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RECURRENCE FOR WEIGHTED PSEUDO-SHIFT OPERATORS

MOHAMED AMOUCH AND FATIMA-EZZAHRA SADEK

Abstract. We provide a characterization of multiply recurrent operators that act on a Fréchet space. As an application, we extend the weighted shift results established by Costakis and Parissis (2012). We achieve this by characterizing topologically multiply recurrent pseudo-shifts acting on an F-sequence space indexed by an arbitrary countable infinite set. This characterization is in terms of the weights, the OP-basis and the shift mapping. Additionally, we establish that the recurrence and the hypercyclicity of pseudo-shifts are equivalent.

1. Introduction and preliminary results

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of all integers and positive integers, respectively. Throughout this paper, unless stated otherwise, X refers to an F-sequence space indexed by an arbitrary countable infinite set I, that is, a subspace of the space $\mathbb{K}^I$ of all scalar families $(x_i)_{i\in I}$ endowed with its natural product topology.

The bilateral weighted backward (respectively, forward) shift $B_{\mathbf{w}}$ on a bilateral sequence space is defined as follows:

$$B_{\mathbf{w}}(x_k)_{k\in\mathbb{Z}} := (w_{k+1}x_{k+1})_{k\in\mathbb{Z}},$$

respectively,

$$B_{\mathbf{w}}(x_k)_{k\in\mathbb{Z}} := (w_{k-1}x_{k-1})_{k\in\mathbb{Z}},$$

where the weight sequence $\mathbf{w} = (w_k)_{k\in\mathbb{Z}}$ of the bilateral weighted shift is assumed to be a bounded sequence of positive real numbers. The unilateral weighted backward and forward shifts on a sequence space indexed by $\mathbb{N}$ are defined similarly with $x_0 = w_0 = 0$.

As an extension of weighted shifts, K.-G. Grosse-Erdmann introduced in [13] the concept of a weighted pseudo-shift T that acts on X, defined by the existence of a sequence of non-zero scalars $(w_i)_{i\in I}$, called the weight sequence, and an injective mapping $f : I \to I$ such that $T(x_i)_{i\in I} = (w_i x_{f(i)})_{i\in I}$. Since X is an F-space, the continuity of T follows from the closed graph theorem.

Recall that an operator T acting on a Fréchet space Y is hypercyclic if there is some x such that the orbit of x under T, that is, the set $\mathrm{Orb}(T, x) := \{T^n x : n \geq 0\}$, is dense in Y. Such a vector x is called a hypercyclic vector. The set of all

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hypercyclic vectors for T is denoted by HC(T). Verifying the hypercyclicity of an operator often involves demonstrating through an equivalent formulation that the orbit of any non-empty open set under the operator is dense in Y. For more background and examples about hypercyclicity, we refer the reader to the books by F. Bayart and E. Matheron [1] and by K.-G. Grosse-Erdmann and A. Peris Manguillot [14].

The concept of hypercyclicity was first introduced by S. Rolewicz in 1969, who showed that the multiple of a backward shift in a Banach space is hypercyclic if and only if the multiplication factor is greater than one. Since then, the study of weighted shifts has provided a wealth of examples and counterexamples in linear dynamics, as they offer a means to investigate the dynamic characteristics of continuous linear operators. In this context, H. N. Salas [17] established a characterization of hypercyclic weighted backward shifts. Later on, K.-G. Grosse-Erdmann [13] expanded upon Salas' results by characterizing hypercyclic weighted pseudo-shifts acting on an arbitrary F-sequence space. In recent years, the study of pseudo-shifts has garnered considerable attention, resulting in a surge of research and numerous studies into the topic. For example, Y. Wang et al. [18] investigated disjoint hypercyclicity and disjoint supercyclicity of tuples of distinct powers of weighted pseudo-shifts. On the other hand, N. Çolakoğlu et al. [7] characterized disjoint hypercyclicity of weighted pseudo-shifts that are raised to the same power. In addition, Ö. Martin et al. [15] have studied the so-called disjoint frequently hypercyclic pseudo-shifts.

Recurrence is one of the fundamental concepts in the theory of dynamical systems. Its study was first initiated in 1890 by H. Poincaré with the Poincaré recurrence theorem [16], and later generalized by H. Furstenberg in 1976 with the multiple recurrence theorem [11]. Recently, these notions were studied systematically in linear dynamics in two fundamental papers by G. Costakis et al. in [8, 9].

According to these works, an operator T is said to be recurrent if, for each non-empty open set U, there exists a positive integer n such that $T^n(U) \cap U \neq \emptyset$. A vector x is called a recurrent vector for T if there exists a strictly increasing sequence of positive integers $(n_k)_k$ such that $T^{n_k}x \to x$ as $k \to \infty$. The set of all recurrent vectors for T is denoted by Rec(T). Notice that T is recurrent if and only if Rec(T) is dense. An operator T is said to be topologically multiply recurrent, or simply multiply recurrent, if, for every positive integer m and every non-empty open set U, there exists $n \in \mathbb{N}$ such that $\bigcap_{l=0}^{m} T^{-ln}U \neq \emptyset$. If $m = 1$, then T satisfies $U \cap T^{-n}U \neq \emptyset$, thus T is recurrent. For a deeper understanding of recurrence, see [2, 3, 4, 6, 5, 8, 9, 12].

To proceed further, we collect several definitions of key notions that will be used in our main result.

Definition 1.1 (OP-basis)**.** Let $(e_i)_{i\in I}$ denote the canonical unit vectors in a topological sequence space X over an arbitrary countable infinite set I. We say that $(e_i)_{i\in I}$ is an *OP-basis* (or Ovsepian–Petczynski basis) if span$\{e_i : i \in I\}$ is a dense subspace of X and the family of coordinate projections $x \mapsto x_i e_i (i \in I)$ on X is equicontinuous.

Remarks 1.2. (1) Every separable Banach space is isomorphic to a Banach sequence space in which $(e_n)_{n\in\mathbb{N}}$ is an OP-basis.

(2) Note that in a Banach sequence space over I, the family of coordinate projections is equicontinuous if and only if $\sup_{i\in I}\|e_i x_i\| < \infty$.

Definition 1.3 (Weighted pseudo-shift [13])**.** Let X and Y be topological sequence spaces over countable infinite sets I and J, respectively. Then a linear operator $T : X \to Y$ is called a *weighted pseudo-shift* if there is a sequence $(w_j)_{j\in J}$ of non-zero scalars and an injective mapping $f : J \to I$ such that

$$T(x_i)_{i\in I} = \left(w_j x_{f(j)}\right)_{j\in J}$$

for $(x_i)_i \in X$. We then write $T = T_{w,f}$. $(w_j)_{j\in J}$ is called the *weight sequence*.

Remarks 1.4. (1) We note that if X and Y are F-sequence spaces, then the continuity of T is a straightforward conclusion through the application of the closed graph theorem.

(2) Every unilateral or bilateral weighted backward shift is a weighted pseudo-shift with w_{n+1} as weight, and $f(n) = n+1$. Likewise, every bilateral weighted forward shift is a weighted pseudo-shift with the weight w_{n-1} and $f(n) = n-1$. In contrast, unilateral weighted forward shifts cannot be considered pseudo-shifts due to their definition in the first component.

(3) If $X = Y$, then the identity operator on X defines a pseudo-shift.

(4) If $T = T_{w,f} : X \to X$ is a weighted pseudo-shift, then for each $n \geq 1$, T^n is also a weighted pseudo-shift. Additionally,

$$T^n\left(x_i\right)_{i\in I} = \left(w_{n,i} x_{f^n(i)}\right)_{i\in I},$$

where

$$f^n(i) = (f \circ f \circ \cdots \circ f)(i) \quad (n\text{-fold}),$$

$$w_{n,i} = w_i w_{f(i)} \cdots w_{f^{n-1}(i)} = \prod_{v=0}^{n-1} w_{f^v(i)}.$$

(5) We consider the inverse $f^{-1} : f(I) \to I$ of the mapping f. We also set

$$w_{f^{-1}(i)} = 0 \quad \text{and} \quad e_{f^{-1}(i)} = 0 \quad \text{if } i \in I\backslash f(I).$$

Then, for all $i \in I$,

$$T_{w,f^{-1}} e_i = w_{f^{-1}(i)} e_{f^{-1}(i)}.$$

In this paper, motivated by Grosse-Erdmann's work [13] and in order to generalize the weighted shift's characterizations given in [9], we study the topological multiple recurrence and the recurrence of weighted pseudo-shifts.

The rest of the paper is structured as follows: Section 2 provides equivalent conditions for topologically multiply recurrent operators that act on a Fréchet space.

In Section 3, we characterize topologically multiply recurrent pseudo-shifts that act on an F-sequence space in terms of the weights, the OP-basis and the shift mapping. Furthermore, we establish that recurrence and hypercyclicity of pseudo-shifts are equivalent.

2. Topologically multiply recurrent operators

This section is devoted to the study of topologically multiply recurrent operators that act on a Fréchet space X. In the following, we recall their definition from [9].

Definition 2.1. An operator T that acts on X is called a *topologically multiply recurrent operator* if, for every non-empty open subset U of X and every positive integer m, there exists a positive integer n such that

$$U \cap T^{-n}U \cap \cdots \cap T^{-mn}U \neq \emptyset.$$

Remark 2.2. In the provided definition, when $m = 1$, we obtain the recurrence of the operator T.

The subsequent example demonstrates that the converse implication does not hold in full generality.

Example 2.3. Let $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the bilateral backward shift with weights $(w_n)_{n\in\mathbb{Z}}$ defined as follows:

$$w_n = \begin{cases} 2 & \text{if } n \leq 0, \\ \frac{1}{1+n} & \text{if } n > 0. \end{cases}$$

T is recurrent but not topologically multiply recurrent. Indeed, since $w_n > 0$, we have

$$\lim_{k\to+\infty} \prod_{j=0}^{k} \frac{1}{w_{-j}} = \lim_{k\to+\infty} \prod_{j=0}^{k} \frac{1}{2} = \lim_{k\to+\infty} \left(\frac{1}{2}\right)^k = 0$$

and

$$\lim_{k\to+\infty} \prod_{j=1}^{k} w_j = \lim_{k\to+\infty} \prod_{j=1}^{k} \frac{1}{1+j} = \lim_{k\to+\infty} \frac{1}{(1+k)!} = 0.$$

Then, according to [10, Theorem 3.2] and [9, Proposition 5.1], T is recurrent. However, for all strictly increasing sequences of positive integers $(n_k)_k$, we have

$$\lim_{k\to+\infty} \prod_{j=1}^{n_k} w_j = \lim_{k\to+\infty} \prod_{j=0}^{n_k} \frac{1}{w_{-j}} < \infty.$$

Then, by [9, Proposition 5.3], T is not topologically multiply recurrent.

The following theorem characterizes sequentially the topological multiple recurrence of an operator.

Theorem 2.4. *Let T be an operator that acts on a Fréchet space X. Then, T is topologically multiply recurrent if and only if, for every $m \in \mathbb{N}$ and every $x \in X$, there exist a sequence of vectors $(x_k)_k$ and a strictly increasing sequence of positive integers $(n_k)_k$ such that, for every $l \in \{1, 2, \ldots, m\}$,*

$$x_k \longrightarrow x \quad \textit{and} \quad T^{l n_k}(x_k) \longrightarrow x \qquad \textit{as } k \longrightarrow \infty.$$

Proof. Let $m \in \mathbb{N}$ and $x \in X$. Consider the open ball $B\left(x, \frac{1}{2}\right)$. Since T is topologically multiply recurrent, there exists a positive integer n_1 such that

$$B\left(x, \frac{1}{2}\right) \cap T^{-n_1} B\left(x, \frac{1}{2}\right) \cap \cdots \cap T^{-mn_1} B\left(x, \frac{1}{2}\right) \neq \emptyset.$$

Thus, there exists a vector x_1 in X such that, for every $l \in \{1, 2, \ldots, m\}$,

$$\|x - x_1\| < \frac{1}{2} \quad \text{and} \quad \|x - T^{ln_1} x_1\| \leq \frac{1}{2}.$$

Continuing inductively, we construct $(x_k)_k$ in X and $(n_k)_k$ in $\mathbb{N}$ such that, for every $l \in \{1, 2, \ldots, m\}$,

$$\|x - x_k\| < \frac{1}{2^k} \quad \text{and} \quad \|x - T^{ln_k} x_k\| < \frac{1}{2^k}.$$

As k goes to infinity, we obtain, for every $l \in \{1, 2, \ldots, m\}$,

$$x_k \longrightarrow x \quad \text{and} \quad T^{ln_k}(x_k) \longrightarrow x.$$

Conversely, let U be a non-empty open subset of X and $m \in \mathbb{N}$. Then there exist $x \in U$ and $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$. Since there are $(x_k)_k$ in X and $(n_k)_k$ in $\mathbb{N}$ such that

$$x_k \longrightarrow x \quad \text{and} \quad T^{ln_k}(x_k) \longrightarrow x \quad \text{as } k \longrightarrow \infty \qquad \text{for each } l \in \{1, 2, \ldots, m\},$$

there exists $k \in \mathbb{N}$ such that

$$x_k \in B(x, \varepsilon) \subset U \quad \text{and} \quad T^{ln_k} x_k \in B(x, \varepsilon) \subset U \qquad \text{for each } l \in \{1, 2, \ldots, m\}.$$

Hence, $x \in \bigcap_{l=0}^{m} T^{-ln_k} U$. Finally, T is a topologically multiply recurrent operator. □

3. Weighted pseudo-shifts and recurrence

Throughout this section, X is an F-sequence space over an arbitrary set I in which $(e_i)_{i\in I}$ forms an OP-basis. Before providing the main result, we need to discuss the following concept.

Definition 3.1 (Run-away sequence)**.** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of maps acting on I. $(f_n)_{n\in\mathbb{N}}$ is called a *run-away sequence* if, for each pair of finite subsets $I_0 \subset I$ and $J_0 \subset I$, there exists an $n_0 \in \mathbb{N}$ such that $f_n(J_0) \cap I_0 = \emptyset$ for every $n \geq n_0$.

The following illustrations represent maps for which the run-away property holds for their sequences of iterates.

Examples 3.2. (1) Let f be the translation map defined on the set of positive integers $\mathbb{N}$ by $f(j) = j + 1$. Then, $(f^n)_{n\in\mathbb{N}}$ is a run-away sequence. Let I_0 and J_0 be two finite subsets of $\mathbb{N}$. Notice that $f^n(J_0) = \{n + j : j \in J_0\}$. By setting $n_0 = \max(I_0) + 1$, we obtain $f^n(J_0) \cap I_0 = \emptyset$ for every $n \geq n_0$. As a result, the sequence $(f^n)_{n\in\mathbb{N}}$ is indeed run-away.

(2) Let $f(j) = 2j$ be a map defined on the set of strictly positive integers $\mathbb{N}\setminus\{0\}$. Let I_0 and J_0 be two finite subsets of $\mathbb{N}\setminus\{0\}$. Observe that $f^n(J_0) = \{2^n j : j \in J_0\}$. Then, $f^n(J_0) \cap I_0 = \emptyset$ for every $n \geq \max(I_0)$. Hence, $(f^n)_{n\in\mathbb{N}}$ is a run-away sequence.

The next examples illustrate that the run-away property may not hold for f^n.

Examples 3.3. (1) Consider the identity map $\mathrm{Id}_{\mathbb{Z}}$ defined on the set of integers $\mathbb{Z}$. Then $(\mathrm{Id}_{\mathbb{Z}}^n)_{n\in\mathbb{N}}$ is not a run-away sequence. This is because there exists a nonempty finite subset $I_0 \subset \mathbb{Z}$ with $\mathrm{Id}_{\mathbb{Z}}^n(I_0) \cap I_0 = I_0 \neq \emptyset$ for every $n \in \mathbb{N}$.

(2) Let us define $f(j) = j^2$ on the set of positive integers $\mathbb{N}$. Then $(f^n)_{n\in\mathbb{N}}$ is not a run-away sequence. Indeed, by setting $I_0 = \{1\}$, we observe that $f^n(I_0) \cap I_0 = I_0 \neq \emptyset$ for all $n \in \mathbb{N}$. Hence, the sequence $(f^n)_{n\in\mathbb{N}}$ is not run-away.

In the upcoming proposition, we will present a characterization of the run-away property.

Proposition 3.4. *Let f be a map acting on I. f has no periodic points if and only if $(f^n)_{n\in\mathbb{N}}$ is a run-away sequence.*

Proof. Suppose that f has no periodic points. Let I_0 be a finite subset of I. Then for any $i \in I$ there exists a positive integer $n_0 \in \mathbb{N}$ with $f^n(i) \notin I_0$ for $n \geq n_0$. This shows that $(f^n)_{n\in\mathbb{N}}$ is a run-away sequence.

Conversely, for the sake of contradiction, let us assume that f admits a periodic point. This implies the existence of $(x, k) \in I \times \mathbb{N}$ such that $f^k(x) = x$.

Set $I_0 = \{x\}$. Since $(f^n)_{n\in\mathbb{N}}$ is run-away, there exists a positive integer n_0 such that $f^n I_0 \cap I_0 = \emptyset$, for every $n \geq n_0$. If $k \geq n_0$, then $f^k I_0 \cap I_0 = \{x\} \neq \emptyset$, which contradicts the run-away assumption. If $k < n_0$, there exists a positive integer p such that $pk \geq n_0$. However, $f^{pk}(x) = x$, and thus $f^{pk} I_0 \cap I_0 = \{x\} \neq \emptyset$. $\square$

The following theorem represents our main result.

Theorem 3.5. *Let T be a weighted pseudo-shift that acts on X with weights $(w_i)_{i\in I}$. If the mapping $f : I \to I$ has no periodic points, then the following assertions are equivalent:*

(1) *T is topologically multiply recurrent.*
(2) *For every $\varepsilon > 0$ and all strictly positive integers m and N_0, there exists an integer $n > N_0$ such that, for every $i \in I$ and every $l \in \{1, 2, \ldots, m\}$,*

$$\left\| w_{ln,i}^{-1} e_{f^{ln}(i)} \right\| < \varepsilon \quad \textit{and} \quad \left\| \prod_{v=1}^{ln} w_{f^{-v}(i)} e_{f^{-ln}(i)} \right\| < \varepsilon.$$

Proof. Let us first prove that (1) implies (2). Fix a positive integer m, $\varepsilon > 0$, and also choose a positive integer N_0. Given a finite subset $I_0 \subset I$, by the equicontinuity of the coordinate projections in X, there exists some $\mu > 0$ such that, for every $x \in X$ and every $i \in I_0$,

$$\|x_i e_i\| < \frac{\varepsilon}{2} \quad \text{if } \|x\| < \mu. \tag{3.1}$$

Since the mapping $f : I \to I$ has no periodic points, by Proposition 3.4, there exists an integer $N \in \mathbb{N}$ such that, for every $l \in \{1, 2, \ldots, m\}$ and every $n > N$,

$$f^{ln}(I_0) \cap I_0 = \emptyset. \tag{3.2}$$

It follows that

$$f^{-ln}\left(I_0 \cap f^{ln}(I)\right) \cap I_0 = \emptyset. \tag{3.3}$$

Consider the open ball $B\left(\sum_{i\in I_0} e_i,\mu\right)$. Since T is topologically multiply recurrent, there exists a positive integer $n>\max(N,N_0)$ such that

$$B\left(\sum_{i\in I_0} e_i,\mu\right)\cap T^{-n}\left(B\left(\sum_{i\in I_0} e_i,\mu\right)\right)\cap\cdots\cap T^{-mn}\left(B\left(\sum_{i\in I_0} e_i,\mu\right)\right)\neq\emptyset.$$

Thus, by virtue of Theorem 2.4, there exist $x\in X$ and $n>\max(N,N_0)$ such that, for every $l\in\{1,2,\dots,m\}$,

$$\left\|x-\sum_{i\in I_0} e_i\right\|<\mu \quad\text{and}\quad \left\|T^{ln}x-\sum_{i\in I_0} e_i\right\|<\mu. \tag{3.4}$$

By (3.1), (3.2), and the first inequality in (3.4), it follows that

$$\|x_ie_i\|<\frac{\varepsilon}{2}\quad\text{if } i\notin I_0.$$

Then, by (3.2) we have that, for every $i\in I_0$ and every $l\in\{1,2,\dots,m\}$,

$$\|x_{f^{ln}(i)}e_{f^{ln}(i)}\|<\frac{\varepsilon}{2}. \tag{3.5}$$

By continuous inclusion of X into $\mathbb{K}^I$ we have

$$\sup_{i\in I_0}|x_i-1|\leq\frac{1}{2}\quad\text{and}\quad\sup_{i\in I_0}\left|y_i^{(l)}-1\right|\leq\frac{1}{2}, \tag{3.6}$$

where $y^{(l)}:=T^{ln}x=(w_{ln,i}x_{f^{ln}(i)})_{i\in I}$.

Consequently, by the second inequality we have, for every $i\in I_0$ and every $l\in\{1,2,\dots,m\}$,

$$x_{f^{ln}(i)}\neq 0\quad\text{and}\quad\left|w_{ln,i}x_{f^{ln}(i)}-1\right|\leq\frac{1}{2}. \tag{3.7}$$

It follows from (3.5) and (3.7) that, for every $i\in I_0$ and every $l\in\{1,2,\dots,m\}$,

$$\begin{aligned}\left\|w_{ln,i}^{-1}e_{f^{ln}(i)}\right\|&=\left\|\frac{1}{w_{ln,i}x_{f^{ln}(i)}}x_{f^{ln}(i)}e_{f^{ln}(i)}\right\|\\&\leq\left\|x_{f^{ln}(i)}e_{f^{ln}(i)}\right\|+\left\|\left(\frac{1}{w_{ln,i}x_{f^{ln}(i)}}-1\right)x_{f^{ln}(i)}e_{f^{ln}(i)}\right\|<\varepsilon.\end{aligned}$$

Hence the first inequality of (2) holds.

Now, by (3.1) and the second inequality in (3.4) we obtain that, for every $i\notin I_0$ and every $l\in\{1,2,\dots,m\}$,

$$\left\|\prod_{v=0}^{ln-1} w_{f^v(i)}x_{f^{ln}(i)}.e_i\right\|<\frac{\varepsilon}{2}. \tag{3.8}$$

Notice that

$$e_{f^{-ln}(i)}=0\quad\text{if } i\in I\backslash f^{ln}(I).$$

Then, by (3.3) and (3.8) we conclude that, for every $i\in I_0$ and every $l\in\{1,2,\dots,m\}$,

$$\left\|\prod_{v=0}^{ln-1} w_{f^v(f^{-ln}(i))}x_{f^{ln}(f^{-ln}(i))}e_{f^{-ln}(i)}\right\|=\left\|\prod_{v=1}^{ln} w_{f^{-v}(i)}x_ie_{f^{-ln}(i)}\right\|<\frac{\varepsilon}{2}. \tag{3.9}$$

From the first inequality in (3.6) we have

$$2\,|x_i| \geq 1 \quad \text{for each } i \in I_0. \tag{3.10}$$

Now, (3.9) and (3.10) imply that, for every $i \in I_0$ and every $l \in \{1, 2, \dots, m\}$,

$$\begin{aligned}\left\| \left(\prod_{v=1}^{ln} w_{f^{-v}(i)}\right) e_{f^{-ln}(i)} \right\| &= \left\| \frac{1}{2x_i} 2 \left(\prod_{v=1}^{ln} w_{f^{-v}(i)}\right) x_i e_{f^{-ln}(i)} \right\| \\ &\leq \left\| 2 \left(\prod_{v=1}^{ln} w_{f^{-v}(i)}\right) x_i e_{f^{-ln}(i)} \right\| < \varepsilon.\end{aligned}$$

This completes the proof of the implication (1) $\Rightarrow$ (2).

Conversely, fix $i \in I$, $m \in \mathbb{N}$ and suppose that there is an increasing sequence of positive integers $(n_k)_k$ satisfying both inequalities in (2). We consider the linear mapping $S_n : X \to X$ given by

$$S_n(e_i) = \left(\prod_{v=0}^{n-1} w_{f^v(i)}\right)^{-1} e_{f^n(i)} \quad (i \in I).$$

According to the first inequality of (2) we have

$$y_k := \sum_{l=1}^{m} S_{ln_k}(e_i) = \sum_{l=1}^{m} \left(\prod_{v=0}^{ln_k - 1} w_{f^v(i)}\right)^{-1} e_{f^{ln_k}(i)} \underset{k \to +\infty}{\longrightarrow} 0.$$

Set $x_k := y_k + e_i$. We have

$$x_k - e_i = y_k \underset{k \to +\infty}{\longrightarrow} 0. \tag{3.11}$$

On the other hand, by virtue of the second inequality of (2) we have, for any $l \in \{1, 2, \dots, m\}$,

$$T^{ln_k} e_i = \left(\prod_{v=1}^{ln_k} w_{f^{-v}(i)}\right) e_{f^{-ln_k}(i)} \underset{k \to +\infty}{\longrightarrow} 0.$$

Notice that by hypothesis we obtain that, for any $1 \leq s < l \leq m$,

$$T^{ln_k} S_{sn_k} e_i = \left(\prod_{v=0}^{sn_k - 1} w_{f^v(i)}\right)^{-1} \left(\prod_{v=1}^{ln_k} w_{f^{-v}(f^{sn_k}(i))}\right) e_{f^{(s-l)n_k}(i)} \underset{k \to +\infty}{\longrightarrow} 0$$

and

$$T^{sn_k} S_{ln_k} e_i = \left(\prod_{v=0}^{ln_k - 1} w_{f^v(i)}\right)^{-1} \left(\prod_{v=1}^{sn_k} w_{f^{-v}(f^{ln_k}(i))}\right) e_{f^{(l-s)n_k}(i)} \underset{k \to +\infty}{\longrightarrow} 0.$$

Since $T^{ln_k} S_{ln_k} e_i = e_i$, we conclude that, for any $l = 1, \dots, m$,

$$T^{ln_k} y_k = T^{ln_k} \left(\sum_{j=1}^{m} S_{jn_k}(e_i)\right) \underset{k \to +\infty}{\longrightarrow} e_i.$$

Finally,

$$T^{ln_k}x_k - e_i = T^{ln_k}e_i + T^{ln_k}y_k - e_i \underset{k\to+\infty}{\longrightarrow} 0. \tag{3.12}$$

According to (3.11), (3.12) and Theorem 2.4, it follows that T is topologically multiply recurrent. □

The next example shows that the equivalence stated in the previous theorem fails without the assumption of no periodic points for f nor without the run-away assumption for the sequence of maps f^n.

Example 3.6. Let $T_{w,f} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the weighted pseudo-shift with weights $(w_i)_{i\in\mathbb{Z}}$, where $w_i = \lambda$ for each $i \in \mathbb{Z}$ such that $|\lambda| = 1$ and the shift mapping $f := Id_{\mathbb{Z}}$. According to Examples 3.3, $(f^n)_{n\in\mathbb{N}}$ is not run-away. Furthermore, $T_{w,f}$ is topologically multiply recurrent, yet it does not fulfill the inequalities stated in (2) of the previous theorem. Indeed, let $m \in \mathbb{N}$ and $\varepsilon > 0$ and let U be a non-empty open subset of the complex space $\ell^2(\mathbb{Z})$ and $x \in U$. Since $|\lambda^l| = 1$ for any $l \in \{1, 2, \dots, m\}$, the set $\{n \geq 0 : \forall l \in \{1, 2, \dots, m\} |\lambda^{ln} - 1| < \varepsilon\}$ has bounded gaps. Then, there exists a strictly increasing sequence of positive integers $(n_k)_k$ such that $\lambda^{ln_k} \to 1$ as $k \to \infty$ for any $l \in \{1, 2, \dots, m\}$. Hence, $T_{w,f}^{ln_k}x \to x$ as $k \to \infty$, which means that $\bigcap_{l=1}^m T_{w,f}^{ln_k}U \cap U \neq \emptyset$. Then, $T_{w,f}$ is topologically multiply recurrent.

However, for every $n \in \mathbb{N}$ we have

$$\begin{cases} \left\| \left(\prod_{v=0}^{n-1} w_{f^v(i)} \right)^{-1} e_{f^n(i)} \right\| = \left| \left(\prod_{i=0}^{n-1} w_i \right)^{-1} \right| = 1, \\ \left\| \left(\prod_{v=1}^{n} w_{f^{-v}(i)} \right) e_{f^{-n}(i)} \right\| = \left| \prod_{i=1}^{n} w_i \right| = 1. \end{cases}$$

In the following, we consider the special case of Theorem 3.5 where $m = 1$. This leads us to a characterization of recurrent or hypercyclic pseudo-shifts in terms of conditions on the weights which extends [9, Proposition 5.1] and [13, Theorem 5].

Corollary 3.7. *Let T be weighted pseudo-shift that acts on X with weights $(w_i)_{i\in I}$. If the mapping $f : I \to I$ has no periodic points, then the following assertions are equivalent:*

(1) *T is recurrent.*
(2) *T is hypercyclic.*
(3) *For every $\varepsilon > 0$ and every strictly positive integer N_0, there exists an integer $n > N_0$ such that, for every $i \in I$,*

$$\|w_{n,i}^{-1}e_{f^n(i)}\| < \varepsilon \quad \text{and} \quad \|w_{n,f^{-n}(i)}e_{f^{-n}(i)}\| < \varepsilon.$$

Proof. Suppose that T is recurrent. Arguing as in the proof of Theorem 3.5 with $m = 1$, we obtain that for every $\varepsilon > 0$ and every N_0 there exists $n > N_0$ such that, for any $i \in I$,

$$\begin{cases} \|w_{n,i}^{-1}e_{f^n(i)}\| < \varepsilon, \\ \|w_{n,f^{-n}(i)}e_{f^{-n}(i)}\| < \varepsilon. \end{cases}$$

Hence we have the proof of (1) ⇒ (3). By [13, Theorem 5] we have that (3) implies (2). Finally the implication (2) ⇒ (1) holds trivially and this completes the proof of the equivalence of statements (1)–(3) of the theorem. □

A bilateral weighted backward shift is an example of pseudo-shift with the strictly increasing map $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = n + 1$. Using this observation, the characterization of topologically multiply recurrent backward shifts in terms of the weights established by G. Costakis and I. Parissis [9] is now a consequence of Theorem 3.5.

Proposition 3.8 ([9, Proposition 5.3]). *Let $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be a bilateral weighted shift with weight sequence $(w_n)_{n\in\mathbb{Z}}$. The following are equivalent:*

(1) *T is topologically multiply recurrent.*
(2) *For every $m \in \mathbb{N}$, the operator $T \oplus T^2 \oplus \cdots \oplus T^m$ is hypercyclic on X^m.*
(3) *For every $m \in \mathbb{N}$, the operators $T, T^2, \ldots, T^m$ are densely d-hypercyclic.*
(4) *For every $m, q \in \mathbb{N}$ and every $\varepsilon > 0$, there exists a positive integer $n := n(m, q, \varepsilon)$ such that, for every integer j with $|j| \le q$ and every $l = 1, \ldots, m$,*

$$\prod_{i=0}^{\ln -1} w_{j+i} > \frac{1}{\varepsilon} \quad \textit{and} \quad \prod_{i=1}^{\ln} w_{j-i} < \varepsilon.$$

Proof. Let m and q be positive integers and let $\varepsilon > 0$. Suppose that T is topologically multiply recurrent. Then there exists a positive integer n such that, for every integer j with $|j| \le q$ and every $l = 1, \ldots, m$,

$$\prod_{i=0}^{\ln -1} w_{j+i} > \frac{1}{\varepsilon} \quad \text{and} \quad \prod_{i=1}^{\ln} w_{j-i} < \varepsilon.$$

Notice that

$$\begin{cases} \left\| \left(\prod_{v=0}^{ln-1} w_{f^v(i)} \right)^{-1} e_{f^{ln}(i)} \right\| = \left(\prod_{i=0}^{ln-1} w_{j+i} \right)^{-1} < \varepsilon, \\ \left\| \left(\prod_{v=1}^{ln} w_{f^{-v}(i)} \right) e_{f^{-ln}(i)} \right\| = \prod_{i=1}^{ln} w_{j-i} < \varepsilon. \end{cases}$$

□

If each $i \in I$ lies outside $f^n(I)$ for all sufficiently large n, then the following statement can be considered as a special case of Theorem 3.5.

Theorem 3.9. *Let X be an F-sequence space over I, in which $(e_i)_{i\in I}$ is an OP-basis. Let $T = T_{w,f} : X \to X$ be a weighted pseudo-shift with weight sequence $(w_i)_{i\in I}$, so that each $i \in I$ lies outside $f^n(I)$ for all sufficiently large n. If the* *mapping $f : I \to I$ has no periodic points, then the following assertions are equivalent:*

1. *T is topologically multiply recurrent.*
2. *For every positive integer m and every $\varepsilon > 0$, there exists an increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that, for every $i \in I$ and every $l \in \{1, \dots, m\}$,*

$$\left\| \left(\prod_{v=0}^{ln_k-1} w_{f^v(i)} \right)^{-1} e_{f^{ln_k}(i)} \right\| < \varepsilon.$$

Proof. Suppose that T is topologically multiply recurrent. Since each $i \in I$ lies outside $f^n(I)$ for all sufficiently large n, it follows that, for every $i \in I$, there exists a positive integer N_i such that $e_{f^{-n}(i)} = 0$ when $n > N_i$. Then the inequality

$$\left\| \prod_{v=1}^{ln} w_{f^{-v}(i)} e_{f^{-ln}(i)} \right\| < \varepsilon$$

is always satisfied. Hence by virtue of Theorem 3.5 the result follows. □

Remark 3.10. Note that every unilateral weighted shift is topologically multiply recurrent.

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Mohamed Amouch
Department of Mathematics, Chouaib Doukkali University, Faculty of Sciences, El Jadida, Morocco
amouch.m@ucd.ac.ma

Fatima-ezzahra Sadek✉
Department of Mathematics, Chouaib Doukkali University, Faculty of Sciences, El Jadida, Morocco
sadek.fatimaezzahra@yahoo.fr

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