

A GENERALIZED BERNOULLI DIFFERENTIAL EQUATION

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ABSTRACT. We study a generalized form of the Bernoulli differential equation, employing a generalized conformable derivative. We first establish a generalized variant of Gronwall's inequality, which is essential for assessing the stability of generalized differential equation systems, and offer insights into the qualitative behavior of the trivial solution of the proposed equation. We then present and prove the main results concerning the solution of the generalized Bernoulli differential equation, complemented by illustrative examples that highlight the advantages of this generalized derivative approach. Furthermore, we introduce a finite difference method as an alternative technique to approximate the solution of the generalized Bernoulli equation and demonstrate its validity through practical examples.

1. INTRODUCTION

Fractional calculus, a branch of mathematics exploring differentiation and integration operators of generalized orders, emerged nearly simultaneously with traditional calculus. While its inception paralleled that of classical calculus, fractional calculus has proven its versatility across a myriad of applications. Notable works such as [3, 5, 13, 7, 32, 21] serve as testament to its widespread utility. However, tackling systems of fractional differential equations has presented unique hurdles, prompting the exploration of diverse methodologies, as evidenced by studies such as [15, 17, 34].

The global fractional derivatives, which collect information on an interval and keep track of the history of the process, can be said to possess a certain memory. Which makes it possible to model non-local and distributed responses that commonly appear in natural and physical phenomena, although it is known that they have certain limitations. In [27], a conformable fractional derivative is defined, offering advantages in its own right. More recently, a non-conformable local fractional derivative was introduced in [19]. The conformable fractional derivative, serving as a local operator, sets itself apart from non-local counterparts like Caputo or Riemann–Liouville. Viewing fractional local derivatives as a new perspective, they

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have demonstrated utility in various applications by several authors, as evidenced in [36, 14, 19, 3, 18, 24, 28].

This paper relies on the use of new differential operators, which depend on a general kernel function $T(t, \alpha)$. These operators encompass several local derivatives that have been introduced and studied in various sources. This new tool is powerful because it allows us to model a phenomenon from two perspectives: by considering different kernels and by varying the order associated with each kernel as shown in several studies [22, 29, 31, 33].

It is known that one of the most paradigmatic nonlinear equations is the Bernoulli differential equation, introduced by Jacob Bernoulli in his work [8]. This equation can be viewed as a manifestation of the principle of conservation of energy in fluids. In this paper, we will investigate Bernoulli's equation using the generalized conformable derivative operator. To begin, we establish a generalized form of Gronwall's inequality and derive stability conditions for systems of generalized differential equations utilizing the generalized conformable derivative approach. Subsequently, we determine solubility and stability conditions for the proposed generalized Bernoulli equation. Finally, we provide examples illustrating particular cases of this Bernoulli equation viewed from different perspectives.

There are cases in which the generalized differential equation becomes too complicated to solve by classical methods, and other alternatives must be analyzed. Numerical methods are a very useful tool when we must solve this type of equations. Finite difference methods replace the derivative operator with an appropriate quotient in differences, which allows us to approximate the solution quite efficiently. Here, we use a finite difference method to estimate the solution of the generalized Bernoulli equation and showcase the results obtained through examples.

2. PRELIMINARIES

In this section, we present a definition of a generalized conformable derivative, introduced in [16], as well as a fractional integration operator, given in [14], together with some of its most important properties that will be useful in the next section.

In [16] the definition is given as follows. Given $s \in \mathbb{R}$, we denote by $\lceil s \rceil$ the *upper integer part* of s , i.e., the smallest integer greater than or equal to s .

Definition 2.1. Given an interval $I \subseteq \mathbb{R}$, $f : I \rightarrow \mathbb{R}$, $\alpha \in \mathbb{R}^+$ and a positive continuous function $T(t, \alpha)$ on I , the derivative $G_T^\alpha f$ of f of order α at the point $t \in I$ is defined by

$$G_T^\alpha f(t) = \lim_{h \rightarrow 0} \frac{1}{h^{\lceil \alpha \rceil}} \sum_{k=0}^{\lceil \alpha \rceil} (-1)^k \binom{\lceil \alpha \rceil}{k} f(t - khT(t, \alpha)).$$

If $a = \inf\{t \in I\}$ (respectively, $b = \sup\{t \in I\}$), then $G_T^\alpha f(a)$ (respectively, $G_T^\alpha f(b)$) is defined with $h \rightarrow 0^-$ (respectively, $h \rightarrow 0^+$) instead of $h \rightarrow 0$ in the limit.

The derivative we are considering generalizes many of the properties of the local derivatives existing so far. It also allows the computation of higher-order

derivatives and is not limited only to functions defined on the positive half-line. It is important to emphasize that the choice of the kernel $T(t, \alpha)$ leads to different practical applications. Thanks to the generality of the theoretical results obtained, we can state that they do not depend on the choice of the kernel. In the same direction, different applications of the generalized derivatives are shown, as well as their relations with other types of local derivatives (conformable or not), as shown in [2]:

- (1) If $\alpha \in (0, 1]$ and $T(t, \alpha) = t^{1-\alpha}$, then the conformable fractional derivative defined in [27] is obtained.
- (2) If $\alpha \in (0, 1]$ and $T(t, \alpha) = k(t)^{1-\alpha}$, then the general conformable fractional derivative defined in [4] is derived.
- (3) If $\alpha \in (0, 1]$ and $T(t, \alpha) = e^{t^{-\alpha}}$, then the non-conformable fractional derivative defined in [19] is obtained.
- (4) If $\alpha \in (0, 1]$ and $T(t, \alpha) = \left(t + \frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}$, then we obtain the beta-derivative defined in [6].

If we choose the function $T(t, \alpha) = t^{\lceil \alpha \rceil - \alpha}$, then we obtain the following case of the function G_T^α , which is a conformable derivative.

Definition 2.2. Let I be an interval $I \subseteq (0, \infty)$, $f : I \rightarrow \mathbb{R}$ and $\alpha \in \mathbb{R}^+$. The conformable derivative $G^\alpha f$ of f of order α at the point $t \in I$ is defined by

$$G^\alpha f(t) = \lim_{h \rightarrow 0} \frac{1}{h^{\lceil \alpha \rceil}} \sum_{k=0}^{\lceil \alpha \rceil} (-1)^k \binom{\lceil \alpha \rceil}{k} f(t - kht^{\lceil \alpha \rceil - \alpha}).$$

Note that, if $\alpha = n \in \mathbb{N}$ and f is smooth enough, then Definition 2.2 coincides with the classical definition of the n -th derivative.

The conformable derivative T_α that was defined in [27] is a particular case of G^α when $\alpha \in (0, 1]$ and $T(t, \alpha) = t^{1-\alpha}$. See [1, 25, 26] for more information on T_α .

The following results in [16] contain some basic properties of the derivative G_T^α .

Theorem 2.3. Let I be an interval $I \subseteq \mathbb{R}$, $f : I \rightarrow \mathbb{R}$ and $\alpha \in \mathbb{R}^+$.

- (1) If there exists $D^{\lceil \alpha \rceil} f$ at the point $t \in I$, then f is G_T^α -differentiable at t and $G_T^\alpha f(t) = T(t, \alpha)^{\lceil \alpha \rceil} D^{\lceil \alpha \rceil} f(t)$.
- (2) If $\alpha \in (0, 1]$, then f is G_T^α -differentiable at $t \in I$ if and only if f is differentiable at t ; in this case, we have $G_T^\alpha f(t) = T(t, \alpha) f'(t)$.

Theorem 2.4. Let I be an interval $I \subseteq \mathbb{R}$, $f, g : I \rightarrow \mathbb{R}$ and $\alpha \in \mathbb{R}^+$. Assume that f, g are G_T^α -differentiable functions at $t \in I$. Then the following statements hold:

- (1) $af + bg$ is G_T^α -differentiable at t for every $a, b \in \mathbb{R}$, and $G_T^\alpha(af + bg)(t) = aG_T^\alpha f(t) + bG_T^\alpha g(t)$.
- (2) If $\alpha \in (0, 1]$, then fg is G_T^α -differentiable at t and $G_T^\alpha(fg)(t) = f(t)G_T^\alpha g(t) + g(t)G_T^\alpha f(t)$.
- (3) If $\alpha \in (0, 1]$ and $g(t) \neq 0$, then f/g is G_T^α -differentiable at t and $G_T^\alpha\left(\frac{f}{g}\right)(t) = \frac{g(t)G_T^\alpha f(t) - f(t)G_T^\alpha g(t)}{g(t)^2}$.
- (4) $G_T^\alpha(\lambda) = 0$ for every $\lambda \in \mathbb{R}$.

- (5) $G_T^\alpha(t^p) = \frac{\Gamma(p+1)}{\Gamma(p-[\alpha]+1)} t^{p-[\alpha]} T(t, \alpha)^{[\alpha]}$ for every $p \in \mathbb{R} \setminus \mathbb{Z}^-$.
 (6) $G_T^\alpha(t^{-n}) = (-1)^{[\alpha]} \frac{\Gamma(n+[\alpha])}{\Gamma(n)} t^{-n-[\alpha]} T(t, \alpha)^{[\alpha]}$ for every $n \in \mathbb{Z}^+$.

Theorem 2.5. Let $\alpha \in (0, 1]$; let g be a G_T^α -differentiable function at t and f a differentiable function at $g(t)$. Then $f \circ g$ is G_T^α -differentiable at t , and $G_T^\alpha(f \circ g)(t) = f'(g(t)) G_T^\alpha g(t)$.

In [14], an integral operator is defined in the following way. Let I be an interval $I \subseteq \mathbb{R}$, $a, t \in I$ and $\alpha \in \mathbb{R}$. The integral operator $J_{T,a}^\alpha$ is defined for every locally integrable function f on I as

$$J_{T,a}^\alpha(f)(t) = \int_a^t \frac{f(\omega)}{T(\omega, \alpha)} d\omega.$$

The following results appear in [14].

Proposition 2.6. Let I be an interval $I \subseteq \mathbb{R}$, $a \in I$, $0 < \alpha \leq 1$ and f a differentiable function on I such that f' is a locally integrable function on I . Then we have, for all $t \in I$,

$$J_{T,a}^\alpha(G_T^\alpha(f))(t) = f(t) - f(a).$$

Proposition 2.7. Let I be an interval $I \subseteq \mathbb{R}$, $a \in I$ and $\alpha \in (0, 1]$. We have

$$G_T^\alpha(J_{T,a}^\alpha(f))(t) = f(t)$$

for every continuous function f on I and $a, t \in I$.

In [27], the integral operator $J_{T,a}^\alpha$ is defined with T given by $T(t, \alpha) = t^{1-\alpha}$, and [27, Theorem 3.1] shows that

$$G^\alpha J_{t^{1-\alpha},a}^\alpha(f)(t) = f(t)$$

for every continuous function f on I , $a, t \in I$ and $\alpha \in (0, 1]$. Hence, Proposition 2.7 extends to any T this important equality.

The following result summarizes some elementary properties of the integral operator $J_{T,a}^\alpha$.

Theorem 2.8. Let I be an interval $I \subseteq \mathbb{R}$, $a, b \in I$ and $\alpha \in \mathbb{R}$. Suppose that f, g are locally integrable functions on I , and $k_1, k_2 \in \mathbb{R}$. Then we have

- (1) $J_{T,a}^\alpha(k_1 f + k_2 g)(t) = k_1 J_{T,a}^\alpha f(t) + k_2 J_{T,a}^\alpha g(t)$;
- (2) if $f \geq g$, then $J_{T,a}^\alpha f(t) \geq J_{T,a}^\alpha g(t)$ for every $t \in I$ with $t \geq a$;
- (3) $|J_{T,a}^\alpha f(t)| \leq J_{T,a}^\alpha |f|(t)$ for every $t \in I$ with $t \geq a$;
- (4) $\int_a^b \frac{f(\omega)}{T(\omega, \alpha)} d\omega = J_{T,a}^\alpha f(t) - J_{T,b}^\alpha f(t) = J_{T,a}^\alpha f(t)(b)$ for every $t \in I$.

Remark 2.9. The above results generalize Proposition 1, Proposition 2 and Theorem 1 of [20], respectively, obtained with $0 < \alpha \leq 1$.

The following propositions are presented in order to serve as a basis for future research.

Given $a, b \in I$ ($b > a$), let us denote by $F_{a,b}$ the usual inner product in $L^2[a, b]$,

$$F_{a,b}(f, g) = \int_a^b f(t)g(t) dt.$$

Proposition 2.10 ([9]). *Let I be an interval $I \subseteq \mathbb{R}$, $a, b \in I$ with $a < b$ and $\alpha \in \mathbb{R}$. The adjoint of $J_{T,a}^\alpha$ in $L^2[a, b]$ with respect to the inner product $F_{a,b}$ is the operator*

$$A_{T,a,b}^\alpha(f)(t) = \frac{1}{T(t, \alpha)} \int_t^b f(s) ds.$$

For theoretical completeness we show the following result.

Proposition 2.11. *Let I be an interval $I \subseteq \mathbb{R}$, $a, b \in I$ with $a < b$ and $\alpha \in \mathbb{R}$. Then $J_{T,a}^\alpha$ is a Hilbert–Schmidt integral operator on $L^2[a, b]$, and so, a continuous and compact operator.*

Proof. Let us denote by χ_A the characteristic function of the set A (i.e., $\chi_A(t) = 1$ if $t \in A$ and $\chi_A(t) = 0$ otherwise). Then

$$J_{T,a}^\alpha(f)(t) = \int_a^t \frac{f(\omega)}{T(\omega, \alpha)} d\omega = \int_a^b k(t, \omega) f(\omega) d\omega,$$

with

$$k(t, \omega) = \chi_{[a,t]}(\omega) \frac{1}{T(\omega, \alpha)}.$$

We have

$$\begin{aligned} \int_a^b \int_a^b |k(t, \omega)|^2 d\omega dt &\leq \int_a^b \int_a^b \frac{1}{T(\omega, \alpha)^2} d\omega dt \\ &= \int_a^b \frac{b-a}{T(\omega, \alpha)^2} d\omega < \infty, \end{aligned}$$

since $T(\omega, \alpha)$ is a positive continuous function on I . Thus, $J_{T,a}^\alpha$ is a Hilbert–Schmidt integral operator on $L^2[a, b]$, and so, a continuous and compact operator. $\square$

3. GENERALIZED GRONWALL'S INEQUALITY

Next, we prove a generalized version of Gronwall's inequality which will be useful in the study of the stability of systems of generalized differential equations. A version of this inequality was proved in [1].

Theorem 3.1. *Let r , c , d and k be continuous functions on the interval $[a, b]$, $d, k \geq 0$, and $\alpha \in (0, 1]$, such that*

$$r(t) \leq c(t) + d(t)J_{T,a}^\alpha(kr)(t)$$

for all $t \in [a, b]$. Then we have

$$r(t) \leq c(t) + d(t)J_{T,a}^\alpha(cke^{-J_{T,a}^\alpha(dk)})(t)e^{J_{T,a}^\alpha(dk)(t)}$$

for every $t \in [a, b]$.

Proof. Let us define $R(t) = J_{T,a}^\alpha(kr)(t)$. Thus, $R(a) = 0$,

$$r(t) \leq c(t) + d(t)R(t)$$

for all $t \in [a, b]$, and Proposition 2.7 gives

$$G_T^\alpha R(t) = k(t)r(t) \leq c(t)k(t) + d(t)k(t)R(t) \quad (3.1)$$

for every $t \in [a, b]$, since $k \geq 0$. Let us define $E(t) = e^{-J_{T,a}^\alpha(dk)(t)}$. Theorem 2.5 and Proposition 2.7 give

$$\begin{aligned} G_T^\alpha E(t) &= G_T^\alpha(e^{-J_{T,a}^\alpha(dk)})(t) \\ &= G_T^\alpha(-J_{T,a}^\alpha(dk))(t)e^{-J_{T,a}^\alpha(dk)(t)} \\ &= -d(t)k(t)E(t). \end{aligned}$$

Since R and E are G_T^α -differentiable on $[a, b]$, Theorem 2.4 and (3.1) give

$$\begin{aligned} G_T^\alpha(ER)(t) &= E(t)G_T^\alpha R(t) + R(t)G_T^\alpha E(t) \\ &= E(t)G_T^\alpha R(t) - d(t)k(t)R(t)E(t) \\ &\leq c(t)k(t)E(t) + d(t)k(t)R(t)E(t) \\ &\quad - d(t)k(t)R(t)E(t) = c(t)k(t)E(t) \end{aligned}$$

for every $t \in [a, b]$, since $E \geq 0$. Since $E(t)R(t)$ is differentiable on $[a, b]$ by Theorem 2.3, we have that Theorem 2.8 and Proposition 2.6 give

$$\begin{aligned} J_{T,a}^\alpha(ckE)(t) &\geq J_{T,a}^\alpha(G_T^\alpha(ER))(t) \\ &= E(t)R(t) - E(a)R(a) = E(t)R(t) \end{aligned}$$

for any $t \in [a, b]$. Thus, since $d \geq 0$, $E > 0$,

$$\begin{aligned} r(t) &\leq c(t) + d(t)R(t) \leq c(t) + d(t) \frac{J_{T,a}^\alpha(ckE)(t)}{E(t)} \\ &= c(t) + d(t)J_{T,a}^\alpha(cke^{-J_{T,a}^\alpha(dk)})(t)e^{J_{T,a}^\alpha(dk)(t)} \end{aligned}$$

for every $t \in [a, b]$. □

The argument in the proof of Theorem 3.1 also gives the following converse inequality.

Theorem 3.2. *Let r , c , d and k be continuous functions on the interval $[a, b]$, $d, k \geq 0$, and $\alpha \in (0, 1]$, such that*

$$r(t) \geq c(t) + d(t)J_{T,a}^\alpha(kr)(t)$$

for all $t \in [a, b]$. Then we have

$$r(t) \geq c(t) + d(t)J_{T,a}^\alpha(cke^{-J_{T,a}^\alpha(dk)})(t)e^{J_{T,a}^\alpha(dk)(t)}$$

for every $t \in [a, b]$.

If $c \geq 0$, then the conclusion of Theorem 3.1 can be simplified.

Theorem 3.3. *Let r , c , d and k be continuous functions on the interval $[a, b]$, $c, d, k \geq 0$, and $\alpha \in (0, 1]$, such that*

$$r(t) \leq c(t) + d(t)J_{T,a}^\alpha(kr)(t)$$

for all $t \in [a, b]$. Then we have

$$r(t) \leq c(t) + d(t)J_{T,a}^\alpha(ck)(t)e^{J_{T,a}^\alpha(dk)(t)}$$

for every $t \in [a, b]$.

Proof. Theorem 3.1 gives

$$r(t) \leq c(t) + d(t)J_{T,a}^\alpha(cke^{-J_{T,a}^\alpha(dk)})(t)e^{J_{T,a}^\alpha(dk)(t)}$$

for every $t \in [a, b]$. Since $d \geq 0$, it suffices to prove that

$$J_{T,a}^\alpha(cke^{-J_{T,a}^\alpha(dk)})(t) \leq J_{T,a}^\alpha(ck)(t) \quad (3.2)$$

for any $t \in [a, b]$.

Since $dk \geq 0$, Theorem 2.8 gives $J_{T,a}^\alpha(dk) \geq 0$, so $e^{-J_{T,a}^\alpha(dk)} \leq 1$. Since $ck \geq 0$, we have $cke^{-J_{T,a}^\alpha(dk)} \leq ck$, and Theorem 2.8 gives (3.2). $\square$

The following result appears in [14].

Lemma 3.4. *Let $\alpha_1, \alpha_2, \dots, \alpha_n \in (0, 1]$, $x = (x_1, \dots, x_n)$, $I \subseteq \mathbb{R}$ an interval, $\Omega \subseteq \mathbb{R}^n$ an open set, $t_0 \in I$ and $x_0 \in \Omega$. Let $F = (F_1, \dots, F_n) : I \times \Omega \rightarrow \mathbb{R}^n$ be in (C, Lip) on some open neighborhood of the point (t_0, x_0) , and consider the initial value problem*

$$G_T^{\alpha_j} x_j = F_j(t, x), \quad 1 \leq j \leq n, \quad x(t_0) = x_0. \quad (3.3)$$

Then there exists $h > 0$ such that (3.3) has a unique solution on the interval $[t_0 - h, t_0 + h] \cap I$. Furthermore, if $\Omega = \mathbb{R}^n$ and F is in (C, Lip) on $J \times \mathbb{R}^n$ for each compact interval $J \subseteq I$, then (3.3) has a unique solution on I .

Let $\alpha \in (0, 1]$, $a \in \mathbb{R}$, $t_0 \geq a$, $x_0 \in \mathbb{R}^n$, and $F : [a, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ in (C, Lip) on $[a, \infty) \times \mathbb{R}^n$. Let us consider the initial value problem

$$G_T^\alpha x(t) = F(t, x), \quad x(t_0) = x_0. \quad (3.4)$$

Lemma 3.4 guarantees that (3.4) has a unique solution on $[a, \infty)$. The study of boundedness of solutions of a differential equation, either generalized or not, plays an important role in qualitative theory. In addition, the qualitative behavior of solutions plays an important role in many real-world phenomena related to applied research. Based on the previous results, we can obtain stability results for the solutions of generalized differential equations.

Definition 3.5. If $F(t, 0) = 0$ for every $t \geq a$, then the trivial solution $x \equiv 0$ of (3.4) is said to be *stable* if for any $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that if $|x_0| < \delta$, then $|x(t)| < \varepsilon$ for every $t \geq t_0$; it is *uniformly stable* if there exists $\delta = \delta(\varepsilon) > 0$ such that if $|x_0| < \delta$, then $|x(t)| < \varepsilon$ for every $t \geq t_0 \geq a$.

Propositions 2.6 and 2.7 give the following result.

Proposition 3.6. *Let $\alpha \in (0, 1]$, $a \in \mathbb{R}$, $t_0 \geq a$, $x_0 \in \mathbb{R}^n$, and $F : [a, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ in (C, Lip) on $[a, \infty) \times \mathbb{R}^n$. Then the problem (3.4) is equivalent to*

$$x(t) = x_0 + J_{T,t_0}^\alpha F(s, x(s))(t).$$

Theorem 3.7. *Let $\alpha \in (0, 1]$, $a \in \mathbb{R}$ and $F : [a, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ in (C, Lip) on $[a, \infty) \times \mathbb{R}^n$. Assume that $F(t, 0) = 0$ for every $t \geq a$, $|F(t, x)| \leq k(t)|x|$ for every $t \geq a$, $x \in \mathbb{R}^n$, and there exists a continuous function k such that*

$$\int_a^\infty \frac{k(t)}{T(t, \alpha)} dt < \infty.$$

Then the trivial solution $x \equiv 0$ of (3.4) is uniformly stable.

Proof. Let us define

$$N = \int_a^\infty \frac{k(t)}{T(t, \alpha)} dt < \infty.$$

Theorem 2.5, Proposition 2.6 and $k \geq 0$ give, for every $t \geq t_0 \geq a$,

$$\begin{aligned} J_{T, t_0}^\alpha (ke^{-J_{T, t_0}^\alpha(k)})(t) e^{J_{T, t_0}^\alpha(k)(t)} &= J_{T, t_0}^\alpha (G_T^\alpha(-e^{-J_{T, t_0}^\alpha(k)}))(t) e^{J_{T, t_0}^\alpha(k)(t)} \\ &= (e^{-J_{T, t_0}^\alpha(k)(t_0)} - e^{-J_{T, t_0}^\alpha(k)(t)}) e^{J_{T, t_0}^\alpha(k)(t)} \\ &= (1 - e^{-J_{T, t_0}^\alpha(k)(t)}) e^{J_{T, t_0}^\alpha(k)(t)} \\ &= e^{J_{T, t_0}^\alpha(k)(t)} - 1 \\ &\leq e^{J_{T, a}^\alpha(k)(t)} - 1 \\ &\leq e^N - 1. \end{aligned} \tag{3.5}$$

Fix $\varepsilon > 0$ and let $x = x(t)$ be the solution of the Cauchy problem (3.4), where x_0 satisfies

$$|x_0| < \delta = \delta(\varepsilon) = e^{-N} \varepsilon. \tag{3.6}$$

Proposition 3.6 gives

$$x(t) = x_0 + J_{T, t_0}^\alpha F(s, x(s))(t)$$

for every $t \geq t_0$. Thus, Theorem 2.8 gives

$$\begin{aligned} |x(t)| &\leq |x_0| + |J_{T, t_0}^\alpha F(s, x(s))(t)| \\ &\leq |x_0| + J_{T, t_0}^\alpha |F(s, x(s))|(t) \\ &\leq |x_0| + J_{T, t_0}^\alpha (k|x|)(t) \end{aligned}$$

for every $t \geq t_0$. Hence, Theorem 3.1, (3.5) and (3.6) give

$$\begin{aligned} |x(t)| &\leq |x_0| + |x_0| J_{T, t_0}^\alpha (ke^{-J_{T, t_0}^\alpha(k)})(t) e^{J_{T, t_0}^\alpha(k)(t)} \\ &\leq |x_0| + |x_0| (e^N - 1) = |x_0| e^N < \varepsilon \end{aligned}$$

for all $t \geq t_0$. Since δ does not depend on t_0 , the trivial solution of (3.4) is uniformly stable. $\square$

4. THE GENERALIZED BERNOULLI DIFFERENTIAL EQUATION

The Bernoulli differential equation stands out as one of the most crucial non-linear equations, playing a pivotal role in solving intricate mathematical problems. Its practical significance is evident in its ability to describe and analyze phenomena across various fields such as physics, biology, and engineering. Specifically, it enables the examination of flow behaviors in systems where pressure, velocity, and

height play crucial roles. Moreover, it serves as a cornerstone for studying system dynamics, enzyme kinetics, and population dynamics [10, 11, 35].

The Bernoulli differential equation has garnered recent attention within the realm of global fractional operators, as evidenced by studies such as [12]. This section builds upon prior research by examining the Bernoulli differential equation through the lens of local operators, including both conformable and non-conformable approaches. Furthermore, this research raises the prospect of comparing the effectiveness of both global and local methodologies in practical applications, as illustrated, for instance, in the study [23].

In this section we introduce the generalized Bernoulli differential equation, defined as

$$G_T^\alpha y + p(t)y = q(t)y^n, \quad n \neq 0, 1. \quad (4.1)$$

We will start by presenting the basic results of the stability theory for this equation.

First, we will pose the problem for a much more general system than (4.1):

$$G_T^\alpha y(t) = f(t, y(t)), \quad (4.2)$$

$$y(t_0) = y_0, \quad (4.3)$$

where $f \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, $t_0 > 0$. It is further assumed that for $(t_0, y_0) \in \text{int}(\mathbb{R}_+ \times \mathbb{R})$ the initial value problem (4.2)–(4.3) has a solution $y(t) \in C^\alpha(I)$ for all $t > t_0 > 0$. In addition, it is assumed that $f(t, 0) = 0$ for all $t > t_0 > 0$.

Several types of stability can be discussed for solutions of differential equations (of integer or fractional order) describing dynamical systems. The most important type concerns the stability of solutions near an equilibrium point. This can be discussed by Lyapunov's theory. In simple terms, if solutions that start near an equilibrium point y_e stay close to y_e for all t then y_e is Lyapunov stable. More strongly, if y_e is Lyapunov stable and all solutions starting near y_e converge to y_e , then y_e is asymptotically stable.

The concept of exponential stability, applicable to linear equations or systems, ensures a minimum rate of decay. This implies an assessment of how rapidly the solutions converge.

The stability of a solution of generalized differential equations can be defined in exactly the same way. The following results form part of a theory of stability for equations of the type of (4.2). First, we state a lemma that will serve as a basis for obtaining the desired stability results.

Lemma 4.1 ([9]). *Let I be an interval with $[t_0, \infty) \subseteq I$ and $\alpha \in (0, 1]$. Let p be a continuous function on $[t_0, \infty)$, and consider the linear equation*

$$G_T^\alpha y + p(t)y = 0.$$

- (1) *If $\liminf_{t \rightarrow \infty} J_{T, t_0}^\alpha(p)(t) > -\infty$, then the trivial solution $y(t) \equiv 0$ is stable.*
- (2) *If $\lim_{t \rightarrow \infty} J_{T, t_0}^\alpha(p)(t) = \infty$, then $y(t) \equiv 0$ is asymptotically stable.*

Now consider the initial value problem

$$G_T^\alpha y + p(t)y = 0, \quad y(t_0) = y_0, \quad (4.4)$$

and note that the trivial solution $y(t) \equiv 0$ of (4.4) is stable if and only if

$$J_{T,0}^\alpha(p)(+\infty) = +\infty. \quad (4.5)$$

By means of the properties of the integral operator, we can then see that the general solution of (4.4) is

$$y(t) = y_0 e^{-J_{T,t_0}^\alpha p(t)},$$

from which (4.5) follows easily.

We are now in a position to establish the main results on the stability of equations of type (4.2).

Theorem 4.2. *Consider the differential equation*

$$G_T^\alpha y + p(t)y = g(t, y), \quad t > 0, \quad (4.6)$$

where $p(t)$ is a positive and continuous function such that $0 < p \leq p(t)$, and g is a continuous function with $g(t, 0) \equiv 0$ for all t . If

$$\limsup_{z \rightarrow 0} \sup_{t > 0} \frac{|g(t, z)|}{z} = 0, \quad (4.7)$$

then the trivial solution $y(t) \equiv 0$ is an asymptotically stable solution.

Proof. Choose any $\varepsilon \in (0, p)$. From (4.7), there exists $\delta > 0$ such that

$$\sup_{t > 0} |g(t, z)| \leq \varepsilon |z| \quad \text{for all } |z| \leq \delta.$$

Let $y_0 \in (0, \delta)$. Since no two solutions can intersect, we have $y(t) > 0$ (or $y(t) < 0$) for all $t > 0$. Consider first that $y(t) > 0$; taking $y(t_0) = y_0$ we obtain from (4.6) $-p(t)y + g(t, y) \leq -py + \varepsilon y < 0$, i.e., the solution $y(t)$ is decreasing. Therefore, we have $G_T^\alpha y(t) \leq -(p - \varepsilon)y$, but this is a particular case of (4.4), so we have $0 < y(t) \leq y_0 e^{-J_{T,t_0}^\alpha (p - \varepsilon)(t)} \rightarrow 0$ as $t \rightarrow +\infty$. Similarly, we can prove the $y(t) \rightarrow 0$ as $t \rightarrow +\infty$ if $y_0 \in (-\delta, 0)$. Hence the trivial solution $y(t) \equiv 0$ is asymptotically stable. $\square$

The previous theorem allows us to establish conditions to ensure the asymptotic stability of the Bernoulli equation (4.1). In this way we have the following corollary.

Corollary 4.3. *Under the assumption of Theorem 4.2 on the function $p(t)$, let $q(t)$ be a bounded function; then the trivial solution $y(t) \equiv 0$ of (4.1) is asymptotically stable.*

The following result shows how the integral operator $J_{T,a}^\alpha$ can be applied in order to solve generalized differential equations.

Theorem 4.4. *Let p and q be continuous functions on an interval $I \subseteq \mathbb{R}$, $a \in I$, $\alpha \in (0, 1]$ and $n \in \mathbb{R} \setminus \{1\}$, and consider the generalized Bernoulli differential equation*

$$G_T^\alpha y + p(t)y = q(t)y^n. \quad (4.8)$$

Then the following statements hold.

(1) For each $C \in \mathbb{R}$, the function

$$y(t) = [e^{-(1-n)J_{T,a}^\alpha(p)(t)}((1-n)J_{T,a}^\alpha(qe^{(1-n)J_{T,a}^\alpha(p)})(t) + C)]^{1/(1-n)} \quad (4.9)$$

is a solution of the Bernoulli generalized equation (4.8).

(2) For each $t_0 \in I$ and $y_0 \in \mathbb{R}$, let $y(t; t_0, y_0)$ be the function in (4.9) with

$$C = y_0^{1-n} e^{(1-n)J_{T,a}^\alpha(p)(t_0)} - (1-n)J_{T,a}^\alpha(qe^{(1-n)J_{T,a}^\alpha(p)})(t_0).$$

If $(y_0^{1-n})^{1/(1-n)} = y_0$ and $y(t; t_0, y_0)$ is defined in some right or left neighborhood $U \subseteq I$ of t_0 , then $y(t; t_0, y_0)$ is a solution of (4.8) on U satisfying the initial condition $y(t_0; t_0, y_0) = y_0$.

(3) For each $t_0 \in I$ and $y_0 > 0$, there exists a unique solution of (4.8) satisfying the initial value $y(t_0) = y_0$, given by $y(t; t_0, y_0)$.

(4) If $n \in (1, \infty) \cap \mathbb{Q}$ and $1/(1-n) = r/s$ with s an odd integer, then for each $t_0 \in I$ and $y_0 \in \mathbb{R}$ there exists a unique solution of (4.8) satisfying the initial value $y(t_0) = y_0$, given by $y(t; t_0, y_0)$.

Proof. For each $C \in \mathbb{R}$, let us define the function

$$z(t) = e^{-(1-n)J_{T,a}^\alpha(p)(t)}((1-n)J_{T,a}^\alpha(qe^{(1-n)J_{T,a}^\alpha(p)})(t) + C).$$

Therefore, $y = z^{1/(1-n)}$. Theorems 2.4 and 2.5 and Proposition 2.7 give

$$\begin{aligned} G_T^\alpha z &= (1-n)G_T^\alpha(-J_{T,a}^\alpha(p))(t) e^{-(1-n)J_{T,a}^\alpha(p)(t)} \\ &\quad \times ((1-n)J_{T,a}^\alpha(qe^{(1-n)J_{T,a}^\alpha(p)})(t) + C) \\ &\quad + (1-n)e^{-(1-n)J_{T,a}^\alpha(p)(t)} G_T^\alpha(J_{T,a}^\alpha(qe^{(1-n)J_{T,a}^\alpha(p)}))(t) \\ &= -(1-n)p(t) e^{-(1-n)J_{T,a}^\alpha(p)(t)} \\ &\quad \times ((1-n)J_{T,a}^\alpha(qe^{(1-n)J_{T,a}^\alpha(p)})(t) + C) \\ &\quad + (1-n)e^{-(1-n)J_{T,a}^\alpha(p)(t)} q(t) e^{(1-n)J_{T,a}^\alpha(p)(t)} \\ &= -(1-n)p(t)z + (1-n)q(t). \end{aligned}$$

Hence, Theorem 2.5 gives

$$\begin{aligned} G_T^\alpha y &= \frac{1}{1-n} z^{n/(1-n)} G_T^\alpha z \\ &= \frac{1}{1-n} z^{n/(1-n)} (-(1-n)p(t)z + (1-n)q(t)) \\ &= y^n (-p(t)y^{1-n} + q(t)) \\ &= -p(t)y + q(t)y^n, \end{aligned}$$

and y is a solution of the Bernoulli generalized equation (4.8).

If $(y_0^{1-n})^{1/(1-n)} = y_0$ and $y(t; t_0, y_0)$ is defined in some right or left neighborhood $U \subseteq I$ of t_0 , then it is clear that $y(t; t_0, y_0)$ is a solution of (4.8) on U satisfying the initial condition $y(t_0; t_0, y_0) = y_0$.

By Theorem 2.3, the generalized differential equation (4.8) is equivalent to

$$y' = \frac{-1}{T(t, \alpha)} p(t)y + \frac{1}{T(t, \alpha)} q(t)y^n =: f(t, y).$$

Since $y_0 > 0$, the function $f(t, y)$ is continuous in a neighborhood V of (t_0, y_0) and, also, it is Lipschitz in the second variable in V . Therefore, Picard's theorem gives that there exists a unique solution of (4.8) satisfying the initial value $y(t_0) = y_0$. Since $y_0 > 0$, we have $(y_0^{1-n})^{1/(1-n)} = y_0$, and the second item of this theorem gives that this solution is $y(t; t_0, y_0)$.

Assume now that $n \in (1, \infty) \cap \mathbb{Q}$ and $1/(1-n) = r/s$ with s an odd integer. For each $y_0 \in \mathbb{R}$, the function $f(t, y)$ is continuous in a neighborhood V of (t_0, y_0) ; since $n > 1$, f is Lipschitz in the second variable in V . Hence, Picard's theorem gives that there exists a unique solution of (4.8) with $y(t_0) = y_0$. Since $1/(1-n) = r/s$ with s an odd integer, we have $(y_0^{1-n})^{1/(1-n)} = y_0$ for every $y_0 \in \mathbb{R}$, and the second item of this theorem gives that this solution is $y(t; t_0, y_0)$. $\square$

Remark 4.5. The results obtained here, relative to the Bernoulli equation, generalize and complete those in [30], which were derived using the conformable derivative of [27].

4.1. Examples. The generalized derivative can be considered as a good tool for solving certain types of problems. Therefore, by treating the Bernoulli differential equation with this approach, we gain a greater ability to study and solve such problems, as will be shown in the following examples.

Consider the functions $p(t) = q(t) = 1$, the interval $I = [0.5, 2]$ and $n = 2$. According to equation (4.8), we obtain

$$G_T^\alpha y + y = y^2. \quad (4.10)$$

Now consider the function $T(t, \alpha) = t^{1-\alpha}$ and $C = -1$; then by Theorem 4.4 we obtain the solution

$$y(t) = \left(1 - 2e^{\frac{t^\alpha - (0.5)^\alpha}{\alpha}}\right)^{-1}.$$

The solutions for different values of α are shown in Figure 1.

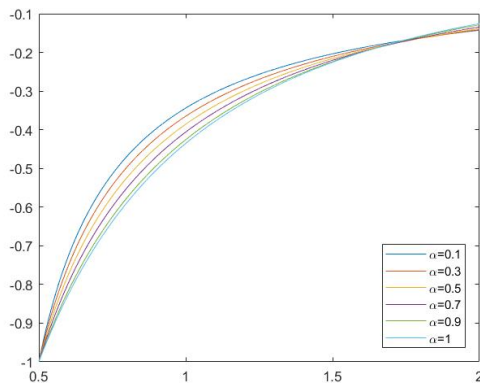


FIGURE 1. Solution of equation (4.10) with $T(t, \alpha) = t^{1-\alpha}$ for $\alpha = 0.1, 0.3, 0.5, 0.7, 0.9, 1$.

As we can see, the solutions approach the curve associated with the value of $\alpha = 1$, which corresponds to the case of the Bernoulli equation with derivative of order one. This is due to the fact the kernel $T(t, \alpha) = t^{1-\alpha}$ is associated with a generalized conformable derivative; however, this does not happen in all cases, as we will show in the next example.

Now if we choose the functions $p(t) = q(t) = t$, the interval $I = [-1, 1]$ and $n = 2$, replacing these values in equation (4.8) we obtain

$$G_T^\alpha y + ty = ty^2. \quad (4.11)$$

Then, if we choose the kernel $T(t, \alpha) = t^{-\alpha}$ and $C = -1$, by Theorem 4.4 we obtain

$$y(t) = \left(1 - 2e^{\frac{t^{(\alpha+2)} + e^{0.5(\alpha+2)}}{\alpha+2}} \right)^{-1}.$$

The function

$$y^*(t) = \left(1 - 2e^{\frac{t^2 + e^{(0.5)^2}}{2}} \right)^{-1} \quad (4.12)$$

is the solution associated to the Bernoulli differential equation with the classical derivative. We wanted to show both solutions to demonstrate how, in this example, the solutions of the generalized equation are approaching the solution associated with $\alpha = 0$. But they differ from the solution of the equation using the classical derivative, because a non-conformable kernel has been used. In Figure 2 the equation (4.12) and solutions associated to (4.11) for different values of α are shown.

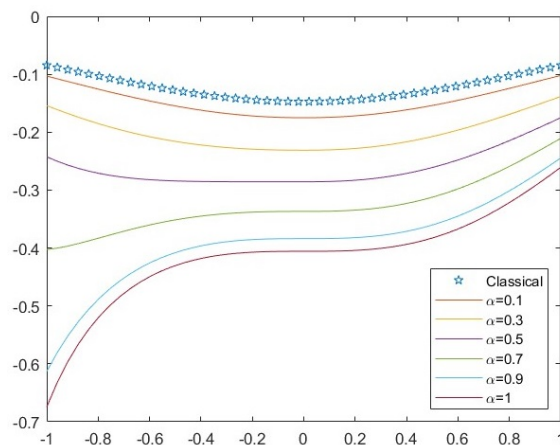


FIGURE 2. Equation (4.12) and solution of equation (4.11) with $T(t, \alpha) = t^{-\alpha}$ for $\alpha = 0.1, 0.3, 0.5, 0.7, 0.9, 1$.

As we can see, one of the advantages of considering the generalized derivative is precisely the freedom to choose the kernel in such a way that it is appropriate for the study and solution of the problem under consideration. In addition, it generalizes the definitions of the operators mentioned at the beginning of this paper.

5. A FINITE DIFFERENCE METHOD FOR THE GENERALIZED BERNOULLI DIFFERENTIAL EQUATION

Numerical methods are usually employed to solve differential equations for which there is no exact solution. In other cases, even when an exact solution exists, difficulties may arise, such as integrals that cannot be expressed in terms of elementary functions. That is why the objective of the numerical method presented below is to provide an approximate solution in cases where, even with an exact solution, the solution is not easy to visualize. The second example, given by equation (5.6), is one of these cases.

The finite difference method is often used to solve differential equations of various orders with boundary conditions. In general, first the differential operators are replaced by appropriate quotients in differences and a matrix is constructed, which is usually tridiagonal. Then the boundary conditions that form the first and last row of the matrix are added. Finally, the system of equations is solved by an appropriate method that minimizes the approximation error. We have intended to follow this line of thought to develop our method. In this case, since we do not have any information at the right end of the interval, we cannot proceed as in the usual finite differences.

So, the strategy will be to transform the process to an iterative method based on the idea of the fixed point theory, then use the two-point finite difference formula to find the first value on the right using the initial condition as the initial value, then the three-point difference formula to generate the next three values and achieve a better order of convergence, and then finally move to the five-point difference formula with an order of convergence $\mathcal{O}(h^4)$ to finish the iteration.

It is worth mentioning that the method can be easily extended to other types of non-linearities, and it can also be divided into three independent methods that work with different degrees of convergence. By switching from forward to backward difference, one can decide whether to solve a nonlinear equation or not, although it is necessary to choose between speed and convergence.

The equation

$$\begin{cases} G_T^\alpha y + p(t)y = q(t)y^n, & a \leq t \leq b, \\ y(a) = \beta \end{cases} \quad (5.1)$$

is, according to Theorem 2.3, equivalent to

$$y' + \frac{1}{T(t, \alpha)} p(t)y = \frac{1}{T(t, \alpha)} q(t)y^n.$$

First, we select an integer $N > 0$ and divide the interval $[a, b]$ into $N + 1$ equal subintervals whose endpoints are the mesh points $t_i = a + ih$ for $i = 0, 1, \dots, N + 1$, where $h = (b - a)/(N + 1)$ and $y_i = y(t_i)$.

Since the only information we do have is the $y_0 = y(a) = \beta$ value given by the initial condition, we will start by using the two-point forward difference formula to determine y_1 , with truncation error $\mathcal{O}(h)$

$$y'(t_i) = \frac{y(t_i + h) - y(t_i)}{h} - \frac{h}{2} y''(\xi_0) \quad (t_i < \xi_0 < t_i + h).$$

By making $y(t_{i+1}) = y(t_i + h)$, the equation (4.8) can be written as

$$\frac{1}{h} [y(t_{i+1}) - y(t_i)] + \frac{p(t_i)}{T(t_i, \alpha)} y(t_i) = \frac{q(t_i)}{T(t_i, \alpha)} y(t_i)^n,$$

or

$$y(t_{i+1}) = \frac{-hp(t_i)}{T(t_i, \alpha)} y(t_i) + \frac{hq(t_i)}{T(t_i, \alpha)} y(t_i)^n + y(t_i). \quad (5.2)$$

Starting at $i = 0$, the equation (5.2) defines an iterative process that converges to the solution; this is the well-known Euler method.

We can also use the finite difference backward formula (replacing h by $-h$). Then the iterative process is defined as follows:

$$y(t_i) - y(t_{i-1}) + \frac{hp(t_i)}{T(t_i, \alpha)} y(t_i) - \frac{hq(t_i)}{T(t_i, \alpha)} y(t_i)^n = 0. \quad (5.3)$$

At each step of the iterative process, we must solve a nonlinear equation to determine $y(t_i)$. The `fsolve` command in Matlab R2022b can be used for this purpose, taking the initial condition as the starting approximation, which is then updated using the previous point.

The equation (5.3) improves the convergence with respect to the equation (5.2) in exchange for consuming more time. So we must decide between a better convergence or a shorter execution time.

We have already found a solution to our problem, but recall that our goal was to use one of the above equations to find y_1 . Using the three-point midpoint difference formula, with truncation error $\mathcal{O}(h^2)$, we can find y_2 :

$$y'(t_i) = \frac{1}{2h} [y(t_i + h) - y(t_i - h)] - \frac{h^2}{6} y^{(3)}(\xi_1) \quad (t_i - h < \xi_1 < t_i + h),$$

and since we know y_0 and y_1 and taking $i = 1$ to start, the equation (5.1) becomes

$$\frac{1}{2h} [y(t_{i+1}) - y(t_{i-1})] + \frac{p(t_i)}{T(t_i, \alpha)} y(t_i) = \frac{q(t_i)}{T(t_i, \alpha)} y(t_i)^n,$$

or

$$y(t_{i+1}) = \frac{-2hp(t_i)}{T(t_i, \alpha)} y(t_i) + \frac{2hq(t_i)}{T(t_i, \alpha)} y(t_i)^n + y(t_{i-1}). \quad (5.4)$$

With (5.4) we again obtain an iterative process that converges to the solution, with better results than previously found. Now let us try with the three-point backward difference formula

$$y'(t_i) = \frac{1}{2h} [3y(t_i) - 4y(t_{i-1}) + y(t_{i-2})] - \frac{h^2}{3} y^{(3)}(\xi_1) \quad (t_i < \xi_1 < t_i + 2h).$$

The process is determined as follows:

$$3y(t_i) - 4y(t_{i-1}) + y(t_{i-2}) + \frac{hp(t_i)}{T(t_i, \alpha)} y(t_i) - \frac{hq(t_i)}{T(t_i, \alpha)} y(t_i)^n = 0. \quad (5.5)$$

Starting at $i = 2$ we must again solve for $y(t_i)$ at each iteration and the same effects as above will be obtained.

Finally, let us use the initial condition y_0 , take y_1 from the first iteration of (5.2) or (5.3), and y_3 and y_4 from (5.4) or (5.5) to develop a last method from the

backward five-point finite difference formula with truncation error $\mathcal{O}(h^4)$. It is not recommended to use a higher order approximation scheme due to Runge's phenomenon; instead, other techniques including for example irregular grids, Chebyshev polynomials or spectral methods can be used.

The backward five-point difference formula is determined by the formula

$$y'(t_i) = \frac{1}{12h} [25y(t_i) - 48y(t_i - h) + 36y(t_i - 2h) - 16y(t_i - 3h) + 3y(t_i - 4h)] \\ - \frac{h^4}{5} y^{(3)}(\xi_2) \quad (t_i < \xi_2 < t_i + 4h).$$

We then obtain

$$25y(t_i) - 48y(t_{i-1}) + 36y(t_{i-2}) - 16y(t_{i-3}) + 3y(t_{i-4}) \\ + \frac{12hp(t_i)}{T(t_i, \alpha)} y(t_i) - \frac{12hq(t_i)}{T(t_i, \alpha)} y(t_i)^n = 0,$$

which again must be solved using some method for nonlinear equations.

5.1. Some numerical examples. Let us solve again equation (4.10) using this variation of the finite difference method (FDM). Table 1 and Figure 3 show the solution and the error between the finite difference method and the exact solution of equation (4.10) for $\alpha = 0.5$. We have zoomed in on Figure 3 to show the difference between the curves.

Pts	Exact Sol	FDM	Error
0.5	-1	-1	0
0.500299940011998	-0.999152308016293	-0.999152149990093	1.58026200125505e-07
0.500599880023995	-0.998305946878573	-0.998305948338594	1.46002121503841e-09
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1.99970005998800	-0.138412976596618	-0.138413036296161	5.96995426982439e-08
2.000000000000000	-0.138379560668148	-0.138379620351763	5.96836149668878e-08

TABLE 1. Comparison between the FDM and the exact solution of (4.10) with $N = 5000$ interval divisions.

The above comparison shows that the proposed finite difference method is indeed feasible and that it delivers a solution quite close to the exact solution with considerably small error and low run time. In the following example we will show that the method can also be applied to more complicated problems, where it is sometimes difficult to obtain an exact solution.

There are cases in which the functions $p(t)$ and $q(t)$ make the use of Theorem 4.4 difficult because complicated integrals must be solved; herein lies the usefulness of numerical methods. If we choose the functions $p(t) = e^{-t^2}$, $q(t) = t$, $I = [-3, 3]$ and $n = 2$, we get the equation

$$G_T^\alpha y + e^{-t^2} y = ty^2. \quad (5.6)$$

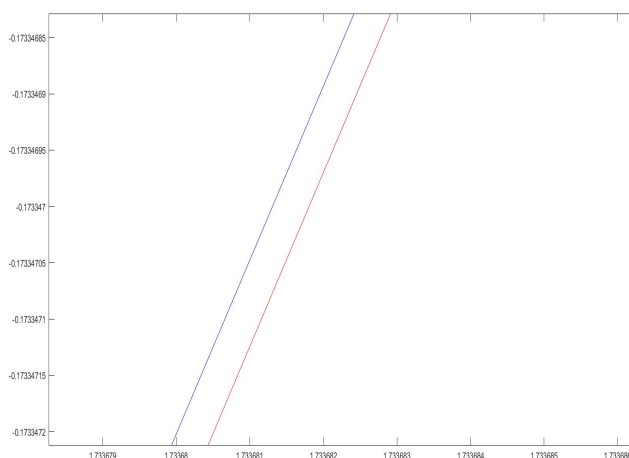


FIGURE 3. Comparison between the FDM and the exact solution of (4.10).

Taking $T(t, \alpha) = e^{(1-\alpha)t}$ and $C = 1$, we can find the solution to the previous equation through the FDM. Since we do not have an exact solution for comparison, we may approximate the integrals in Theorem 4.4 by a numerical integration method. This will allow us to validate again the proposed numerical method, as we do not have an explicit solution of the generalized equation. Table 2 shows both solutions for $\alpha = 0.5$. Graphically, the difference between the two solution curves is hardly noticeable. Figure 4 displays the solutions for different values of α for the FDM.

Pts	Integ	FDM	abs(FDM-Integ)
-3	1	1	0
-2.99880023995201	0.984132376988178	0.983900701320463	0.000231675667714537
-2.99760047990402	0.968775560311577	0.968798608504813	2.30481932365079e-05
$\vdots$	$\vdots$	$\vdots$	$\vdots$
2.99880023995201	0.0115194694289867	0.0115187854523517	6.83976635032849e-07
3.00000000000000	0.0115195756295232	0.0115188916402543	6.83989268873003e-07

TABLE 2. Comparison between FDM and approximation of the integrals in the exact solution.

In this example we can see that the solutions approximate in ascending order the case $\alpha = 1$ with the kernel $T(t, \alpha) = e^{(1-\alpha)t}$ that corresponds to a conformable derivative.

We have been able to compare two different approaches, one in which the derivative is approximated by an appropriate difference quotient and another in which the

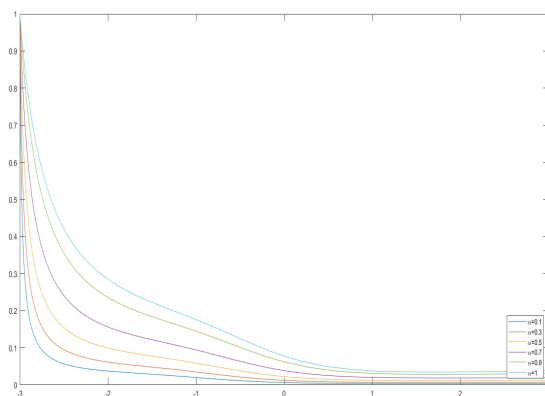


FIGURE 4. Solution of equation (5.6) with $T(t, \alpha) = e^{(1-\alpha)t}$ for $\alpha = 0.1, 0.3, 0.5, 0.7, 0.9, 1$.

integral has been approximated within the explicit solution. The approximation of the integral has been performed by numerical methods using Matlab R2022b. Although it has proved to be an alternative to FDM, it has required much more time to deliver the solution, showing an increase of time as the complexity of the integrals and the number of nodes increase.

The examples shown above prove that the proposed numerical method (FDM) is a valid alternative when complicated integrals appear in the exact solution, thereby broadening the range of problems that we can solve.

6. CONCLUSIONS

In this article we proposed and solved a generalization of the Bernoulli differential equation under the generalized conformable derivative approach. We also found solubility conditions and results about the qualitative behavior of the trivial solution. To this end, a generalization of Gronwall's inequality was proved as well as its reciprocal and a particular case of this inequality. Thereafter, we showed by means of examples how this generalized derivative approach has some advantages over other definitions; for example, this derivative generalizes certain definitions of fractional derivative known in the literature and further. Furthermore, it allows us to choose the kernel of the derivative depending on the problem under consideration, with the goal of solving different problems under different derivatives approaches. We also proposed and tested the reliability of a finite difference method by means of examples. We exposed the case in which the explicit solution involves complex integrals. Finally, we compared the solution obtained by (FDM) with the approximation of the integrals in the explicit solution of the generalized equation.

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