\alpha\in\Gamma} N_\alpha$, where $N_\alpha = N \cap M_\alpha$ for all $\alpha \in \Gamma$ if and only if $m = m_{\alpha_1} + m_{\alpha_2} + \cdots + m_{\alpha_n} \in N$ implies that $m_{\alpha_i} \in N$, where $m_{\alpha_i} \in M_{\alpha_i}$. If I is a homogeneous ideal of a graded ring $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$, then $R/I = \bigoplus_{\alpha\in\Gamma} (R/I)_\alpha$ is a graded ring, where $(R/I)_\alpha := (R_\alpha + I)/I$. A homogeneous ideal P of R is called a prime homogeneous ideal (gr-prime) if P is a proper homogeneous ideal of R with the property that $a, b \in h(R)$ and $ab \in P$ implies either $a \in P$ or $b \in P$. A homogeneous ideal M of R is called a maximal homogeneous ideal (gr-maximal) if it is maximal among proper homogeneous ideals; equivalently, if every nonzero homogeneous element of R/M is invertible.

A graded ring is said to be graded quasi local (gr-quasi local) if it has a unique maximal homogeneous (gr-maximal) ideal and a graded ring A is called a graded-field (gr-field) if every nonzero homogeneous element of R is invertible. Clearly, every field is a graded field, however, the converse is not true in general, see [16, p. 46].

Let R_1 and R_2 be two graded rings. Then $R = R_1 \times R_2$ is a graded ring with homogeneous elements $h(R) = \bigcup_{\alpha\in\Gamma} R_\alpha$, where $R_\alpha = (R_1)_\alpha \times (R_2)_\alpha$ for all $\alpha \in \Gamma$. It is well known that an ideal of $R_1 \times R_2$ is of the form $I_1 \times I_2$ for some ideals I_1 of R_1 and I_2 of R_2. Also it is easily seen that $I_1 \times I_2$ is a homogeneous ideal of $R_1 \times R_2$ if and only if I_1, I_2 are homogeneous ideals of R_1 and R_2, respectively.

Let R be a graded ring, and let $tq(R)$ denote the total ring of quotients of R and H the saturated multiplicative set of regular homogeneous elements of R. Then, by extending some definitions to the case where rings are with zero divisors, R_H, called the homogeneous total ring of quotients of R, is a ring graded by $\langle\Gamma\rangle$, where $R_H = \bigoplus_{\alpha\in\langle\Gamma\rangle} (R_H)_\alpha$ with

$$(R_H)_\alpha = \left\{ \frac{r}{s} \mid r \in R_\beta, s \text{ a regular element of } R_\gamma \text{ and } \beta - \gamma = \alpha \right\}.$$

If R is a graded integral domain (an integral domain graded by Γ), then R_H is called the homogeneous quotient field of R. Clearly, every nonzero homogeneous element of R_H is invertible and $(R_H)_0$ is a field.

We will be using the following definition (which agrees with the classical one if R is a graded integral domain). A graded R-module E is said to be a *torsion R-graded module* if, for each homogeneous $e \in E$, there exists $a \in R \setminus \{0\}$ such that $ae = 0$. We will also use the following standard definitions. A *regular homogeneous element* of a graded ring R is a non-zero-divisor homogeneous element; a graded R-module E is *gr-divisible* if, for each homogeneous $e \in E$ and each regular homogeneous element a of R there exists $f \in E$ such that $e = af$; a graded A-module E is a *torsion-free* (graded A-*module*) if whenever $a \in h(A)$ and $e \in E$ with $ae = 0$ implies that either $a = 0$ or $e = 0$. Lastly, as usual, for any Γ-graded ring A, h-Spec(R) denotes the set of prime homogeneous ideals of R, h-Z(R) denotes the set of all homogeneous zero-divisors of R and h-Reg(R) denotes the set of regular homogeneous elements of R.

3. Some remarks of valuation-like properties

Recall from the introduction that a ring R is called an almost valuation ring if, for any two elements a and b in R, there exists a positive integer n such that either a^n divides b^n (in R) or b^n divides a^n. Following [1], a ring R is said to be a valuation ring if, for all $a, b \in R$ such that $\{a, b\} \nsubseteq Z(R)$, either $Ra \subseteq Rb$ or $Rb \subseteq Ra$ as defined in Huckaba's book [9]; but it is not equivalent to the definition of "valuation ring" used by Kaplansky [14, p. 35], as Kaplansky's definition omits the above stipulation that $\{a, b\} \nsubseteq \mathrm{Z}(R)$. It is clear that any "valuation ring" in the sense of [14] is an AV-ring, but a "valuation ring" in the sense of [1] is not necessary an AV-ring, as shown by the following example.

Recall that a ring R is said to be a total quotient ring if each regular element of R is unit (each nonunit element of R is in $Z(R)$). Obviously, every total quotient ring is a valuation ring in the sense of [1, 9].

Example 3.1. Let A be a non-AV-ring. Set $M = \bigoplus_{I \in \Delta} A/I$, where Δ is the set of all ideals of A. Let x be a nonunit element of A, then $I_0 = xA \in \Delta$. So $x\,(r_I)_{I\in\Delta} = 0$, where $r_I = 0$ for every $I \neq I_0$ and $r_{I_0} = 1 + I_0$, hence $x \in Z(M)$. Consequently, $Z(M) = A \setminus U(A)$. Consider the trivial ring extension $R = A \ltimes M$. So $Z(R) = \{(a, m) \mid a \text{ is a nonunit element of } A, m \in M\}$ by [9, Theorem 25.3]. Therefore R is a valuation ring in the sense of [1] since R is a total quotient ring; but according to [15, Theorem 2.1] R is not a AV-ring since A is not.

Note that any valuation ring in the sense of Kaplansky [14] is a valuation ring in the sense of [1, 9]; but the converse is not true in general as the following example clarifies.

Example 3.2. Consider a non-valuation ring A in the sense of [14] (for instance, a non-quasi-local ring). Set $R = A \ltimes M$ the trivial ring extension of A by M with $M = \bigoplus_{I\in\Delta} A/I$. Then R is a valuation ring in the sense of [1] (cf. [Example 3.1]) but not a valuation ring in the sense of [14], since A is not [13, Lemma 2.2].

Remark 3.3. Note that the concept of valuation rings in the both senses coincide in the case where the ring A is an integral domain. Then let F be a finite field and X an indeterminate over F. Put $H := F(X)$, the quotient field of $F[X]$, and let Y be an analytic indeterminate over H. Set $D := H + Y^3H[[Y]]$. Then by [7, Example 2.20] D is an almost valuation domain, which is not a valuation domain.

Hence the valuation ring in Huckaba's sense [1] is unrelated to the almost valuation ring. We can clarify the meaning of each implication by the following figure; none of the implications are reversible:

$$\begin{tikzcd}
 & \text{Valuation ring in [14]} \arrow[dl] \arrow[dr] & \\
\text{Almost valuation ring} & & \text{Valuation ring in [1]}
\end{tikzcd}$$

4. Main results

There are two reasons why we begin by extending the definiton of a valuation ring in Kaplansy's sense [14, p. 35] to the case of Γ-graded rings. First, Definition 4.1 will be our framework of this study (unless otherwise stated, a gr-valuation ring is as defined there). Second, Definition 4.1 will be used in a proof later in this paper. That theme will motivate the choice of the contexts studied in our later results, which, for the most part, seek to identify situations admitting positive transfer results for the other graded ring-theoretic properties being considered here.

Definition 4.1. A graded ring $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is said to be a *graded valuation ring* (gr-valuation ring) if every homogeneous elements $a, b \in R$, either $aR \subseteq bR$ or $bR \subseteq aR$.

Obviously, every valuation graded ring is a gr-valuation ring. The converse is not true in general, as shown by the following construction.

Example 4.2. Let R be a gr-valuation ring. Pick a homogeneous element $x \in R$ with $\deg(x) \neq 0$, then $(1 + x + x^3)R$ is not comparable with $(1 + x^2)R$ under inclusion. Hence R is a gr-valuation ring which is not valuation.

We say R is a graded almost valuation ring (gr AV-ring) if for every homogeneous elements $a, b \in R$, there exists an integer $n \geq 1$ such that $a^n R \subseteq b^n R$ or $b^n R \subseteq a^n R$; equivalently, if for any two homogeneous elements $a, b \in R$, there exists an integer n such that a^n divides b^n or b^n divides a^n. Obviously, the concepts of "gr AV-rings" and "AV-rings" coincide when the ring is trivially graded. We next clarify the situation for nontrivially graded rings.

Proposition 4.3. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a nontrivially graded ring. Then R is never an AV-ring. In particular, if R is a nontrivially graded AV-ring, then R is never an AV-ring.*

Proof. Choose a nonzero homogeneous element x of R with a nonzero degree. Since R is nontrivially graded, then $1 + x^2 + x^3$ and $x + x^2$ are nonhomogeneous elements of R and so are nonunits of R. Therefore R is not quasi-local. Hence, by [10, Proposition 2.2], R is not a AV-ring, as desired. ☐

The following result is straightforward.

Proposition 4.4. *Every gr-valuation ring $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ is a gr-AV ring.*

The converse of Proposition 4.4 fails. The obvious example is a nonvaluation AV-ring [10, Example 2.5]; also, we can quote the failure of Anderson–Zafrullah's theorem beyond the context of integral domains [15, Example 2.3] which are trivially graded; for nontrivially graded examples see Examples 4.17, 4.25 and 4.26.

Proposition 4.5. *Let R be a gr AV-ring and I be a homogeneous ideal of R. Then R/I is a gr AV-ring. In particular, if D is a gr-AVD and I is a non-prime homogeneous ideal of D then D/I is a gr AV-ring with nonzero zero-divisors.*

Proof. Put $\widetilde{R} := R/I$. Let x, y be two homogeneous elements in R/I. Pick two homogeneous elements $a, b \in R$ such that $x = a + I$ and $y = b + I$. Since R is a gr AV-ring, there exists an integer $n \geq 1$ such that either $a^nR \subseteq b^nR$ or $b^nR \subseteq a^nR$. If $a^nR \subseteq b^nR$, then

$$x^n\widetilde{R} = (a^n + I)(R/I) = (a^nR + I)/I \subseteq (b^nR + I)/I = y^n\widetilde{R}.$$

Similarly, if $b^nR \subseteq a^nR$, then $y^n\widetilde{R} \subseteq x^n\widetilde{R}$. The proof is complete. □

Recall that for a proper homogeneous ideal I of a graded ring R, the graded radical of I will be designated by $\mathrm{Gr}(I) = \{x = \sum_{g\in\Gamma} x_g \in R :$ for each $g \in \Gamma$, there exists $n_g \in \mathbb{N}$ such that $x_g^{n_g} \in I\}$. It is straightforward to see that $\mathrm{Gr}(I)$ is always a homogeneous ideal of R. Note that, if x is a homogeneous element, then $x \in \mathrm{Gr}(I)$ if and only if $x^n \in I$ for some positive integer n (see [18]). Now, we determine the gr-almost valuation ring in terms of graded radical ideals.

Proposition 4.6. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a gr AV-ring. Then $I \subseteq \mathrm{Gr}(J)$ or $J \subseteq \mathrm{Gr}(I)$, for every homogeneous ideals I and J in R. In particular, the graded radical ideals of R are linearly ordered.*

Proof. Assume that R is a gr AV-ring. Let I and J be two homogeneous ideals of R. If $I \not\subseteq \mathrm{Gr}(J)$ and $J \not\subseteq \mathrm{Gr}(I)$, then there exist homogeneous elements $y \in I \setminus \mathrm{Gr}(J)$ and $x \in J \setminus \mathrm{Gr}(I)$ respectively. Since R is a gr AV-ring, there exist a positive integer n such that either $x^n \mid y^n$ or $y^n \mid x^n$. We may assume, without loss of generality, that $x^n \mid y^n$. Then $y^n = rx^n$ for some $r \in R$, so $y^n \in J$. Consequently, $y \in \mathrm{Gr}(J)$, which is a contradiction. Hence $I \subseteq \mathrm{Gr}(J)$ or $J \subseteq \mathrm{Gr}(I)$. □

The following result is an immediate corollary of Proposition 4.6.

Corollary 4.7. *Let R be a gr AV-ring. Then the prime homogeneous ideals of R are linearly ordered. In particular, R has a unique maximal homogeneous ideal.*

Recall that an overring of a ring R is a subring of the total quotient ring of R that contains R. An overring T of R is called a homogeneous overring of R if $T \subseteq R_H$ and $T = \bigoplus_{\alpha\in\langle\Gamma\rangle}(T \cap (R_H)_\alpha)$; that is T is a graded subring of R_H. Clearly, for any homogeneous ideal I of R, the subset $(I : I) = \{x \in R_H \mid xI \subseteq I\}$ is a homogeneous overring.

Proposition 4.8. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a gr AV-ring and T be a homogeneous overring of R. Then T is a gr AV-ring.*

Proof. Let x and y be two homogeneous elements of T. Then $x = a/s$ and $y = b/t$ for some homogeneous $a, b \in R$ and regular homogeneous elements $s, t \in R$. Since R is a gr AV-ring, there exists an integer $n \geq 1$ such that $(at)^n R \subseteq (bs)^n R$ or $(bs)^n R \subseteq (at)^n R$. It follows that there is a $r \in R$ such that $(at)^n = (bs)^n r$ or $(bs)^n = (at)^n r$, and so $x^n = (a/s)^n = (b/t)^n (r/1) = y^n (r/1)$ or $y^n = x^n (r/1)$. Therefore, $x^n T \subseteq y^n T$ or $y^n T \subseteq x^n T$, which leads to the fact that T is a gr AV-ring. $\square$

We next give an example of a gr AV-ring for the reader's convenience.

Example 4.9. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded ring in which every homogeneous element is either a unit or nilpotent. Then R is a gr AV-ring. In particular, if we consider the graded trivial ring extension $R = K \ltimes E$ with K is a field and E is an K-vector space graded by $\mathbb{Z}_2$ via $R_0 = K \ltimes 0$ and $R_1 = 0 \ltimes E$. Then R is a gr AV-ring.

Recall from [4, p. 288] that an extension $R \subseteq T$ of a ring is said to be a root extension of R if, for every $x \in T$, there exists an integer $n \geq 1$ such that $x^n \in R$. Analogously, we can define this notion in the setting of Γ-graded rings as follows. Let $T = \bigoplus_{\alpha \in \Gamma} T_\alpha$ be a graded ring and R a homogeneous subring of T. Then the extension $R \subseteq T$ is called a graded root extension (gr-root extension) if for every homogeneous element $x \in T$, there exists an integer $n \geq 1$ such that $x^n \in R$. Obviously, if $R \subseteq T$ is a root extension, then it is a gr-root extension; the converse is not true in general: the example is given by the polynomial rings $R := K[X] \subseteq L[X] =: T$, where $K \subsetneq L$ are finite fields and R, T are $\mathbb{Z}_+$-graded with $\deg(aX^n) = n$ for every $0 \neq a \in L$ and $n \in \mathbb{Z}_+$ (cf. [2, p. 550]). The following theorem characterizes gr-AV rings (cf. [10, Theorem 3.4]).

Theorem 4.10. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded quasi-local ring with maximal homogeneous ideal M. Set*

$$\mathrm{Rad}_V(M) = \Big\{ x = \sum_{g \in \Gamma} x_g \in V \mid \forall g \in \Gamma, \exists n_g \in \mathbb{N} : x_g^{n_g} \in M \Big\},$$

where V is a gr AV-homogeneous overring of R such that M is a homogeneous ideal of V and $\mathrm{Rad}_V(M)$ is the maximal homogeneous ideal of V. Then R is a gr AV-ring if and only if V is a gr-root extension of R.

Proof. If $V = R$, then the assertions are clear in this case. Hence, we may assume that $V \neq R$ and R is a gr AV-ring. Pick a homogeneous element $x_g \in V \backslash R$. Then $x_g = a/b$ for some homogeneous a, $b \in R$, where b is a regular homogeneous element and $\deg(a) - \deg(b) = g$. If $x_g \in \mathrm{Rad}_V(M)$, then $x_g^n \in M \subset R$ for some $n \geq 1$. Now, assume that $x_g \notin \mathrm{Rad}_V(M)$. Since $\mathrm{Rad}_V(M)$ is the maximal homogeneous ideal of V, x_g is a unit of V, and so a is a regular homogeneous element of R. Since R is a gr AV-ring, there exists an integer $n \geq 1$ such that $a^n R \subseteq b^n R$ or $b^n R \subseteq a^n R$. In the first case, if $a^n R \subseteq b^n R$, then $x_g^n \in R$. If $x_g^{-n} \in M$, then $x_g^{-1} \in \mathrm{Rad}_V(M)$, which is a contradiction. Hence x_g^{-n} is a unit of R, and so $x_g^n \in R$. Therefore V is a gr-root extension of R.

Conversely, assume that V is a gr-root extension of R and a, b are homogeneous elements of R. Since V is a graded almost valuation ring, there is an $n \geq 1$ such that $a^nV \subseteq b^nV$ or $b^nV \subseteq a^nV$. Assume that $a^nV \subseteq b^nV$ for some $n \geq 1$. Then there exists $y \in V$ such that $a^n = b^n y$, from which it is easy to check that y is homogeneous in V. Since V is a gr-root extension of R, there is an $m \geq 1$ such that $y^m \in R$, and so $a^{mn} = b^{mn}y^m \in b^{mn}R$. Hence, we get $a^{mn}R \subseteq b^{mn}R$, and consequently R is a graded almost valuation ring. □

Let $R \subseteq S$ be an extension of graded rings. It is obvious that h-Reg$(S) \cap R \subseteq$ h-Reg(R). However, the reverse inequality need not hold. Thus, it is not required that h-Reg$(S) \cap R =$ h-Reg(R). (To put it another way, it is not necessary that S is a torsion-free graded R-module, in the usual sense of the term). This equality is frequently desired because it is equivalent to the statement that the universal mapping property of graded rings of fractions allows the inclusion map $R \hookrightarrow S_H$ to extend to a (unique, necessarily injective) graded ring homomorphism $R_H \to S_H$, in which case we use that injection to view $R_H \subseteq S_H$. It is natural to wonder if there are relevant kinds of base graded rings R and graded ring extensions $R \subseteq S$ that admit an embedding of R_H into S_H in this way. We begin with a closely similar result, following the next useful remark.

Remark 4.11. A homogeneous element $x \in$ h-Reg(R) if and only if $xr = 0$ implies that $r = 0$ for every homogeneous element $r \in R$.

Proof. If $x \in$ h-Reg(R), then naturally $xr = 0$ implies that $r = 0$ for every homogeneous element $r \in R$. Conversely, assume that if $xr = 0$, then $r = 0$ for every homogeneous element $r \in R$. Let $r = \sum_{g\in\Gamma} r_g \in R$ such that $xr = 0$, then $x(\sum_{g\in\Gamma} r_g) = \sum_{g\in\Gamma} xr_g = 0$, which implies that $xr_g = 0$ for every $g \in \Gamma$ and so all $r_g = 0$. Consequently, $r = 0$; that is, $x \in$ h-Reg(R). □

Lemma 4.12. *Let $R \subseteq S$ be a gr-root extension of rings with S (and hence R) being a gr-reduced ring. Then* h-Reg$(S) \cap R =$ h-Reg(R) *and $R_H \subseteq S_H$ is a gr-root extension.*

Proof. For the first assertion, we need only show that if $x \in$ h-Reg(R), then $x \in$ h-Reg(S). Suppose that this fails, by Remark 4.11, we may pick a nonzero homogeneous element $y \in S$, such that $xy = 0$. Since $R \subseteq S$ is a gr-root extension, there exists an integer $n \geq 1$ such that $y^n \in R$. Then $x^ny^n = (xy)^n = 0$. As h-Reg(R) is a multiplicatively closed set, $x^n \in$ h-Reg(R), and so $y^n = 0$. Since S is gr-reduced, $y = 0$, the desired contradiction. This completes the proof of the first assertion.

In light of the first assertion, it follows from the above comments that we may view $R_H \subseteq S_H$. It remains only to show that this is a gr-root extension. Given a homogeneous $u \in S_H$, we must find an integer $k \geq 1$ such that $u^k \in R_H$. Write $u = a/z$, with $a \in h(S)$ and $z \in$ h-Reg(S). Since $R \subseteq S$ is a root extension, there exist integers $n \geq 1$ and $m \geq 1$ such that $a^n \in R$ and $z^m \in R$. Note that $(a^n)^m \in R^m = R$. Moreover, since h-Reg(S) is a multiplicatively closed set, z^m and $(z^m)^n$ are elements of h-Reg(S) (and of R). Consequently, by the first assertion,

$(z^m)^n \in$ h-Reg(R). Since

$$u^{nm} = \frac{(a^n)^m}{(z^m)^n} \in S_H,$$

it follows that $u^{\mathrm{nm}} \in R_H$. Therefore, taking $k := nm$ completes the proof. ☐

While going further into the "gr-root extension" hypothesis, we next continue the project of generalizing some ring-theoretic observations and results from [11] and [10] to the graded ring-theoretic context.

Theorem 4.13. *The following statements hold.*

(1) *Let $R \subseteq S$ be a gr-root extension of rings. Then R is a gr AV-ring if and only if S is a gr AV-ring.*

(2) *Let $R \subseteq S$ be a gr-root extension of rings. Then the following conditions are equivalent:*

  (a) *R is a gr AV-ring;*

  (b) *S is a gr AV-ring;*

  (c) *T is a gr AV-ring for each graded ring T such that $R \subseteq T \subseteq S$.*

If, in addition, S is a gr-reduced ring, then the above conditions (a)–(c) *are equivalent to:*

  (d) *T is a gr AV-ring, for each graded ring T such that $R \subseteq T \subseteq S_H$.*

Proof. (1) Consider two nonzero homogeneous elements $x, y \in S$. As $R \subseteq S$ is a gr-root extension, there exists an integer $n \geq 1$ such that $x^n, y^n \in R$. Since R is a gr AV-ring, there exists an integer $m \geq 1$ such that either x^{nm} divides y^{nm} in R or y^{nm} divides x^{nm} in R. Therefore, either x^{nm} divides y^{nm} in S or y^{nm} divides x^{nm} in S. Thus, S is a gr AV-ring. Conversely, consider two nonzero homogeneous elements $x, y \in R$. Then $x, y \in S$. Since S is a gr AV-ring, there exists a positive integer k such that either $x^k = ay^k$ or $y^k = bx^k$ for some elements $a, b \in S$ (i.e, a, b are homogeneous in S). Since $R \subseteq S$ is a gr-root extension, there exists a positive integer n such that $a^n, b^n \in R$. Hence, either $x^{kn} = a^n y^{kn}$ or $y^{kn} = b^n x^{kn}$. Since $a^n, b^n \in R$, this completes the proof that R is a gr AV-ring.

(2) For any graded rings extensions $R \subseteq A \subseteq B \subseteq S$, it is clear that the graded ring extension $A \subseteq B$ inherits the "gr-root extension" property from $R \subseteq S$. Hence, the equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) follow from (a) and (b).

Next, assume that S is gr-reduced. Then by Lemma 4.12 (and the discussion preceding it), we have $(R \subseteq) R_H \subseteq S_H$. Since (d) $\Rightarrow$ (c) trivially, it will suffice to show that (b) $\Rightarrow$ (d).

Assume that S is a gr AV-ring (and that S is gr-reduced). Our task is to show that if T is a graded ring such that $R \subseteq T \subseteq S_H$, then T is a gr AV-ring. To that end, consider two nonzero homogeneous elements $x, y \in T$. Then $x, y \in S_H$. As S_H inherits the "gr AV-ring" property from S by Proposition 4.8, there exists an integer $n \geq 1$ and $\alpha, \beta \in S_H$ (which are homogeneous) such that either $x^n = \alpha y^n$ or $y^n = \beta x^n$. Without loss of generality, we assume that $x^n = \alpha y^n$. Since $R_H \subseteq S_H$ is a gr-root extension by Lemma 4.12, there exists an integer $m \geq 1$ such that $\alpha^m \in R_H$. Thus, $\alpha^m = c/d \in R_H$, for some homogeneous elements $c \in R$ and $d \in$ h-Reg(R). As S is a gr AV-ring, there exist homogeneous

$u, v \in S$ and an integer $k \geq 1$ such that either $c^k = ud^k$ or $d^k = vc^k$. Again as $R \subseteq S$ is a gr-root extension, there exists a positive integer p such that u^p, $v^p \in R \subseteq T$. Therefore, either $x^{nmkp} = u^p y^{nmkp}$ or $y^{mmkp} = v^p x^{nmkp}$. (The handling of the latter possibility involved a somewhat subtle additional use of Lemma 4.12. In detail, if $d^k = vc^k$, one obtains $x^{nmkp} = y^{nmkp}/v^p$ en route to showing that $y^{nmkp} = v^p x^{nmkp}$, and this use of "fractional" notation is legitimate since $v^p \in \text{h-Reg}(S) \cap R = \text{h-Reg}(R)$, the underlying point being that $vc^k = d^k \in \text{h-Reg}(R) \subseteq \text{h-Reg}(S)$ ensures that v is an element of the multiplicatively closed set h-Reg(S).) Thus, either $x^{nmkp}T \subseteq y^{nmkp}T$ or $y^{nmkp}T \subseteq x^{nmkp}T$. This proves that T is a gr AV-ring. The proof is complete. □

Example 4.14. Let R be a graded integral domain but not a gr AV-domain; it is nonetheless the case that $S = R_H$ is the homogeneous quotient field of R is a gr AV-domain. Thus, the equivalence (a) ⇔ (b) in Theorem 4.13 (2) cannot be expected to hold for an arbitrary graded extension of domains (let alone an arbitrary pair of graded rings) $R \subseteq S$. What led to that equivalence holding in Theorem 4.13 (2) was the assumption that $R \subseteq S$ is a gr-root extension.

Remark 4.15. Note that for a ring A and an A-module E, the trivial ring extension $R = A \ltimes E$ is naturally graded via $\mathbb{N}$ where the subgroups are defined, in [3], as follows: $R_0 = A \ltimes 0$, $R_1 = 0 \ltimes E$ and $R_n = 0$ for $n \geq 2$, which can simply be viewed as a $\mathbb{Z}_2$-grading since $(0 \ltimes E)^2 = 0$.

The following proposition characterizes when $R = A \ltimes E$ is a gr-AV ring if R is trivially graded.

Proposition 4.16. *With the notation of Remark* 4.15*, let A be a ring, E an A-module, and the trivial ring extension $R = A \ltimes E$. Then $R = A \ltimes E$ is a gr-AV ring if and only if A is an AV-ring.*

Proof. Let $x, y \in A$; then $(x, 0), (y, 0)$ are homogeneous elements (of a zero degree) in R. Since R is a gr AV-ring, there exists a positive integer n such that $(x, 0)^n \in R(y, 0)^n$ or $(y, 0)^n \in R(x, 0)^n$. Thus, $x^n \in Ay^n$ or $y^n \in Ax^n$, and hence A is an AV-ring. Conversely, let x and y be two homogeneous elements in R. Two cases are then possible.

Case 1. $\deg(x) = \deg(y) = 0$. Then $x = (a, 0)$ and $y = (b, 0)$ for $a, b \in A$. Since A is an AV-ring, there exists a positive integer n such that either $a^n A \subseteq b^n A$ or $b^n A \subseteq a^n A$. We may assume, without loss of generality, that $a^n A \subseteq b^n A$. Then $a^n = b^n c$ for some $c \in A$, we have

$$x^n = (a, 0)^n = (b^n c, 0) = (c, 0)(b^n, 0) = (c, 0)y^n \in y^n R.$$

Case 2. Either $\deg(x) = 1$ or $\deg(y) = 1$. Without loss of generality, assume that $\deg(x) = 1$, then $x = (0, e)$ for some $e \in E$. So $x^2 = (0, 0) \in y^2 R$, as desired. □

Using the notation of Remark 4.15 and Proposition 4.16, we can easily construct straightforward examples of gr AV-rings which are not gr-valuation rings.

Example 4.17. (1) Let A be an AV-ring which is not a valuation ring (for instance, see Remark 3.3) and M an A-module. Set $R = A \ltimes M$ the trivial ring extension. Then R is a gr-AV ring since A is a AV-ring by Proposition 4.16; as A is not valuation ring, there exist two elements $x, y \in A$ such that xA and yA are not comparable under inclusion. Hence the two homogeneous ideals $(x,0)R$ and $(y,0)R$ of R are not comparable under inclusion. Hence R is a gr AV-ring which is not a gr-valuation ring.

(2) Let A be a valuation ring. Set $R = A \ltimes M$ the trivial ring extension with $M = A \oplus A$. Then R is a gr AV-ring since A is an AV-ring. On the other hand, the two homogeneous ideals $(0,a)R$ and $(0,b)R$ of R are not comparable under inclusion with $a = (0,1)$ and $b = (1,0)$, consequently R is not a gr-valuation ring.

Beyond the trivial context of graded rings, the following theorem shows that, for a useful kind of condition, one can characterize when the graded trivial ring extension $R = A \ltimes E$ is a gr-AV ring.

Theorem 4.18. *Let A be a gr-AV ring and $aE = a^2E$ for all $a \in h(A)$. Then $R = A \ltimes E$ is a gr-AV ring.*

Proof. Let $\alpha := (a,x)$ and $\beta := (b,y) \in h(R)$. Since $a, b \in h(A)$ and A is a gr-AV ring, there exists $n \geq 1$ such that either $Aa^n \subseteq Ab^n$ or $Ab^n \subseteq Aa^n$. Without loss of generality, we may assume that $Aa^n \subseteq Ab^n$. Then we can write $a^n = b^n c$ for some $c \in h(A)$. Then we have $a^{2n} = b^{2n}c^2$. Now we will show that $R\alpha^{2n} \subseteq R\beta^{2n}$. To show that the inclusion, we must show that $(a,x)^{2n} = (b,y)^{2n}(c^2, e)$ for some $e \in A$. First note that $a^{2n-1}E = a^{2n}E$ and $b^{2n-1}E = b^{2n}E$ since $aE = a^2E$ and $bE = b^2E$. As $2na^{2n-1}x = a^{2n-1}2nx \in a^{2n-1}E = a^{2n}E$, we can find $m \in E$ such that $2na^{2n-1}x = a^{2n}m = b^{2n}(c^2m)$. Similarly, we can write $2nb^{2n-1}c^2y = b^{2n}m'$ for some $m' \in E$. This gives $2na^{2n-1}x - 2nb^{2n-1}c^2y = b^{2n}(c^2m - m')$. Now, put $e = c^2m - m' \in E$. Then we conclude that $2na^{2n-1}x = 2nb^{2n-1}c^2y + b^{2n}e$ and thus $(a,x)^{2n} = (b,y)^{2n}(c^2,e)$. This implies that $R(a,x)^{2n} \subseteq R(b,y)^{2n}$ and hence R is a gr-AV ring. □

Recall from [12] that an A-module E is said to be a von Neumann regular module if for each $m \in E$, $Am = aE = a^2E$ for some $a \in A$. The authors in [12] showed that a finitely generated A-module E is a von Neumann regular module if and only if $aE = a^2E$ for all $a \in A$.

Corollary 4.19. *Let E be a finitely generated von Neumann regular module. Suppose that A is a graded ring, E is a graded A-module and $R = A \ltimes E$ is the graded trivial extension. Then R is a gr-AV ring if and only if A is a gr-AV ring.*

Proof. This follows from the previous theorem. □

Recall from the introduction that a graded R-module E is *gr-divisible* if, for each homogeneous $e \in E$ and each regular homogeneous element a of R, there exists $f \in E$ such that $af = e$.

Proposition 4.20. *Let E be a graded R-module. Then E is gr-divisible if and only if, for every regular element $a \in h(R)$ and every $e \in E$, the equation $af = e$ has a solution in E.*

Proof. If $af = e$ has a solution f in E for each regular homogeneous element $a \in R$ and each element $e \in E$, then E is trivially a gr-divisible module. Conversely, assume that E is gr-divisible and let a be a regular homogeneous element of R and $e = \sum_{g\in\Gamma} e_g \in E$; then there exists $f_g \in E$ such that $af_g = e_g$ for each $g \in \Gamma$, and hence $a \sum_{g\in\Gamma} f_g = e$, as desired. □

Our new result studies the possible transfer of the gr AV-ring property between a graded ring A and a graded trivial ring extension $A \ltimes E$. Recall from [16] that a graded ring $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ is said to be a crossed product if R_α contains a unit for every $\alpha \in \Gamma$.

Theorem 4.21. *Let A be a graded ring, E a nonzero graded A-module and $R := A \ltimes E$ the graded trivial ring extension. Then:*

(1) *If R is a gr AV-ring, then so is A.*
(2) *Suppose that* h-Z(A) = h-Nil(A) *and E is a gr-divisible A-module. Then R is a gr AV-ring if and only if A is a gr AV-ring.*
(3) *Suppose that $\mathbb{Q} \subseteq A$,* h-Z(A) = h-Nil(A) *and E is a torsion-free graded A-module. Further, assume that the grading monoid Γ is a group and A is a crossed product. Then R is an gr AV-ring if and only if A is a gr AV-ring and E is a gr divisible A-module.*
(4) *Let (A, M) be a graded quasi-local ring (with maximal homogeneous ideal M) and let E be a graded A-module such that $M = \mathrm{Gr}(\mathrm{Ann}(E))$. Then R is a gr AV-ring if and only if A is a gr AV-ring.*

Proof. (1) Let a, b be two homogeneous elements of A; then $(a, 0), (b, 0)$ are homogeneous elements of R. Since R is a gr AV-ring, there exists a positive integer n such that $(a, 0)^n \in R(b, 0)^n$ or $(b, 0)^n \in R(a, 0)^n$. Thus, $a^n \in Ab^n$ or $b^n \in Aa^n$, and hence A is a gr AV-ring.

(2) Assume that h-Z(A) = h-Nil(A) and E is a gr divisible A-module. By (1), it is only required to prove that if A is a gr AV-ring, then R is a gr AV-ring. Which means that if $\alpha := (a, e)$ and $\beta := (b, f)$ are homogeneous elements of R, then there exists a positive integer n such that either $\alpha^n \in \beta^n R$ or $\beta^n \in \alpha^n R$. Then two cases are possible. In the first case, a and b are each regular homogeneous elements of A. Then, since A is a gr AV-ring, there exists a positive integer n such that either $a^n A \subseteq b^n A$ or $b^n A \subseteq a^n A$. We may assume, without loss of generality, that $a^n A \subseteq b^n A$. Then $a^n = b^n c$ for some $c \in A$. Since E is a gr-divisible A-module and b^n is a regular homogeneous element of A, there exists $d \in E$ such that $b^n d = na^{n-1}e - ncb^{n-1}f$ according to Proposition 4.20. Therefore (with $a^0 := 1$), we have

$$\begin{aligned}\alpha^n = (a, e)^n = \left(a^n, na^{n-1}e\right) &= \left(b^n c, b^n d + ncb^{n-1}f\right)\\ &= \left(b^n, nb^{n-1}f\right)(c, d) = \beta^n(c, d) \in \beta^n R,\end{aligned}$$

as desired. In the remaining case, either a or b is a homogeneous zero-divisor in A. Without loss of generality, $a \in$ h-Z(A) = h-Nil(A). Then there exists a positive integer n such that $a^n = 0$. Hence $\alpha^{n+1} = \left(a^{n+1}, (n+1)a^n e\right) = (0, 0) \in \beta^{n+1}R$, as desired.

(3) By (1) and (2), we need only prove that if R is a gr AV-ring (along with the hypotheses that $\mathbb{Q} \subseteq A$ and E is a torsion-free graded A-module), then E is a gr-divisible A-module; that is, if e is a nonzero homogeneous element of E and a is a regular homogeneous element of A, then $e \in aE$. Assume that $\deg(a) = h_1$ and $\deg(e) = h_2$. Since Γ is a group and A is crossed product, we can choose a unit homogeneous element $x \in A_{h_2-h_1}$. Note that (xa, e) is a homogeneous element of R with $\deg(xa, e) = h_2$, since R is a gr AV-ring, there exists a positive integer n such that either $(a, 0)^n \in (xa, e)^n R$ or $(xa, e)^n \in (a, 0)^n R$ (keep in mind, $(a, 0)$ is a homogeneous element for any homogeneous element $a \in A$). There are two cases. The first case, $(a, 0)^n = (xa, e)^n(c, f)$ for some $(c, f) \in R$. Then $(a, 0)^n = (a^n, 0) = (xa, e)^n(c, f) = \left(x^n a^n, nx^{n-1}a^{n-1}e\right)(c, f) = \left(x^n a^n c, x^n a^n f + nx^{n-1}a^{n-1}ce\right)$, so that $x^n a^n c = a^n$ and $x^n a^n f + nx^{n-1}a^{n-1}ce = 0$. These facts can be rewritten as $a^n(x^n c - 1) = 0$ and $a^{n-1}(x^n af + nx^{n-1}ce) = 0$. Since a^n and a^{n-1} are each regular homogeneous elements of A (h-Reg(A) is a multiplicatively closed set), the hypothesis that E is a torsion-free graded A-module gives first that $x^n c = 1$ and next that $x^n af + nx^{n-1}ce = 0$. Multiplying by $n^{-1} \in \mathbb{Q} \subseteq A$ and $(x^{-1})^{n-1} \in A$, we get $ce = n^{-1}(-xaf) = a\left(-n^{-1}xf\right) \in aE$ and so $e \in aE$ since c is unit, as desired. In the remaining case, $(xa, e)^n = (a, 0)^n(d, g)$ for some $(d, g) \in R$. Simplifying and equating second coordinates, we have $a^n g = nx^{n-1}a^{n-1}e$. As above, the hypotheses that E is a torsion-free graded A module and a is a regular homogeneous element give that $ag = nx^{n-1}e$. It follows that $e = a\left(n^{-1}(x^{-1})^{n-1}g\right) \in aE$, as desired.

(4) By (1), we need only prove that if A is a gr AV-ring, along with the hypothesis that $M = \mathrm{Gr}(\mathrm{Ann}(E))$, then R is a gr AV-ring. Let $\alpha = (a, e), \beta = (b, f) \in h(R)$. If a is a homogeneous unit in A, then α is a homogeneous unit in R [6, Proposition 5], so that $\beta^1 = \beta \in R = \alpha R = \alpha^1 R$. Similarly, if b is a homogeneous unit in A, then $\alpha^1 \in \beta^1 R$. Thus, we may assume, without loss of generality, that $a \in M$ and $b \in M$. Then, since $M \subseteq \mathrm{Gr}(\mathrm{Ann}(E))$, there exist positive integers m and p such that $a^m \in \mathrm{Ann}(E)$ and $b^p \in \mathrm{Ann}(E)$. As A is a gr AV-ring, there exists a positive integer n such that either $a^n \in b^n A$ or $b^n \in a^n A$. We may assume, without loss of generality, that $a^n \in b^n A$. Taking $(mp+1)^{\text{th}}$ powers and rewriting, we get $a^{(mp+1)n} = b^{(mp+1)n} r$ for some $r \in A$. Then, since $(mp+1)n - 1 \geq \max(m, p)$,

$$\begin{aligned}
\alpha^{(mp+1)n} &= (a, e)^{(mp+1)n} = \left(a^{(mp+1)n}, (mp+1)na^{(mp+1)n-1}e\right) \\
&= \left(a^{(mp+1)n}, 0\right) = \left(b^{(mp+1)n}r, 0\right) \\
&= \left(b^{(mp+1)n}r, (mp+1)nb^{(mp+1)n-1}f\right) \\
&= \left(b^{(mp+1)n}, (mp+1)nb^{(mp+1)n-1}f\right)(r, 0) \\
&= (b, f)^{(mp+1)n}(r, 0) \in \beta^{(mp+1)n}R,
\end{aligned}$$

as desired. This completes the proof. □

If A is a graded integral domain, Theorem 4.21 (2) specializes to the following result.

Corollary 4.22. *Let A be a graded integral domain, E a graded A-module, and $R = A \ltimes E$ the graded trivial ring extension.*

(1) *Assume that E is a gr-divisible A-module. Then R is a gr-AV ring if and only if A is a gr AV-ring.*
(2) *Assume that E is a K-vector space, where K is the homogeneous quotient field of fractions of A. Then R is a gr-AV ring if and only if A is a gr AV-ring.*

If (A, M) is a graded quasi-local ring and E a graded A-module, then the corollary below gives another application of Theorem 4.21 (4).

Corollary 4.23. *Let (A, M) be a graded quasi-local ring, E a graded A-module, and $R := A \ltimes E$ the graded trivial ring extension. Assume that $M^n E = 0$ for some positive integer $n \geq 1$. Then R is a gr AV-ring if and only if A is a gr AV-ring.*

Proof. Obviously, if $E = 0$ then the assertions are clear since $R \cong A$. Thus, we may henceforth assume, without loss of generality, that $E \neq 0$. Then we have $\mathrm{Ann}(E) \neq A$. Let x be a homogeneous element of M; then $x^n E = 0$, so $x^n \in \mathrm{Ann}(E)$, which implies that $x \in \mathrm{Gr}(\mathrm{Ann}(E))$. Thus $\mathrm{Gr}(\mathrm{Ann}(E)) = M$, and so an application of Theorem 4.21 (4) completes the proof. □

Remark 4.24. Let (A, M) be a graded quasi-local ring and M be the only prime homogeneous ideal of A. Therefore $M = \text{h-Nil}(A)$. Then, every homogeneous element of A is unit or nilpotent. Hence, by Example 4.9, A is always a gr AV-ring. Note that, if A has a unique prime homogeneous ideal M, then $A \ltimes E$ has also a unique prime homogeneous ideal $M \ltimes E$. Therefore $A \ltimes E$ is always a gr AV-ring.

Our next example illustrates part (2) of Theorem 4.21.

Example 4.25. Let D be a gr-AVD, K the homogeneous quotient field of D and $E = K \times K$. Then $R := D \ltimes E$ is a gr AV-ring which is neither a graded integral domain nor a graded valuation ring.

Proof. By Corollary 4.22 R is a gr AV-ring. It is straightforward to see that R is not a graded integral domain. On the other hand, the two homogeneous ideals $(0, x)R$ and $(0, y)R$ of R are not comparable under inclusion with $x = (0, 1)$ and $y = (1, 0)$. Hence R is not a graded valuation ring. □

The integral closure of an almost valuation domain is a valuation domain, as shown by Anderson and Zafrullah in [4, Theorem 5.6]; but beyond the context of integral domains, N. Mahdou, A. Mimouni, and M. A. Salam Moutui explained in [15] the failure of this characterization. Afterwards in [8, Theorem 2.3], C. Bakkari, N. Mahdou, and A. Riffi provided an analog characterization of the gr-AVDs in relation to the graded theory. The following example shows how Bakkari, Mahdou, and Riffi's theorem fails outside of the framework of graded integral domains.

Example 4.26. Let (A, M) be a valuation domain which is not a field, and E a non-simple A-module such that $ME = 0$. With the notation of Remark 4.15, let $R = A \ltimes E$ be the trivial ring extension, and $\bar{R}$ the integral closure of R in the total quotient ring of R. Then we have as follows:

(1) R is a gr AV-ring.
(2) $\bar{R} = R$ is not a gr-valuation ring.

Proof. (1) By Proposition 4.16, since A is an AV-ring.

(2) First, it is straightforward to see that $Z(R) = \{(a, m) \mid a \text{ is a nonunit element of } A, m \in E\}$. Hence R is a total quotient ring. Thus R is a fortiori integrally closed in its total quotient ring. However, since A is not a field we conclude that $M \neq 0$, E has a proper submodule N since it is not a simple module. Let $a(\neq 0) \in M$, as $aE = 0$, $aR \ltimes N$ is a homogeneous ideal of R by [3, Theorem 3.3]. Therefore we can see the two homogeneous ideals $aR \ltimes N$ and $0 \ltimes E$ are not comparable and so R is not a gr-valuation ring, as desired. □

Recall from the introduction that a ring R is called an *almost Bezout ring* if, for any two elements a and b in R, there exists a positive integer n such that the ideal (a^n, b^n) is principal. Now we introduce the notion of graded almost Bezout rings.

Definition 4.27. A graded ring R is said to be a *graded almost Bezout ring* (gr-AB ring) if, for each $a, b \in h(R)$, there exist $n \geq 1$ and $x \in h(R)$ such that $(a^n, b^n) = (x)$.

Theorem 4.28. *Let R be a graded integral domain. Then R is a gr-AV domain if and only if R is a gr-AB domain and R is a gr-local ring.*

Proof. ($\Rightarrow$): Suppose that R is a gr-AV domain. Then by Corollary 4.7, (R, M) is a gr-local ring. Let $a, b \in h(R)$. Then there exists $n \geq 1$ such that $a^nR \subseteq b^nR$ or $b^nR \subseteq a^nR$. This implies that $(a^n, b^n) = (b^n)$ or $(a^n, b^n) = (a^n)$. Thus, R is a gr-AB domain.

($\Leftarrow$): Let R be a gr-AB domain and R be a gr-local ring, where M is the unique homogeneous maximal ideal of R. Let $a, b \in h(R)$. Since M is unique homogeneous maximal ideal, we may assume that $a, b \in M$. Since R is a gr-AB domain, there exist $n \geq 1$ and $x \in h(R)$ such that $(a^n, b^n) = (x)$. Then there exist $r, s \in R$ such that $x = ra^n + sb^n$. Let $r = \sum r_g$ and $s = \sum s_g$. Then $x = \sum a^n r_g + \sum b^n s_g$. Then there exists $g, h \in G$ such that $x = r_g a^n + s_h b^n$, where $\deg(x) = \deg(r_g a^n) = \deg(s_h b^n)$. Since $a^n \in (x)$, we can write $a^n = y(r_g a^n + s_h b^n)$. As $a^n, x \in h(R)$, we may assume that $y \in h(R)$. Similarly, we can find $z \in h(R)$ such that $b^n = z(r_g a^n + s_h b^n)$. This gives that $r_g a^n + s_h b^n = (r_g a^n + s_h b^n)(r_g y + s_h z)$. Since R is a graded domain, we have $r_g y + s_h z = 1$. Since $y, z \in h(R)$ and M is unique homogeneous maximal ideal, either y or z is unit. Indeed, if y, z are nonunit, then $y, z \in M$, which implies that $r_g y + s_h z = 1 \in M$, a contradiction. Without loss of generality, we may assume that y is a unit of R. Since $a^n = y(r_g a^n + s_h b^n)$, we have $x = r_g a^n + s_h b^n \in y^{-1}a^n \in Ra^n$. This implies that $b^n \in Rx \subseteq Ra^n$ and so we have $Rb^n \subseteq Ra^n$. Hence, R is a gr-AV domain. □

The following theorem studies the possible transfer of the gr-AB ring property between a ring A and a graded trivial ring extension $A \ltimes E$.

Theorem 4.29. *Let A be a graded ring, E a nonzero graded A-module and $R = A \ltimes E$ the graded trivial extension. Then:*

(1) *If R is a gr-AB ring, then so is A.*
(2) *Suppose that* $\text{h-Z}(A) = \text{h-Nil}(A)$ *and E is a gr-divisible A-module. Then R is a gr-AB ring if and only if A is a gr-AB ring.*
(3) *Let (A, M) be a gr-local (with unique homogeneous maximal ideal M of A) and let $M = \mathrm{Gr}(\mathrm{Ann}(E))$. Then R is a gr-AB ring if and only if A is a gr-AB ring.*

Proof. (1) Assume that R is a gr-AB ring. Let $a, b \in h(A)$. Then $\alpha := (a, 0)$ and $\beta := (b, 0)$ are homogeneous in R. Then there exist a homogeneous $x := (c, y) \in h(R)$ and $n \geq 1$ such that $R(a, 0)^n + R(b, 0)^n = Rx$. This gives $Aa^n + Ab^n = Ax$, that is, A is a gr-AB ring.

(2) Suppose that $\text{h-Z}(A) = \text{h-Nil}(A)$ and E is a gr-divisible module. If R is a gr-AB ring, then by (1), A is a gr-AB ring. Now, suppose that A is a gr-AB ring and $(a, m), (b, m')$ are homogeneous in R. Then $a, b \in h(A)$. Let $a \in \text{h-Z}(A)$. Then there exists $n \geq 1$ such that $a^n = 0$. This gives $(a, m)^{n+1} = (a^{n+1}, (n + 1)a^n m) = (0, 0)$, that is, $R(a, m)^{n+1} \subseteq R(b, m')^{n+1}$. If $b \in \text{h-Z}(A)$, one can similarly show that $R(b, m')^{n+1} \subseteq R(a, m)^{n+1}$ for some $n \geq 1$. Now, suppose that a, b are regular homogeneous elements of A. Put $\alpha := (a, m)$ and $\beta := (b, m')$. We will show that there exist $n \geq 1$ and $\gamma \in h(R)$ such that $R\alpha^n + R\beta^n = R\gamma$. Since A is a gr-AB ring, there exist $n \geq 1$ and $x \in h(A)$ such that $Aa^n + Ab^n = Ax$. This gives $x = ra^n + sb^n$ for some $r, s \in A$. Let $z \in R\alpha^n + R\beta^n$. Then $z = (ca^n + db^n, e)$ for some $c, d \in A$ and $e \in E$. Since $ca^n + db^n \in Aa^n + Ab^n = Ax$, we can write $ca^n + db^n = xy$ for some $y \in A$. Note that x is a regular homogeneous element of A since a^n is regular and $a^n \in Aa^n \subseteq Ax$. Since E is a gr-divisible module and x is regular, by Proposition 4.18, there exists $f \in E$ such that $e = xf$. This gives $z = (ca^n + db^n, e) = (xy, xf) = (x, 0)(y, f)$. Then we have $R\alpha^n + R\beta^n \subseteq R(x, 0)$. Now, we will show that the reverse inclusion holds. Since E is gr-divisible and a^n is regular, we can write $-nra^{n-1}m - nsb^{n-1}m' = a^n m''$ for some $m'' \in E$. Thus $(x, 0) = (ra^n + sb^n, 0) = (a, m)^n(r, m'') + (b, m')(s, 0)$, and this implies that $R(x, 0) \subseteq R\alpha^n + R\beta^n$. Thus we have $R\alpha^n + R\beta^n = R(x, 0)$, that is, R is a gr-AB ring.

(3) We need only show that if (A, M) is a gr-local and gr-AB ring such that $\mathrm{Gr}(\mathrm{Ann}(E)) = M$, then R is a gr-AB ring. Let $\alpha := (a, x)$ and $\beta := (b, y) \in h(R)$. If a or b is unit, then α or β is unit. Thus we have $R\alpha + R\beta = R(1, 0)$, which completes the proof. Now, assume that a, b are nonunits homogeneous elements of A, that is, $a, b \in M = \mathrm{Gr}(\mathrm{Ann}(E))$. Then there exists $n \geq 1$ such that $a^n E = b^n E = 0$. Since $a^{n+1}, b^{n+1} \in h(A)$ and A is a gr-AB ring, there exists $k \geq 1$ such that $A(a^{n+1})^k + A(b^{n+1})^k = Ac$ for some $c \in h(A)$. On the other hand, note that

$$\begin{aligned} R\alpha^{(n+1)k} + R\beta^{(n+1)k} &= R(a^{(n+1)k}, (n+1)ka^{(n+1)k-1}x) \\ &\quad + R(b^{(n+1)k}, (n+1)kb^{(n+1)k-1}y) \end{aligned}$$

$$= R(a^{(n+1)k}, 0) + R(b^{(n+1)k}, 0)$$
$$= R(a^{n+1}, 0)^k + R(b^{n+1}, 0)^k.$$

Thus we have $R\alpha^{(n+1)k} + R\beta^{(n+1)k} = R(c, 0)$, which completes the proof. □

If A is a graded integral domain and E is a gr-divisible A-module, Theorem 4.29 (2) specializes to the following result.

Corollary 4.30. *Let A be a graded integral domain and E a gr-divisible A-module. Then $A \ltimes E$ is a gr-AB ring if and only if A is a gr-AB domain.*

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