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DEPTH AND STANLEY DEPTH OF POWERS OF THE PATH IDEAL OF A CYCLE GRAPH

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ABSTRACT. Let $J_{n,m} := (x_1x_2\cdots x_m, x_2x_3\cdots x_{m+1},\ldots,x_{n-m+1}\cdots x_n, x_{n-m+2}\cdots x_nx_1,\ldots,x_nx_1\cdots x_{m-1})$ be the m-path ideal of the cycle graph of length n in the ring $S=K[x_1,\ldots,x_n]$. Let $d=\gcd(n,m)$. We prove that $\operatorname{depth}(S/J_{n,m}^t)\leq d-1$ for all $t\geq n-1$. We show that $\operatorname{sdepth}(S/J_{n,n-1}^t)=\operatorname{depth}(S/J_{n,n-1}^t)=\max\{n-t-1,0\}$ for all $t\geq 1$. Also, we give some bounds for $\operatorname{depth}(S/J_{n,m}^t)$ and $\operatorname{sdepth}(S/J_{n,m}^t)$, where $t\geq 1$.

Introduction

Let K be a field and $S=K[x_1,\ldots,x_n]$ the polynomial ring over K. The study of the edge ideals associated to graphs is a classical topic in combinatorial commutative algebra. Conca and De Negri generalized the definition of an edge ideal and first introduced the notion of a m-path ideal in [6]. In the recent years, several algebraic and combinatorial properties of path ideals have been studied extensively. However, little is known about the powers of m-path ideals.

Following our previous work [3], the aim of our paper is to investigate the depth and the Stanley depth (sdepth) of the quotient rings associated to powers of the m-path ideal of a cycle. For the definition of the sdepth invariant see Section 2.

For $n\geq m\geq 1$, the *m-path ideal of the path graph* of length n is

$$I_{n,m}=(x_1x_2\cdots x_m, x_2x_3\cdots x_{m+1},\ldots,x_{n-m+1}\cdots x_n)\subset S.$$

The *m-path ideal of the cycle graph* of length n is

$$J_{n,m}=I_{n,m}+(x_{n-m+2}\cdots x_nx_1, x_{n-m+3}\cdots x_nx_1x_2,\ldots,x_nx_1\cdots x_{m-1}).$$

In [3] we proved that

$$\operatorname{depth}(S/I_{n,m}^t)=\varphi(n,m,t):=\begin{cases} n-t+2-\left\lfloor\frac{n-t+2}{m+1}\right\rfloor-\left\lceil\frac{n-t+2}{m+1}\right\rceil, & t\leq n+1-m;\\ m-1, & t>n+1-m.\end{cases}$$

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Also, we proved that

$$\operatorname{sdepth}(S/I_{n,m}^t) \geq \operatorname{depth}(S/I_{n,m}^t) \quad \text{and} \quad \operatorname{sdepth}(I_{n,m}^t) \geq \operatorname{depth}(I_{n,m}^t).$$

The scope of our paper is to obtain similar results for powers of the ideal $J_{n,m}$. Let $n > m \geq 2$ and $t \geq 1$. For $m = 2$, Mihn, Trung and Vu [10] proved that

$$\operatorname{depth}(S/J_{n,2}^t) = \left\lceil \frac{n-t+1}{3} \right\rceil \quad \text{for all } 2 \leq t < \left\lceil \frac{n+1}{2} \right\rceil.$$

Let $d = \gcd(n,m)$ and let $t_0 \leq n-1$ be maximal with the property that there exists an integer α such that $mt_0 = \alpha n + d$. In Theorem 2.5, we prove that if $d = 1$ then

$$\operatorname{sdepth}(S/J_{n,m}^t) = \operatorname{depth}(S/J_{n,m}^t) = 0 \quad \text{for all } t \geq t_0.$$

Also, we prove that if $d > 1$ then

$$\operatorname{depth}(S/J_{n,m}^t) \leq d-1 \quad \text{and} \quad \operatorname{sdepth}(S/J_{n,m}^t) \leq n - \frac{n}{d} \quad \text{for all } t \geq t_0.$$

In Corollary 2.8, we prove that if n is odd, then

$$\operatorname{sdepth}(S/J_{n,n-2}^t) = \operatorname{depth}(S/J_{n,n-2}^t) = 0 \quad \text{for all } t \geq \frac{n-1}{2}.$$

Also, we prove that if n is even, then

$$\operatorname{depth}(S/J_{n,n-2}^t) \leq 1 \quad \text{and} \quad \operatorname{sdepth}(S/J_{n,n-2}^t) \leq \frac{n}{2} \quad \text{for all } t \geq n-1.$$

In Theorem 2.10, we prove that

$$\operatorname{depth}(S/J_{n,m}^t) \leq \varphi(n-1,m,t)+1.$$

In Theorem 3.1, we prove that

$$\operatorname{sdepth}(S/J_{n,n-1}^t) = \operatorname{depth}(S/J_{n,n-1}^t) = \begin{cases} n-t-1, & t \leq n-1; \\ 0, & t \geq n. \end{cases}$$

In Theorem 3.4, we show that if $n = mt-1$ then

$$\operatorname{sdepth}(S/J_{n,m}^s) = \operatorname{depth}(S/J_{n,m}^s) = 0 \quad \text{for all } s \geq t.$$

Also, for $n \geq mt$, we prove that

$$\operatorname{sdepth}(S/J_{n,m}^t), \operatorname{depth}(S/J_{n,m}^t) \geq \varphi(n-1,m,t).$$

In the last section, we provide a code in CoCoA that computes $\operatorname{depth}(S/J_{n,m}^t)$.

1. Preliminaries

First, we recall the well-known depth lemma; see, for instance, [15, Lemma 2.3.9].

Lemma 1.1 (Depth lemma). *If $0 \to U \to M \to N \to 0$ is a short exact sequence of modules over a local ring S, or a Noetherian graded ring with S_0 local, then*

1. $\operatorname{depth} M \geq \min\{\operatorname{depth} N, \operatorname{depth} U\}$.
2. $\operatorname{depth} U \geq \min\{\operatorname{depth} M, \operatorname{depth} N + 1\}$.
3. $\operatorname{depth} N \geq \min\{\operatorname{depth} U - 1, \operatorname{depth} M\}$.

Let M be a $\mathbb{Z}^n$-graded S-module. A *Stanley decomposition* of M is a direct sum $\mathcal{D} : M = \bigoplus_{i=1}^{r} m_i K[Z_i]$ as a $\mathbb{Z}^n$-graded K-vector space, where $m_i \in M$ is homogeneous with respect to $\mathbb{Z}^n$-grading, $Z_i \subset \{x_1, \ldots, x_n\}$ such that $m_i K[Z_i] = \{u m_i : u \in K[Z_i]\} \subset M$ is a free $K[Z_i]$-submodule of M. We define $\operatorname{sdepth}(\mathcal{D}) = \min_{i=1,\ldots,r} |Z_i|$ and $\operatorname{sdepth}(M) = \max\{\operatorname{sdepth}(\mathcal{D}) \mid \mathcal{D}$ is a Stanley decomposition of $M\}$. The number $\operatorname{sdepth}(M)$ is called the *Stanley depth* of M.

Herzog, Vladoiu and Zheng showed in [9] that $\operatorname{sdepth}(M)$ can be computed in a finite number of steps if $M = I/J$, where $J \subset I \subset S$ are monomial ideals. In [13], Rinaldo gave a computer implementation for this algorithm, in the computer algebra system CoCoA. We say that a $\mathbb{Z}^n$-graded S-module M satisfies the Stanley inequality if

$$\operatorname{sdepth}(M) \geq \operatorname{depth}(M).$$

In [2], J. Apel restated a conjecture firstly given by Stanley in [14], namely that any $\mathbb{Z}^n$-graded S-module M satisfies the Stanley inequality. This conjecture proves to be false, in general, for $M = S/I$ and $M = J/I$, where $0 \neq I \subset J \subset S$ are monomial ideals (see [7]), but remains open for $M = I$.

The explicit computation of the Stanley depth it is a difficult task, even in very particular cases, and it is interesting in itself. Also, although the Stanley conjecture was disproved in the most general setup, it is interesting to find large classes of ideals that satisfy the Stanley inequality. For a friendly introduction to the topic of Stanley depth, we refer the reader to [8].

In [12], Asia Rauf proved the analog of Lemma 1.1 for sdepth:

Lemma 1.2. *If* $0 \to U \to M \to N \to 0$ *is a short exact sequence of* $\mathbb{Z}^n$*-graded* S*-modules, then* $\operatorname{sdepth}(M) \geq \min\{\operatorname{sdepth}(U), \operatorname{sdepth}(N)\}$.

We recall the following well-known result (see, for instance, [15, Lemma 2.3.10]):

Lemma 1.3. *Let* M *be a graded* S*-module and* $f \in \mathfrak{m} = (x_1, \ldots, x_n) \subset S$ *a homogeneous polynomial such that* f *is regular on* M*. Then* $\operatorname{depth}(M/fM) = \operatorname{depth}(M) - 1$.

We also recall the following well-known results. See, for instance, [12, Corollary 1.3], [5, Proposition 2.7], [4, Theorem 1.1], [9, Lemma 3.6] and [12, Corollary 3.3].

Lemma 1.4. *Let* $I \subset S$ *be a monomial ideal and let* $u \in S$ *be a monomial such that* $u \notin I$*. Then*

(1) $\operatorname{sdepth}(S/(I : u)) \geq \operatorname{sdepth}(S/I)$.
(2) $\operatorname{depth}(S/(I : u)) \geq \operatorname{depth}(S/I)$.

Lemma 1.5. *Let* $I \subset S$ *be a monomial ideal and let* $u \in S$ *be a monomial such that* $I = u(I : u)$*. Then*

(1) $\operatorname{sdepth}(S/(I : u)) = \operatorname{sdepth}(S/I)$.
(2) $\operatorname{depth}(S/(I : u)) = \operatorname{depth}(S/I)$.

Lemma 1.6. *Let* $I \subset S$ *be a monomial ideal and let* $S' = S[x_{n+1}]$*. Then*

(1) $\operatorname{sdepth}_{S'}(S'/IS') = \operatorname{sdepth}_S(S/I) + 1$.
(2) $\operatorname{depth}_{S'}(S'/IS') = \operatorname{depth}_S(S/I) + 1$.

Lemma 1.7. *Let $I \subset S$ be a monomial ideal. Then the following assertions are equivalent:*

1. $\mathfrak{m} = (x_1, \dots, x_n) \in \operatorname{Ass}(S/I)$.
2. $\operatorname{depth}(S/I) = 0$.
3. $\operatorname{sdepth}(S/I) = 0$.

Let $2 \leq m < n$ be two integers. We consider the ideal

$$I_{n,m} = (x_1 \cdots x_m, x_2 \cdots x_{m+1}, \dots, x_{n-m+1} \cdots x_n) \subset S.$$

We denote

$$\varphi(n,m,t) := \begin{cases} n - t + 2 - \left\lfloor \frac{n-t+2}{m+1} \right\rfloor - \left\lceil \frac{n-t+2}{m+1} \right\rceil, & t \leq n+1-m; \\ m-1, & t > n+1-m. \end{cases}$$

We recall the main result of [3]:

Theorem 1.8 (See [3, Theorem 2.6])**.** *With the above notation, we have:*

1. $\operatorname{sdepth}(S/I_{n,m}^t) \geq \operatorname{depth}(S/I_{n,m}^t) = \varphi(n,m,t)$ *for any* $1 \leq m \leq n$ *and* $t \geq 1$.
2. $\operatorname{sdepth}(S/I_{n,m}^t) \leq \operatorname{sdepth}(S/I_{n,m}) = \varphi(n,m,1)$.

2. Main results

We consider the following ideal:

$$J_{n,m} = I_{n,m} + (x_{n-m+2} \cdots x_n x_1, x_{n-m+3} \cdots x_n x_1 x_2, \dots, x_n x_1 \cdots x_{m-1}).$$

Let $d = \gcd(n,m)$ and let $t_0 := t_0(n,m)$ be the maximal integer such that $t_0 \leq n-1$ and there exists a positive integer α such that

$$mt_0 = \alpha n + d.$$

Let $t \geq t_0$ be an integer. Let $w = (x_1 x_2 \cdots x_n)^\alpha$, $w_t = w \cdot (x_1 \cdots x_m)^{t-t_0}$, $r := \frac{n}{d}$ and $s := \frac{m}{d}$. If $d > 1$, we consider the ideal

$$U_{n,d} = (x_1, x_{d+1}, \dots, x_{d(r-1)+1}) \cap (x_2, x_{d+2}, \dots, x_{d(r-1)+2}) \cap \cdots \cap (x_d, x_{2d}, \dots, x_{rd}).$$

Firstly, we state the following lemma:

Lemma 2.1. *The map* $\frac{\mathbb{Z}/n\mathbb{Z}}{r \cdot (\mathbb{Z}/n\mathbb{Z})} \xrightarrow{\cdot s} \frac{\mathbb{Z}/n\mathbb{Z}}{r \cdot (\mathbb{Z}/n\mathbb{Z})}$ *is bijective.*

Proof. It follows from the fact that $\gcd(s,r) = 1$. □

As usual, if $J \subset S$ is a monomial ideal, we denote by $G(J)$ the set of minimal monomial generators of J.

Lemma 2.2. *With the above notations, we have:*

1. *If* $d = 1$ *then* $(J_{n,m}^t : w_t) = \mathfrak{m}$ *for all* $t \geq t_0$.
2. *If* $d > 1$ *then* $(J_{n,m}^t : w_t) = U_{n,d}$ *for all* $t \geq t_0$.

Proof. (1) Note that $\widehat{t_0} = \widehat{m}^{-1}$ in $\mathbb{Z}/n\mathbb{Z}$, hence t_0 and α are uniquely defined. We claim that it is enough to show the assertion for $t = t_0$, that is, $(J_{n,m}^{t_0} : w) = \mathfrak{m}$.

Assume that $(J_{n,m}^{t_0} : w) = \mathfrak{m}$ and $t > t_0$. Since $x_j w \in J_{n,m}^{t_0}$ for all $1 \leq j \leq n$, it follows that $x_j w_t = x_j w(x_1 \cdots x_m)^{t-t_0} \in J_{n,m}^t$ for all $1 \leq j \leq n$, and therefore $\mathfrak{m} \subset (J_{n,m}^t : w_t)$. On the other hand, $w_t \notin J_{n,m}^t$ since $\deg(w_t) = mt - 1$ and $J_{n,m}^t$ is generated in degree mt. Hence $(J_{n,m}^t : w_t) = \mathfrak{m}$, and the claim is proved.

Since $J_{n,m}$ is invariant to circular permutations of variables and $w \notin J_{n,m}^{t_0}$, it is enough to show that

$$x_1 w = x_1^{\alpha+1} x_2^{\alpha} \cdots x_n^{\alpha} \in G(J_{n,m}^{t_0}). \tag{2.1}$$

Indeed, one can easily check that

$$x_1 w = \prod_{j=0}^{t_0-1} (x_{\ell(mj+1)} \cdots x_{\ell(mj+m)}),$$

where $\ell(k) \in \{1, \ldots, n\}$ is the unique integer with $k \equiv \ell(k) \pmod n$.

As $x_{\ell(mj+1)} \cdots x_{\ell(mj+m)} \in G(J_{n,m})$ for all $0 \leq j \leq t_0 - 1$, we proved (2.1) and thus (1).

(2) Note that $\deg(w_t) = \alpha n + m(t - t_0) = mt - d$, while $J_{n,m}^t$ is minimally generated by monomials of degree mt. Also, as $w_t = (x_1 \cdots x_m)^{t-t_0} w$ and $x_1 \cdots x_m \in G(J_{n,m})$, we have that

$$(J_{n,m}^{t_0} : w) \subseteq (J_{n,m}^t : w_t). \tag{2.2}$$

Let $u = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \in G(J_{n,m}^t)$. Note that if $u \in G(J_{n,m})$ then $\mathrm{supp}(u)$ contains exactly $s = \frac{m}{d}$ variables whose indices are congruent with j modulo d, where $0 \leq j \leq d - 1$. Therefore, as $d = \gcd(n, m)$, it follows that

$$\begin{aligned} a_1 + a_{d+1} + \cdots + a_{d(r-1)+1} = a_2 + a_{d+2} + \cdots + a_{d(r-1)+2} = \cdots \\ = a_d + a_{2d} + \cdots + a_{rd} = \frac{tm}{d}. \end{aligned} \tag{2.3}$$

Similarly, if we rewrite $w_t = (x_1 \cdots x_n)^{\alpha} (x_1 \cdots x_m)^{t-t_0}$ as $w_t = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ then we have

$$b_1 + b_{d+1} + \cdots + b_{d(r-1)+1} = \cdots = b_d + b_{2d} + \cdots + b_{rd} = \frac{n\alpha}{d} + \frac{m(t-t_0)}{d} = \frac{tm}{d} - 1. \tag{2.4}$$

Let $v \in S$ be a monomial such that $v w_t \in J_{n,m}^t$. It follows that there exists $u \in G(J_{n,m}^t)$ such that $u | v w_t$. From (2.3) and (2.4) it follows that for every $0 \leq j \leq d - 1$ there exists $k_j \in \{1, \ldots, n\}$ with $k_j \equiv j \pmod d$ such that $x_{k_j} | v$. Therefore, $v \in U_{n,d}$ and thus $(J_{n,m}^t : w_t) \subseteq U_{n,d}$. Hence, from (2.2), the identity $(J_{n,m}^t : w_t) = U_{n,d}$ follows from

$$U_{n,d} \subseteq (J_{n,m}^{t_0} : w). \tag{2.5}$$

Let $v = x_{\ell_1} x_{\ell_2} \cdots x_{\ell_d} \in G(U_{n,d})$, where $\ell_j \equiv j \pmod d$. In order to prove (2.5), it suffices to show that $vw \in G(J_{n,m}^{t_0})$.

Since $mt_0 = \alpha n + d$, by dividing with d, it follows that $st_0 = \alpha r + 1$ and therefore $\overline{t_0} = \overline{s}^{-1}$ in $\mathbb{Z}/r\mathbb{Z}$. If $\overline{t_0} = \overline{\ell}$ with $0 \leq \ell \leq r-1$, then we claim that

$$t_0 = \ell + n - r \geq n - r = r(d-1). \tag{2.6}$$

Let $t_0' = \ell + n - r$. Since $\overline{t_0'} = \overline{t_0} = \overline{s}^{-1}$, we can write $\ell s = \beta r + 1$ for some β and thus

$$st_0' = s(\ell + n - r) = \beta r + s(n-r) + 1 = (\beta + s(d-1))r + 1 = \alpha' r + 1,$$

with $\alpha' = \beta + s(d-1)$. Hence $mt_0' = \alpha' r + d$. Since $\overline{t_0'} = \overline{t_0}$ in $\mathbb{Z}/r\mathbb{Z}$ and $t_0 \leq n-1$ is the greatest integer with $mt_0 = \alpha r + d$, it follows that $t_0 = t_0'$. Hence, we proved (2.6).

For simplicity, if $j > n$, we denote by x_j the variable $x_{\ell(j)}$, where $1 \leq \ell(j) \leq n$ such that $j \equiv \ell(j) \pmod{n}$. See also the proof of (1).

Given a monomial $u = x_i x_{i+1} \cdots x_{i+m-1} \in G(J_{n,m})$, we let $x_{\min(u)} = x_i$ and $x_{\max(u)} = x_{i+m-1}$ (with the above convention).

We apply the following algorithm:

(1) We let $u_1 := x_{\ell_d} x_{\ell_d+1} \cdots x_{\ell_d+m-1}$, where $v = x_{\ell_1} \cdots x_{\ell_d}$; see above.
(2) Assume we defined $u_1, \ldots, u_k$, where $1 \leq k \leq t_0 - 1$. If $x_{\max(u_k)} = x_{\ell_j}$ for some $1 \leq j \leq d-1$, then we let $u_{k+1} = x_{\ell_j} x_{\ell_j+1} \cdots x_{\ell_j+m-1}$. Otherwise, we let $u_{k+1} := x_{\max(u_k)+1} \cdots x_{\max(u_k)+m}$.
(3) We repeat step (2) until $k = t_0$.

We claim that $vw = u_1 u_2 \cdots u_{t_0}$. Obviously, $\deg(vw) = mt_0 = \deg(u_1 \cdots u_{t_0})$.

Let k_1 be the minimal index with $\max(u_{k_1}) = d-1$. We claim that $k_1 \leq r$. Indeed, if $k_1 \geq r$ then $u_1 = x_{\ell_d} \cdots x_{\ell_d+m-1}, \ldots, u_r = x_{\ell_d+(r-1)m} \cdots x_{\ell_d+rm-1}$. From Lemma 2.1 and the fact that $m = sd$, it follows that

$$\{\ell_d + m - 1, \ldots, \ell_d + rm - 1\} = \{d-1, 2d-1, \ldots, rd-1\},$$

using the above convention. Since $\ell_{d-1} \equiv (d-1) \pmod{d}$, from all the above, it follows that $\ell_{d-1} = \ell_d + rm - 1$ and hence $k_1 = r$. Note that $u_{k_1} = x_{\ell_{d-1}} \cdots x_{\ell_{d-1}+m-1}$.

Similarly, let k_2 be the minimal index with $\max(u_{k_2}) = d-2$. Using the same line of arguing, it follows that $k_2 \leq 2r$. Inductively, let k_j be the minimal index with $\max(u_{k_j}) = d-j$, for $j \leq d-1$. Then $k_j \leq jr$. In particular, we have that $k_{d-1} \leq (d-1)r \leq t_0$. Also, for $k > k_{d-1}$, from the definition of u_k's, we have that $\max(u_k) \notin \{\ell_1, \ldots, \ell_{d-1}\}$.

Now, from all the above, it is easy to see that $u_1 \cdots u_{t_0} = vw$, as required. □

In the following example, we show how the algorithm given in the proof of Lemma 2.2 (2) works.

Example 2.3. Let $n = 12$ and $m = 8$. Then $d = \gcd(n,m) = 4$, $r = 3$ and $s = 2$. Note that $8 \cdot 11 = 7 \cdot 12 + 4$ and $t_0 = 11$ is the largest integer $\leq n-1 = 11$ with $t_0 m = \alpha n + d$. Also $\alpha = 7$.

We have $w := (x_1 \cdots x_{12})^7$ and $U_{12,4} = (x_1, x_5, x_9) \cap (x_2, x_6, x_{10}) \cap (x_3, x_7, x_{11}) \cap (x_4, x_8, x_{12})$.

Let $v := x_5 x_2 x_{11} x_4 \in U_{12,4}$. Then $\ell_1 = 5$, $\ell_2 = 2$, $\ell_3 = 11$ and $\ell_4 = 4$.

We apply the aforementioned algorithm:

- We let $u_1 := x_4x_5x_6x_7x_8x_9x_{10}x_{11}$.
- Since $\ell_3 = 11$, it follows that $k_1 = 1$ and $u_2 = x_{11}x_{12}x_1x_2x_3x_4x_5x_6$.
- Since $\ell_2 \neq 6$, we let $u_3 = x_7x_8x_9x_{10}x_{11}x_{12}x_1x_2$.
- Since $\ell_2 = 2$, it follows that $k_2 = 3$ and $u_4 = x_2x_3x_4x_5x_6x_7x_8x_9$.
- Since $\ell_1 \neq 9$, we let $u_5 = x_{10}x_{11}x_{12}x_1x_2x_3x_4x_5$.
- Since $\ell_1 = 5$, it follows that $k_3 = 5$ and $u_6 = x_5x_6x_7x_8x_9x_{10}x_{11}x_{12}$.

From now on, the algorithm goes smoothly, and we have: $u_7 = x_1x_2x_3x_4x_5x_6x_7x_8$, $u_8 = x_9x_{10}x_{11}x_{12}x_1x_2x_3x_4$, $u_9 = u_6$, $u_{10} = u_7$ and $u_{11} = u_8$. It is easy to see that $u_1u_2\cdots u_{11} = vw$. Therefore $vw \in J_{12,8}^{11}$, as required.

The following result is elementary. However, we give a proof for the sake of completeness.

Lemma 2.4. *Let $d \geq 1$ and $Z_1 \cup Z_2 \cup \cdots \cup Z_d = \{x_1, \dots, x_n\}$ be a partition, i.e., $|Z_i| > 0$ and $Z_i \cap Z_j = \emptyset$ for all $i \neq j$. Let $P_i = (Z_i) \subset S$ for $1 \leq i \leq d$ and $U := P_1 \cap \cdots \cap P_d$. Then* $\operatorname{depth}(S/U) = d - 1$.

Proof. We use induction on $d \geq 1$. If $d = 1$ then $Z_1 = \{x_1, \dots, x_n\}$ and $U = \mathfrak{m} = (x_1, \dots, x_n)$. Hence, there is nothing to prove.

Without loss of generality, we can assume that $Z_1 \cup \cdots \cup Z_{d-1} = \{x_1, \dots, x_k\}$ for some $k < n$ and $Z_d = \{x_{k+1}, \dots, x_n\}$. From the induction hypothesis, we have that

$$\operatorname{depth}(S_k/(P_1 \cap \cdots \cap P_{d-1})) = d - 2, \text{ where } S_k = K[x_1, \dots, x_k].$$

From [11, Lemma 1.1] it follows that

$$\begin{aligned}&\operatorname{depth}(S/(P_1 \cap \cdots \cap P_d))\\ &\quad = \operatorname{depth}(S_k/(P_1 \cap \cdots \cap P_{d-1})) + \operatorname{depth}(K[x_{k+1}, \dots, x_n]/P_d) + 1 = d - 1,\end{aligned}$$

as required. □

Theorem 2.5. *With the above notations, we have:*

(1) *If $d = 1$ then* $\operatorname{sdepth}(S/J_{n,m}^t) = \operatorname{depth}(S/J_{n,m}^t) = 0$ *for all $t \geq t_0$.*
(2) *If $d > 1$ then* $\operatorname{depth}(S/J_{n,m}^t) \leq d - 1$ *for all $t \geq t_0$.*
(3) *If $d > 1$ then* $\operatorname{sdepth}(S/J_{n,m}^t) \leq \operatorname{sdepth}(S/U_{n,d}) \leq n - \frac{n}{d}$ *for all $t \geq t_0$.*

Proof. (1) From Lemma 2.2 (1), it follows that $\mathfrak{m} \in \operatorname{Ass}(S/J_{n,m}^t)$ for all $t \geq t_0$. Therefore, the required conclusion follows from Lemma 1.7.

(2) From Lemma 2.2 (2), it follows that $(J_{n,m}^t : w_t) = U_{n,d}$. From Lemma 1.4 (2) it follows that

$$\operatorname{depth}(S/J_{n,m}^t) \leq \operatorname{depth}(S/U_{n,d}).$$

On the other hand, from Lemma 2.4 it follows that $\operatorname{depth}(S/U_{n,d}) = d - 1$ and therefore $\operatorname{depth}(S/J_{n,m}^t) \leq d - 1$.

(3) Similarly, from Lemma 1.4 (1) it follows that

$$\operatorname{sdepth}(S/J_{n,m}^t) \leq \operatorname{sdepth}(S/U_{n,d}).$$

On the other hand, since $U_{n,d} = (x_d, \dots, x_{dr}) \cap U'$, where $U' = (x_1, \dots, x_{r(d-1)+1}) \cap \cdots \cap (x_{d-1}, \dots, x_{dr-1})$, from [5, Theorem 1.3] it follows that

$$\operatorname{sdepth}(S/U_{n,d}) \leq \operatorname{sdepth}(S/(x_d, \dots, x_{dr})) = n - r = n - \frac{n}{d},$$

as required. ☐

Our computer experiments in CoCoA, using the code given in Section 4, lead us to propose the following conjecture:

Conjecture 2.6. *We have that*

$$\operatorname{depth}(S/J_{n,m}^t) \geq d - 1 \quad \textit{for all } t \geq 1.$$

Remark 2.7. Let $n > m \geq 2$ be two integers and let $d := \gcd(n, m)$. From Theorem 2.5 we have that $\operatorname{depth}(S/J_{n,m}^t) \leq d-1$ for all $t \geq t_0$. Hence, if Conjecture 2.6 is true, then

$$\lim_{t \to \infty} \operatorname{depth}(S/J_{n,m}^t) = d - 1.$$

Corollary 2.8. *We have that:*

1. *If n is odd, then $\operatorname{sdepth}(S/J_{n,n-2}^t) = \operatorname{depth}(S/J_{n,n-2}^t) = 0$ for all $t \geq \frac{n-1}{2}$.*
2. *If n is even, then $\operatorname{depth}(S/J_{n,n-2}^t) \leq 1$ for all $t \geq n-1$.*
3. *If n is even, then $\operatorname{sdepth}(S/J_{n,n-2}^t) \leq \frac{n}{2}$ for all $t \geq n-1$.*

Proof. (1) Since n is odd, we have $d = \gcd(n, n-2) = 1$. It is easy to see that $t_0 = \frac{n-1}{2}$ and $\alpha = \frac{n-1}{3}$. Hence, from Theorem 2.5 (1) it follows that

$$\operatorname{sdepth}(S/J_{n,n-2}^t) = \operatorname{depth}(S/J_{n,n-2}^t) = 0 \quad \text{for all } t \geq \frac{n-1}{2}.$$

(2) Since n is even, we have $d = \gcd(n, n-2) = 2$. It is easy to see that $t_0 = n-1$ and $\alpha = n-3$. From Theorem 2.5 (2) it follows that

$$\operatorname{depth}(S/J_{n,n-2}^t) = 1 \quad \text{for all } t \geq n-1.$$

(3) As in the proof of (2), from Theorem 2.5 (3) it follows that

$$\operatorname{sdepth}(S/J_{n,n-2}^t) \leq \frac{n}{2} \quad \text{for all } t \geq n-1.$$

Hence, the proof is complete. ☐

Lemma 2.9. *Let $n > m \geq 2$ and $t \geq 1$ be some integers. Then*

$$(J_{n,m}^t, x_n) = (I_{n-1,m}^t, x_n).$$

Proof. The inclusion $\supseteq$ is obvious. The converse inclusion follows from the observation that a minimal monomial generator of $J_{n,m}$ which is not divisible by x_n belongs to $I_{n-1,m}$. ☐

Theorem 2.10. *Let $n > m \geq 2$ and $t \geq 1$ be some integers. Then:*

1. $\operatorname{depth}(S/J_{n,m}^t) \leq \varphi(n-1, m, t) + 1$.
2. *If* $\operatorname{depth}(S/(J_{n,m}^t : x_n)) > \operatorname{depth}(S/J_{n,m}^t)$ *then* $\operatorname{depth}(S/J_{n,m}^t) = \varphi(n-1, m, t)$.

Proof. (1) We consider the short exact sequence

$$0 \to S/(J_{n,m}^t : x_n) \to S/J_{n,m}^t \to S/(J_{n,m}^t, x_n) \to 0. \tag{2.7}$$

From Lemma 2.9 and Theorem 1.8 it follows that

$$\operatorname{depth}(S/(J_{n,m}^t, x_n)) = \operatorname{depth}(S/(I_{n-1,m}^t, x_n)) = \varphi(n-1, m, t).$$

Let $s := \operatorname{depth}(S/(J_{n,m}^t : x_n))$ and $d := \operatorname{depth}(S/J_{n,m}^t)$. From Lemma 1.4 (2), we have that $s \geq d$. Therefore, according to Lemma 1.1, it follows that

$$\varphi(n-1, m, t) \geq \min\{s-1, d\} \geq \min\{d-1, d\} = d-1.$$

Hence, $d \leq \varphi(n-1, m, t) + 1$, as required.

(2) As in the proof of (1), it follows from (2.7) and Lemma 1.1. □

3. Some special cases

We use the notations from the previous section.

Theorem 3.1. *We have that*

$$\operatorname{sdepth}(S/J_{n,n-1}^t) = \operatorname{depth}(S/J_{n,n-1}^t) = \begin{cases} n-t-1, & t \leq n-2 \\ 0, & t \geq n-1. \end{cases}$$

Proof. Since $d = \gcd(n, m) = \gcd(n, n-1) = 1$, it follows that $t_0 := t_0(n, n-1) = n-1$, since $mt_0 = (n-1)^2 = (n-2)n + 1 = \alpha n + d$. Therefore, according to Theorem 2.5 (1), the conclusion follows for $t \geq n-1$. Now, assume $t \leq n-2$.

If $n = 3$, then $t = 1$ and it is an easy exercise to show that

$$\operatorname{sdepth}(S/J_{3,2}) = \operatorname{depth}(S/J_{3,2}) = 1 = n-t-1.$$

Now, assume $n \geq 4$ and $t \leq n-2$. We consider the ideals

$$L_j := (J_{n,n-1}^t : x_n^j) \quad \text{for } 0 \leq j \leq t.$$

By straightforward computations, we have

$$L_j = J_{n,n-1}^{t-j}(J_{n-1,n-2}^j S) \quad \text{for } 0 \leq j \leq t \tag{3.1}$$

and

$$(L_j, x_n) = ((x_1 \cdots x_{n-1})^{t-j}(J_{n-1,n-2}^j S), x_n) \quad \text{for } 0 \leq j \leq t-1. \tag{3.2}$$

We consider the short exact sequences

$$\begin{aligned} 0 \to S/L_1 \to S/L_0 \to S/(L_0, x_n) \to 0 \\ 0 \to S/L_2 \to S/L_1 \to S/(L_1, x_n) \to 0 \\ \vdots \\ 0 \to S/L_t \to S/L_{t-1} \to S/(L_{t-1}, x_n) \to 0. \end{aligned} \tag{3.3}$$

From (3.2), the induction hypothesis and Lemma 1.5 it follows that

$$\operatorname{sdepth}(S/(L_j, x_n)) = \operatorname{depth}(S/(L_j, x_n)) = n-j-2 \quad \text{for all } 0 \leq j \leq t-1. \tag{3.4}$$

On the other hand, from (3.1) we have $L_t = (J^t_{n-1,n-2}S)$. Hence, from the induction hypothesis and Lemma 1.6 it follows that

$$\operatorname{sdepth}(S/L_t) = \operatorname{depth}(S/L_t) = (n-1-t-1)+1 = n-1-t. \tag{3.5}$$

From (3.4), (3.5) and the short exact sequences (3.3) we deduce inductively that

$$\operatorname{sdepth}(S/L_j), \operatorname{depth}(S/L_j) \geq n-1-t \quad \text{for all } 0 \leq j \leq t-1. \tag{3.6}$$

On the other hand, from Lemma 1.4 we have

$$\operatorname{depth}(S/L_0) \leq \operatorname{depth}(S/L_t) \quad \text{and} \quad \operatorname{sdepth}(S/L_0) \leq \operatorname{sdepth}(S/L_t). \tag{3.7}$$

Since $L_0 = J^t_{n,n-1}$, from (3.5), (3.6) and (3.7) it follows that

$$\operatorname{sdepth}(S/J^t_{n,n-1}) = \operatorname{depth}(S/J^t_{n,n-1}) = n-t-1,$$

which completes the proof. $\square$

Remark 3.2. Let $S_{n+m-1} := K[x_1, x_2, \ldots, x_{n+m-1}]$. We note that

$$\frac{S_{n+m-1}}{(I^t_{n+m-1,m}, x_1 - x_{n+1}, x_2 - x_{n+2}, \ldots, x_{m-1} - x_{n+m-1})} \cong \frac{S}{J^t_{n,m}}. \tag{3.8}$$

Assume that $m = n-1$. It is not difficult to see that

$$x_1 - x_{n+1}, x_2 - x_{n+2}, \ldots, x_{n-2} - x_{2n-2} \text{ is a regular sequence on } S_{2n-2}/I^t_{2n-2,n-1}. \tag{3.9}$$

From (3.9), Lemma 1.3, (3.8) and Theorem 1.8 it follows that

$$\operatorname{depth}(S/J^t_{n,n-1}) = \operatorname{depth}(S_{2n-2}/I^t_{2n-2,n-1}) = \varphi(2n-2, n-1, t) - n + 2,$$

from where we deduce the formula given in Theorem 3.1 for $\operatorname{depth}(S/J^t_{n,n-1})$. Unfortunately, this method is not useful in the computation of $\operatorname{sdepth}(S/J^t_{n,n-1})$.

We also mention that the sequence $x_1 - x_{n+1}, x_2 - x_{n+2}, \ldots, x_{n-2} - x_{2n-2}$ is not regular when $n > m+1$, therefore we cannot use (3.8) in order to compute (or at least to give some bounds for) $\operatorname{depth}(S/J^t_{n,m})$.

Lemma 3.3. *Let $m, t \geq 2$ and $n \geq mt-1$ be some integers, and $L = (J^t_{n,m} : (x_1x_2\cdots x_{mt-1}))$. We have that:*

1. *If $n = mt-1$ then $L = \mathfrak{m} = (x_1, x_2, \ldots, x_n)$.*
2. *If $n = mt$ then $L = (x_m, x_{2m}, \ldots, x_{mt})$.*
3. *If $mt < n \leq m(t+1)$ then $L = (x_m, x_{2m}, \ldots, x_{mt}, x_n)$.*
4. *If $n > m(t+1)$ then $L = (x_m, \ldots, x_{mt}, x_n) + V$, where*

$$\begin{aligned} V &= (x_{mt+1}\cdots x_{mt+m}, x_{mt+2}\cdots x_{mt+m+1}, \ldots, x_{n-m}\cdots x_{n-1})^t \\ &\subset K[x_{mt+1}, \ldots, x_{n-1}]. \end{aligned}$$

Moreover, $V \cong I^t_{n-mt-1,m}$.

Proof. (1) As in the proof of Lemma 2.2, we use the convention

$$j = n+j = 2n+j = \cdots \quad \text{for all } 1 \leq j \leq n.$$

We fix $1 \leq i \leq n$ and we define inductively the monomials $u_1 := x_ix_{i+1}\cdots x_{i+m-1}$ and $u_k := x_{m_k+1}x_{m_k+2}\cdots x_{m_k+m}$, where $m_1 = i$ and $m_k = m_{k-1}+m$ for $2 \leq k \leq t$.

Obviously, $u_k \in G(J_{n,m})$ for all $1 \leq k \leq t$, thus $u_1u_2\cdots u_t \in G(J_{n,m}^t)$. On the other hand, it is easy to see that

$$x_i \cdot (x_1x_2\cdots x_{mt-1}) = u_1u_2\cdots u_t.$$

Therefore, $x_i \in L$. Since i was arbitrarily chosen, it follows that $\mathfrak{m} \subset L$. Obviously, $L \neq S$, since $x_1x_2\cdots x_{mt-1} \notin J_{n,m}^t$. Therefore $L = \mathfrak{m}$, as required.

(2) Similarly to (1), we can deduce that $x_m, x_{2m}, \ldots, x_{mt} \in L$. For instance, we have

$$\begin{aligned} &x_m \cdot (x_1x_2\cdots x_{mt-1}) \\ &\qquad = (x_1\cdots x_m)(x_m\cdots x_{2m-1})(x_{2m}\cdots x_{3m-1})\cdots(x_{mt-m}\cdots x_{mt-1}) \in J_{n,m}^t. \end{aligned}$$

Also, it is easy to see that $x_j \notin L$ for any $j \notin \{m, 2m, \ldots, mt\}$. Since

$$(J_{n,m}^t, x_m, x_{2m}, \ldots, x_{mt}) = (x_m, \ldots, x_{mt}),$$

the conclusion follows immediately.

(3) The proof is similar to the proof of (2), with the remark that $x_n \in L$, since

$$x_n(x_1x_2\cdots x_{mt-1}) = (x_nx_1\cdots x_{m-1})(x_m\cdots x_{2m-1})\cdots(x_{(t-1)m}\cdots x_{tm-1}) \in J_{n,m}^t.$$

(4) As in the previous cases, it is easy to see that $(x_m, \ldots, x_{mt}, x_n) \subset L$ and $x_j \notin L$ for any $j \notin \{m, \ldots, mt, n\}$. Also, using similar arguments as in the proof of Lemma 2.9, we deduce that

$$(J_{n,m}^t, x_m, x_{2m}, \ldots, x_{mt}) = (x_m, \ldots, x_{mt}, x_n) + V.$$

Hence, we get the required conclusion. $\square$

Using the above lemma, we are able to prove the following result:

Theorem 3.4. *Let $m, t \geq 2$ and $n \geq mt - 1$ be some integers. We have that:*

1. *If $n = mt - 1$ then $\operatorname{sdepth}(S/J_{n,m}^s) = \operatorname{depth}(S/J_{n,m}^s) = 0$ for all $s \geq t$.*
2. *If $n \geq mt$ then $\operatorname{sdepth}(S/J_{n,m}^t) \geq \varphi(n-1, m, t)$.*
3. *If $n \geq mt$ then $\varphi(n-1, m, t) + 1 \geq \operatorname{depth}(S/J_{n,m}^t) \geq \varphi(n-1, m, t)$.*

Proof. Assume $n = mt - 1$. From Lemma 3.3 (1) it follows that

$$(J_{n,m}^t : w) = \mathfrak{m}, \quad \text{where } w := x_1x_2\cdots x_n. \tag{3.10}$$

Let $w_s = w \cdot (x_1\cdots x_m)^{s-t}$. Since $w_s \notin J_{n,m}^s$, from (3.10) it follows that

$$(J_{n,m}^s : w_s) = \mathfrak{m}.$$

Therefore, $\mathfrak{m} \in \operatorname{Ass}(S/J_{n,m}^s)$ and (1) follows from Lemma 1.7.

Now, assume $n \geq mt$. Let $L_0 = J_{n,m}^t$, $L_j := (L_0 : x_1\cdots x_j)$ for $1 \leq j \leq mt-1$, and $U_j = (L_{j-1}, x_j)$ for $1 \leq j \leq mt - 1$. We consider the short exact sequences

$$0 \to S/L_j \to S/L_{j-1} \to S/U_j \to 0 \quad \text{for } 1 \leq j \leq mt-1. \tag{3.11}$$

Note that, according to Lemma 2.9, we have that

$$(L_0, x_j) \cong (I_{n-1,m}^t, x_n) \quad \text{for all } 1 \leq j \leq mt-1, \tag{3.12}$$

where $I^t_{n-1,m} \subset S' = K[x_1,\ldots,x_{n-1}]$ and the isomorphism is given by the circular permutation of variables which send j to n. On the other hand, we have

$$\begin{aligned} U_j &= (L_{j-1},x_j) = ((J^t_{n,m}:x_1\ldots x_{j-1}),x_j) \\ &= ((J^t_{n,m},x_j):x_1\cdots x_{j-1}) \cong (I^t_{n-1,m}:x_{n-j+1}\cdots x_{n-1}) \end{aligned} \tag{3.13}$$

for all $1 \leq j \leq mt-1$. From (3.12), (3.13), Lemma 1.4 and Theorem 1.8 it follows that

$$\operatorname{depth}(S/U_j) \geq \operatorname{depth}(S'/I^t_{n-1,m}) = \varphi(n-1,m,t) \tag{3.14}$$

and

$$\operatorname{sdepth}(S/U_j) \geq \operatorname{sdepth}(S'/I^t_{n-1,m}) \geq \varphi(n-1,m,t). \tag{3.15}$$

Also, from Lemma 1.4, we have that

$$\operatorname{depth}(S/L_0) \leq \operatorname{depth}(S/L_1) \leq \cdots \leq \operatorname{depth}(S/L_{tm-1}) \tag{3.16}$$

and

$$\operatorname{sdepth}(S/L_0) \leq \operatorname{sdepth}(S/L_1) \leq \cdots \leq \operatorname{sdepth}(S/L_{tm-1}). \tag{3.17}$$

We consider three cases:

(i) If $n=mt$, then according to Lemma 3.3 (2), it follows that

$$\operatorname{sdepth}(S/L_{tm-1}) = \operatorname{depth}(S/L_{tm-1}) = n-t.$$

(ii) If $mt < n \leq m(t+1)$, then according to Lemma 3.3 (3), it follows that

$$\operatorname{sdepth}(S/L_{tm-1}) = \operatorname{depth}(S/L_{tm-1}) = n-t-1.$$

(iii) If $n > m(t+1)$, then according to Lemma 3.3 (4), it follows that

$$\operatorname{sdepth}(S/L_{tm-1}) \geq \operatorname{depth}(S/L_{tm-1}) = (m-1)t + \varphi(n-mt-1,m,t).$$

In all of the above cases (i), (ii) and (iii), it is easy to see that the following inequalities hold:

$$\operatorname{sdepth}(S/L_{tm-1}) \geq \operatorname{depth}(S/L_{tm-1}) \geq \varphi(n-1,m,t). \tag{3.18}$$

From (3.14), (3.15), (3.16), (3.17), (3.18), the short exact sequences (3.11), and Lemmas 1.1 and 1.2, it follows that

$$\operatorname{depth}(S/J^t_{n,m}) \geq \varphi(n-1,m,t) \quad \text{and} \quad \operatorname{sdepth}(S/J^t_{n,m}) \geq \varphi(n-1,m,t).$$

Also, from Theorem 2.10 (1), it follows that $\operatorname{depth}(S/J^t_{n,m}) \leq \varphi(n-1,m,t)+1$. Thus, we complete the proof of (2) and (3). $\square$

Remark 3.5. Note that, in the case (1) of Theorem 3.4, we have that $d = \gcd(n,m) = 1$. However, the result is stronger than the result from Theorem 2.5 (1), since $t_0 = n-1$ is larger than $t = \frac{n-1}{m}$.

4. A CoCoA code

The following code in CoCoA computes $\operatorname{depth}(S/J_{n,m}^t)$ with $S = \mathbb{Q}[x_1, \dots, x_n]$:

$N := 8$; $M := 6$; $T := 4$;
// Here one can choose other values.
Use $R ::= \mathrm{QQ}[x[1..N]]$;
$J := \mathrm{Ideal}(0)$;
For $K := 1$ To N Do
  $U := 1$;
  For $L := 1$ To M Do
    If $(K + L > N)$ Then $U := U * x[K + L - N]$;
    Else $U := U * x[K + L]$; EndIf;
  EndFor;
  $J := J + \mathrm{Ideal}(U)$;
EndFor;
// J is the ideal $J_{n,m}$.
$I := \mathrm{Ideal}(1)$;
For $S := 1$ To T Do $I := I * J$; EndFor;
// I is the ideal $J_{n,m}^t$.
$\mathrm{Depth}(R/I)$;
1;
// $\operatorname{depth}(S/J_{8,6}^4) = 1$, *where* $S := \mathbb{Q}[x_1, \dots, x_8]$.

Note that $d = \gcd(n, m) = \gcd(8, 6) = 1$.

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