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THE NATURAL OPERATORS LIFTING q-FORMS TO p-FORMS ON WEIL BUNDLES

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ABSTRACT. Let q, p, k, m be positive integers with $m \geq k+p+1$, and let A be a Weil algebra with k generators. The (not necessarily regular) natural operators lifting q-forms on an m-dimensional manifold M to p-forms on the Weil bundle $T^A M$ are completely described by means of the so-called excellent maps. As a consequence, we show that any natural operator lifting q-forms on an m-dimensional manifold M to p-forms on $T^A M$ is regular, i.e., it sends smoothly parametrized families into smoothly parametrized families. We apply our general results to the case where T^A is the rth-order tangent bundle $T^r M = J^r_0(\mathbf{R}, M)$ (in particular, the tangent bundle).

1. Introduction

All manifolds and maps considered in this paper are assumed to be smooth (of class C^∞). The theory of Weil functors and the concept of natural operators can be found in [5].

Let $\mathcal{M}f$ be the category of manifolds and maps, $\mathcal{M}f_m$ be the category of m-dimensional manifolds and their submersions, and let $\mathcal{FM}$ be the category of fibred manifolds and their fibred maps. Let A be a Weil algebra and $T^A : \mathcal{M}f \to \mathcal{FM}$ be its Weil functor.

First, in Section 2, we study all (not necessarily regular) $\mathcal{M}f_m$-natural operators $C : \wedge^q T^* \rightsquigarrow \wedge^0 T^* T^A$ transforming q-forms ω on an m-manifold M into maps $C(\omega) : T^A M \to \mathbf{R}$. Namely, using A we define the finite-dimensional real vector space Q^q_A, and for any map $h : Q^q_A \to \mathbf{R}$ and any q-form ω on M we define the smooth map $\omega^{\langle h \rangle} : T^A M \to \mathbf{R}$, and we prove that any C under consideration is of the form $C(\omega) = \omega^{\langle h_C \rangle}$ for some map $h_C : Q^q_A \to \mathbf{R}$. Consequently, we obtain that any such C is regular, i.e., it transforms smoothly parametrized families of q-forms into smoothly parametrized families of maps.

Next, in Section 3 we study all (not necessarily regular) $\mathcal{M}f_m$-natural operators $D : \wedge^q T^* \rightsquigarrow \wedge^p T^* T^A$ transforming q-forms ω on an m-manifold M into p-forms $D(\omega)$ on $T^A M$. Using the well-known fact that $TT^A M \times_{T^A M} \cdots \times_{T^A M} TT^A M = T^B M$ for some new Weil algebra depending on A, we can consider such operators D

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as $\mathcal{M}f_m$-natural operators $\wedge^q T^* \rightsquigarrow \wedge^0 T^* T^B$. By Section 2, they are regular and of the form $D(\omega) = \omega^{\langle h_D \rangle}$ for some $h_D : Q^q_B \to \mathbf{R}$. But for some $h : Q^q_B \to \mathbf{R}$ and some q-form ω on M, the function $\omega^{\langle h \rangle} : T^B M = TT^A M \times_{T^A M} \cdots \times_{T^A M} TT^A M \to \mathbf{R}$ cannot be fiber skew p-linear, i.e., it cannot be a p-form on $T^A M$. That is why we define the concept of excellent maps $h : Q^q_B \to \mathbf{R}$, and deduce that $D(\omega) = \omega^{\langle h \rangle}$ is a p-form on $T^A M$ for any q-form on M if and only if h is excellent.

In Section 4, we try to estimate the space of the excellent maps $Q^q_B \to \mathbf{R}$. We introduce the bigger space of semi-excellent maps $Q^q_B \to \mathbf{R}$, and reduce the description of semi-excellent maps $Q^q_B \to \mathbf{R}$ to solutions of systems of linear equations with coefficients from $\{0, 1\}$ with unknown from $\{0, 1\}$. Consequently, we essentially reduce the problem of the description of all $\mathcal{M}f_m$-natural operators $\wedge^q T^* \rightsquigarrow \wedge^p T^* T^A$ to a rather combinatoric problem.

In the last section, we apply practically our general results when T^A is the rth-order tangent bundle $T^r M = J^r_0(\mathbf{R}, M)$.

The linear $\mathcal{M}f_m$-natural operators $\wedge^q T^* \rightsquigarrow \wedge^p T^* T^A$ are described by J. Debecki [2, 3]. In the present paper, we describe all (not necessarily linear) such natural operators.

This work is also a generalization of [6], where regular $\mathcal{M}f_m$-natural operators $T^* \rightsquigarrow T^* T^A$ are described.

From now on, let k, m, q, p, and r be positive integers. Let $x^1, \dots, x^m$ be the usual coordinates on $\mathbf{R}^m$. Let $y^1, \dots, y^k$ be the usual coordinates on $\mathbf{R}^k$ and let $y^1, \dots, y^k, u^1, \dots, u^p$ be the usual coordinates on $\mathbf{R}^{k+p} = \mathbf{R}^k \times \mathbf{R}^p$. Let $\mathbf{N} = \{0, 1, 2, \dots\}$ be the set of non-negative integers.

2. Lifting q-forms on manifolds to maps on Weil bundles

Let $C^\infty_0(\mathbf{R}^k)$ be the local algebra of germs at 0 of smooth maps $\mathbf{R}^k \to \mathbf{R}$ and let $\mathbf{m}$ be its maximal ideal. For simplicity of notation, for any map $f : \mathbf{R}^k \to \mathbf{R}$ we will denote the germ at $0 \in \mathbf{R}^k$ of f by the same letter f.

Let $\Omega^q_0(\mathbf{R}^k)$ be the $C^\infty_0(\mathbf{R}^k)$-module of germs at 0 of q-forms on $\mathbf{R}^k$. Similarly, for any q-form ω on $\mathbf{R}^k$, we will denote the germ at $0 \in \mathbf{R}^k$ of ω by the same letter ω.

Let $\underline{A}$ be an ideal in $C^\infty_0(\mathbf{R}^k)$ such that $\mathbf{m}^2 \supset \underline{A} \supset \mathbf{m}^{r+1}$, and let

$$A = C^\infty_0(\mathbf{R}^k)/\underline{A} \quad \text{(the factor algebra)}.$$

This factor algebra A is a Weil algebra (of order r).

Given a manifold M, two maps $\gamma, \gamma_1 : \mathbf{R}^k \to M$ have the same A-jet, written $j^A\gamma = j^A\gamma_1$, if

$$g \circ \gamma - g \circ \gamma_1 \in \underline{A} \quad \text{for any map } g : M \to \mathbf{R}.$$

Let

$$T^A M = \{j^A\gamma \mid \gamma : \mathbf{R}^k \to M\}$$

be the space of all A-jets of maps $\mathbf{R}^k \to M$. Then $T^A M$ is a fibred manifold with the base M and the projection $j^A\gamma \mapsto \gamma(0)$. Any map $f : M \to M_1$ of two manifolds induces a fibred map

$$T^A f : T^A M \to T^A M_1, \quad T^A f(v) := j^A(f \circ \gamma), \quad v = j^A \gamma.$$

So, we have the bundle functor $T^A : \mathcal{M}f \to \mathcal{FM}$, the Weil functor of A-jets; see [4, 5, 8].

We have the sub-modules

$$\underline{A} \cdot \Omega_0^q(\mathbf{R}^k) \quad \text{and} \quad d\underline{A} \wedge \Omega_0^{q-1}(\mathbf{R}^k)$$

in the $C_0^\infty(\mathbf{R}^k)$-module $\Omega_0^q(\mathbf{R}^k)$, the one spanned (over $C_0^\infty(\mathbf{R}^k)$) by all $\eta\sigma$ for $\eta \in \underline{A}$ and $\sigma \in \Omega_0^q(\mathbf{R}^k)$, and the one spanned (over $C_0^\infty(\mathbf{R}^k)$) by all $d\eta \wedge \sigma$ for $\eta \in \underline{A}$ and $\sigma \in \Omega_0^{q-1}(\mathbf{R}^k)$, respectively, where d is the exterior derivative ($\Omega_0^0(\mathbf{R}^k) := C_0^\infty(\mathbf{R}^k)$).

Let

$$Q_A^q := \Omega_0^q(\mathbf{R}^k)/\underline{Q}_A^q \quad \text{(the factor module)},$$
$$\underline{Q}_A^q := \underline{A} \cdot \Omega_0^q(\mathbf{R}^k) + d\underline{A} \wedge \Omega_0^{q-1}(\mathbf{R}^k),$$

where $\underline{Q}_A^q$ is the sub-module in the $C_0^\infty(\mathbf{R}^k)$-module $\Omega_0^q(\mathbf{R}^k)$, the one consisting of all elements $\sigma_1 + \sigma_2$ for $\sigma_1 \in \underline{A} \cdot \Omega_0^q(\mathbf{R}^k)$ and $\sigma_2 \in d\underline{A} \wedge \Omega_0^{q-1}(\mathbf{R}^k)$.

One can easily see that the real vector space Q_A^q is finite-dimensional. Any element of $w \in Q_A^q$ is of the form $w = [\sigma]_{\underline{Q}_A^q}$ for a $\sigma \in \Omega_0^q(\mathbf{R}^k)$.

Example 2.1. Let $h : Q_A^q \to \mathbf{R}$ be a map. Given a q-form $\omega \in \Omega^q(M)$ on an m-dimensional manifold M, we define a map $\omega^{\langle h \rangle} : T^A M \to \mathbf{R}$ by

$$\omega^{\langle h \rangle}(v) := h([\gamma^*\omega]_{\underline{Q}_A^q}), \quad v \in T^A M,$$

where $\gamma : \mathbf{R}^k \to M$ is a map such that $v = j^A \gamma$ and where $\gamma^*\omega \in \Omega_0^q(\mathbf{R}^k)$ is the pull-back of ω with respect to γ.

Lemma 2.2. *The value $\omega^{\langle h \rangle}(v)$ is well defined.*

Proof. Suppose $v = j^A\gamma = j^A\gamma_1$, where $\gamma, \gamma_1 : \mathbf{R}^k \to M$. We have to prove that

$$h([\gamma^*\omega]_{\underline{Q}_A^q}) = h([\gamma_1^*\omega]_{\underline{Q}_A^q}).$$

We may assume $M = \mathbf{R}^m$, $\gamma(0) = \gamma_1(0) = 0$, and $\omega = \sum \omega_{i_1,\dots,i_q} dx^{i_1} \wedge \dots \wedge dx^{i_q}$, where $\sum$ is over all integers with $1 \le i_1 < \dots < i_q \le m$.

Let $\gamma = (\gamma^1, \dots, \gamma^m)$ and $\gamma_1 = (\gamma_1^1, \dots, \gamma_1^m)$. Since $j^A\gamma = j^A\gamma_1$, we have

$$\omega_{i_1,\dots,i_q} \circ \gamma - \omega_{i_1,\dots,i_q} \circ \gamma_1 \in \underline{A} \quad \text{and} \quad \gamma^i - \gamma_1^i \in \underline{A}$$

for all integers $i_1, \dots, i_q$ with $1 \le i_1 < \dots < i_q \le m$ and $i = 1, \dots, m$. We have

$$\begin{aligned}
\gamma^*\omega - \gamma_1^*\omega &= \sum \omega_{i_1,\dots,i_q} \circ \gamma \cdot d\gamma^{i_1} \wedge \dots \wedge \gamma^{i_q} - \sum \omega_{i_1,\dots,i_q} \circ \gamma_1 \cdot d\gamma_1^{i_1} \wedge \dots \wedge d\gamma_1^{i_q} \\
&= \sum (\omega_{i_1,\dots,i_q} \circ \gamma - \omega_{i_1,\dots,i_q} \circ \gamma_1) \cdot d\gamma^{i_1} \wedge \dots \wedge d\gamma^{i_q} \\
&\quad + \sum_{s=1}^{q} \sum (-1)^{s+1} \omega_{i_1,\dots,i_q} \circ \gamma_1 \cdot d(\gamma^{i_s} - \gamma_1^{i_s}) \wedge d\gamma_1^{i_1} \wedge \dots \\
&\qquad \wedge d\gamma_1^{i_{s-1}} \wedge d\gamma^{i_{s+1}} \wedge \dots \wedge d\gamma^{i_q} \;.
\end{aligned}$$

Then $\gamma^*\omega - \gamma_1^*\omega \in \underline{Q}^q_A$. So, $[\gamma^*\omega]_{\underline{Q}^q_A} = [\gamma_1^*\omega]_{\underline{Q}^q_A}$. Consequently, $h([\gamma^*\omega]_{\underline{Q}^q_A}) = h([\gamma_1^*\omega]_{\underline{Q}^q_A})$. □

Lemma 2.3. *The map $\omega^{\langle h\rangle} : T^A M \to \mathbf{R}$ is smooth.*

Proof. Let $v_t \in T^A M$ be a smooth curve. There exists a smoothly parametrized family of maps $\gamma_t : \mathbf{R}^k \to M$ such that $v_t = j^A\gamma_t$ for $t \in \mathbf{R}$. Then $[\gamma_t^*\omega]_{\mathbf{m}^{r+1}\cdot\Omega^q_0(\mathbf{R}^k)}$ is a smooth curve in the finite-dimensional real vector space $\Omega^q_0(\mathbf{R}^k)/\mathbf{m}^{r+1}\cdot\Omega^q_0(\mathbf{R}^k)$, the factor space of the $C^\infty_0(\mathbf{R}^k)$-module $\Omega^q_0(\mathbf{R}^k)$ by its sub-module $\mathbf{m}^{r+1}\cdot\Omega^q_0(\mathbf{R}^k)$. Then $[\gamma_t^*\omega]_{\underline{Q}^q_A}$ is a smooth curve in Q^q_A because $[\gamma_t^*\omega]_{\underline{Q}^q_A}$ is the image of the curve $[\gamma_t^*\omega]_{\mathbf{m}^{r+1}\cdot\Omega^q_0(\mathbf{R}^k)}$ with respect to the obvious projection $\Omega^q_0(\mathbf{R}^k)/\mathbf{m}^{r+1}\cdot\Omega^q_0(\mathbf{R}^k) \to Q^q_A$. Then $\omega^{\langle h\rangle}(v_t)$ depends smoothly on t. So, $\omega^{\langle h\rangle} : T^A M \to \mathbf{R}$ is smooth because of the Boman theorem (see [1]). □

Lemma 2.4. *If ω_τ is a smoothly parametrized family of q-forms on M then $(\omega_\tau)^{\langle h\rangle}$ is a smoothly parametrized family of maps $T^A M \to \mathbf{R}$.*

Proof. The proof of this lemma is a simple modification of the proof of the previous one. □

Proposition 2.5. *The correspondence $(-)^{\langle h\rangle} : \wedge^q T^* \rightsquigarrow \wedge^0 T^* T^A$ given by $\omega \mapsto \omega^{\langle h\rangle}$ is a regular $\mathcal{M}f_m$-natural operator in the sense of* [5].

Remark 2.6. The concept of (not necessarily regular) natural operators can be found in [5]. In our situation, the $\mathcal{M}f_m$-naturality (invariance) of $(-)^{\langle h\rangle}$ means that for any $\mathcal{M}f_m$-map $f : M \to M_1$ and q-forms $\omega \in \Omega^1(M)$ and $\omega_1 \in \Omega^q(M_1)$ on M and M_1, respectively, if ω and ω_1 are f-related, then so are $\omega^{\langle h\rangle}$ and $\omega_1^{\langle h\rangle}$. The regularity of $(-)^{\langle h\rangle}$ means that $(-)^{\langle h\rangle}$ transforms smoothly parametrized families of q-forms into smoothly parametrized families of maps.

Proof. The proposition is clear. In particular, because of the canonical character of the construction of $\omega^{\langle h\rangle}$ it follows the $\mathcal{M}f_m$-invariance of $(-)^{\langle h\rangle}$. The regularity of $(-)^{\langle h\rangle}$ is exactly Lemma 2.4. □

The main result of this section is

Theorem 2.7. *If $m \geq k+1$, then any (not necessarily regular) $\mathcal{M}f_m$-natural operator $C : \wedge^q T^* \rightsquigarrow \wedge^0 T^* T^A$ sending q-forms ω on an m-manifold M into maps $C(\omega) : T^A M \to \mathbf{R}$ is of the form $C = (-)^{\langle h\rangle}$ for a map $h = h_C : Q^q_A \to \mathbf{R}$ uniquely determined by C. In particular, any such C is regular.*

The proof of Theorem 2.7 will occupy the rest of this section. We need several lemmas.

Let $C : \wedge^q T^* \rightsquigarrow \wedge^0 T^* T^A$ be a (not necessarily regular) $\mathcal{M}f_m$-natural operator. (Of course, the $\mathcal{M}f_m$-naturality and the regularity of C are explained in Remark 2.6 with C instead of $(-)^{\langle h\rangle}$.)

Assume $m \geq k+1$. Define $\Phi_C : \Omega^q_0(\mathbf{R}^m) \to \mathbf{R}$ by

$$\Phi_C(\omega) = C(\omega)(\kappa^A), \quad \omega \in \Omega^q_0(\mathbf{R}^m),$$

where $\kappa^A := j^A(\iota) \in T_0^A\mathbf{R}^m$, $\iota : \mathbf{R}^k \to \mathbf{R}^m$, $\iota(y_1, \ldots, y_k) = (y_1, \ldots, y_k, 0, \ldots, 0)$, $y_1, \ldots, y_k \in \mathbf{R}$.

Lemma 2.8. *The operator C is determined by Φ_C, i.e., if C_1 is another operator such that $\Phi_{C_1} = \Phi_C$, then $C_1 = C$.*

Proof. By the rank theorem, κ^A has a dense $\mathcal{M}f_m$-orbit in $T^A\mathbf{R}^m$. So, the lemma is clear. □

Lemma 2.9. *If φ is an $\mathcal{M}f_m$-map preserving κ^A, then $\Phi_C(\varphi^*\omega) = \Phi_C(\omega)$ for any $\omega \in \Omega_0^q(\mathbf{R}^m)$.*

Proof. It is an immediate consequence of the invariance of C with respect to φ. □

Lemma 2.10. *For any $\omega \in \Omega_0^q(\mathbf{R}^m)$, we have*

$$\Phi_C(\omega) = \Phi_C((x^1, \ldots, x^k, 0, \ldots, 0)^*\omega).$$

(We recall that $x^1, \ldots, x^m$ denote the usual coordinates on $\mathbf{R}^m$ and $(-)^*$ denotes the pullback.)

Proof. Let t_n, ϵ_n be two sequences of real numbers such that $0 < t_n < \exp(-n)$ and $0 < \epsilon_n < \exp(-n)$ for any n. Let $y_n = (0, \ldots, 0, \frac{1}{n}) \in \mathbf{R}^m$. By the Whitney extension theorem [9], there exists a smooth map $\varphi : \mathbf{R}^m \to \mathbf{R}^m$ such that, for sufficiently large n,

$$\varphi|_{D(y_{2n+1}, \epsilon_{2n+1})} = (x^1, \ldots, x^k, 0, \ldots, 0)$$

and

$$\varphi|_{D(y_{2n}, \epsilon_{2n})} = (x^1, \ldots, x^k, t_{2n}x^{k+1}, \ldots, t_{2n}x^m),$$

where $D(y, \epsilon)$ is the disk $\{x \in \mathbf{R}^m \mid \|x - y\| < \epsilon\}$. Then

$$C(\varphi^*\omega)(j^A(\iota + y_{2n+1})) = C((x^1, \ldots, x^k, 0, \ldots, 0)^*\omega)(j^A(\iota + y_{2n+1}))$$

and

$$\begin{aligned} C(\varphi^*\omega)(j^A(\iota + y_{2n})) &= C((x^1, \ldots, x^k, t_{2n}x^{k+1}, \ldots, t_{2n}x^m)^*\omega)(j^A(\iota + y_{2n})) \\ &= C(\omega)(j^A(\iota + t_{2n}y_{2n})) \end{aligned}$$

for sufficiently large n. The last equality follows from the invariance of C with respect to the $\mathcal{M}f_m$-map $(x^1, \ldots, x^k, t_{2n}x^{k+1}, \ldots, t_{2n}x^m)$. If $n \to \infty$, we obtain

$$C(\omega)(j^A\iota) = C((x^1, \ldots, x^k, 0, \ldots, 0)^*\omega)(j^A\iota),$$

i.e., $\Phi_C(\omega) = \Phi_C((x^1, \ldots, x^k, 0, \ldots, 0)^*\omega)$ as well. □

Lemma 2.11. *If $\sigma_t \in \Omega^q(\mathbf{R}^k)$, $t \in \mathbf{R}$, is a smoothly parametrized family, i.e., the resulting map $\sigma : \mathbf{R}^k \times \mathbf{R} \to \wedge^q T^*\mathbf{R}^k$ is smooth, then the map*

$$\mathbf{R} \ni t \mapsto \Phi_C((x^1, \ldots, x^k)^*\sigma_t) \in \mathbf{R}$$

is smooth.

Proof. Define $\omega \in \Omega^q(\mathbf{R}^m)$ by

$$\omega|_{(x_1,\dots,x_m)} := ((x^1,\dots,x^k)^*\sigma_{x_m})|_{(x_1,\dots,x_m)} \in \wedge^q T^*_{(x_1,\dots,x_m)}\mathbf{R}^m,$$

where $(x_1,\dots,x_m) \in \mathbf{R}^m$. Then

$$(x^1,\dots,x^k,0,\dots,0)^*(\tau_{(0,\dots,0,t)})^*\omega = (x^1,\dots,x^k)^*\sigma_t,$$

where $\tau_{(0,\dots,0,t)} : \mathbf{R}^m \to \mathbf{R}^m$ is the translation by $(0,\dots,0,t) \in \mathbf{R}^m$, $t \in \mathbf{R}$. Then using the previous lemma and the invariance of C with respect to $\tau_{(0,\dots,0,t)}$, we get

$$\begin{aligned}\Phi_C((x^1,\dots,x^k)^*\sigma_t) &= \Phi_C((x^1,\dots,x^k,0,\dots,0)^*(\tau_{(0,\dots,0,t)})^*\omega)\\ &= \Phi_C((\tau_{(0,\dots,0,t)})^*\omega) = C((\tau_{(0,\dots,0,t)})^*\omega)(\kappa^A)\\ &= C(\omega)\circ T^A\tau_{(0,\dots,0,t)}(\kappa^A).\end{aligned}$$

Since $T^A\tau_{(0,\dots,0,t)}(\kappa^A)$ depends smoothly on t, the proof is complete. □

Lemma 2.12. *For any $\omega \in \Omega^q_0(\mathbf{R}^m)$ of the form $\omega = \omega(x^1,\dots,x^k,dx^1,\dots,dx^k)$ and any $\sigma \in \Omega^{q-1}_0(\mathbf{R}^m)$ of the form $\sigma = \sigma(x^1,\dots,x^k,dx^1,\dots,dx^k)$, and any $\eta \in \underline{A}$, we have*

$$\Phi_C(\omega) = \Phi_C(\omega + d\eta \wedge \sigma),$$

where $\eta \circ (x^1,\dots,x^k)$ is (for simplicity) denoted by η.

Proof. Using Lemma 2.10, then Lemma 2.9 with

$$\varphi = (x^1,\dots,x^{m-1},x^m + \eta(x^1,\dots,x^k)),$$

and then Lemma 2.10 again, we get

$$\Phi_C(\omega) = \Phi_C(\omega + dx^m \wedge \sigma) = \Phi_C(\omega + dx^m \wedge \sigma + d\eta \wedge \sigma) = \Phi_C(\omega + d\eta \wedge \sigma).$$

The proof of the lemma is complete. □

Lemma 2.13. *For any $\omega \in \Omega^q_0(\mathbf{R}^m)$ of the form $\omega = \omega(x^1,\dots,x^k,dx^1,\dots,dx^k)$ and any $\omega_1 \in \Omega^q_0(\mathbf{R}^m)$ of the form $\omega_1 = \omega_1(x^1,\dots,x^k,dx^1,\dots,dx^k)$ and any $\eta \in \underline{A}$, we have*

$$\Phi_C(\omega) = \Phi_C(\omega + \eta\cdot\omega_1),$$

where η denotes $\eta \circ (x^1,\dots,x^k)$.

Proof. By the same arguments as in the proof of Lemma 2.12, we have

$$\Phi_C(\omega) = \Phi_C(\omega + x^m\cdot\omega_1) = \Phi_C(\omega + x^m\cdot\omega^1 + \eta\cdot\omega_1) = \Phi_C(\omega + \eta\cdot\omega_1).$$

The proof of the lemma is complete. □

We are now in a position to prove Theorem 2.7.

Proof of Theorem 2.7. Let C be as in Theorem 2.7. We (must) define $h_C : Q^q_A \to \mathbf{R}$ by

$$h_C(w) := \Phi_C(\sigma), \quad w = [\sigma]_{\underline{Q}^q_A}, \quad \sigma \in \Omega^q_0(\mathbf{R}^k) \subset \Omega^q_0(\mathbf{R}^m)$$

(the inclusion is given by the pull-back with respect to the projection $(x^1,\dots,x^k) : \mathbf{R}^m \to \mathbf{R}^k$). By Lemmas 2.12 and 2.13, h_C is well defined. Moreover, h_C is smooth because of Lemma 2.11 and the Boman theorem. By Lemmas 2.8 and 2.10, C is determined by h_C, i.e., if C_1 is another operator in question with $h_{C_1} = h_C$ then $C = C_1$. We can easily see that if $C_1 := (-)^{\langle h_C\rangle}$, then $h_{C_1} = h_C$. Consequently, $C = C_1 = (-)^{\langle h_C\rangle}$. The proof of Theorem 2.7 is complete. □

Corollary 2.14. *Assume additionally $q > k$. If $m \geq k+1$, then any $\mathcal{M}f_m$-natural operator $C : \wedge^q T^* \rightsquigarrow \wedge^0 T^* T^A$ sending q-forms ω on an m-manifold M into maps $C(\omega) : T^A M \to \mathbf{R}$ is a real constant one.*

Proof. If $q > k$, then $\Omega^q_0(\mathbf{R}^k) = (0)$, and hence $Q^q_A = (0)$. Therefore any map $h : Q^q_A \to \mathbf{R}$ is constant, and the corollary follows. □

3. Lifting q-forms on manifolds to p-forms on Weil bundles

Let k, m, q, p, r be positive integers, and let $y^1, \ldots, y^k, u^1, \ldots, u^p$ be the usual coordinates on $\mathbf{R}^{k+p} = \mathbf{R}^k \times \mathbf{R}^p$. Let $A = C^\infty_0(\mathbf{R}^k)/\underline{A}$ be the Weil algebra as in the previous section. Using A we can produce a new Weil algebra B (depending on A) by

$$B := C^\infty_0(\mathbf{R}^{k+p})/\underline{B}, \quad \underline{B} = \langle \underline{A} \circ (y^1, \ldots, y^k),\, u^i u^j \mid i, j = 1, \ldots, p\rangle,$$

where $\underline{B}$ is the ideal in $C^\infty_0(\mathbf{R}^{k+p})$ spanned by the collection consisting of $\eta \circ (y^1, \ldots, y^k)$ for all $\eta \in \underline{A}$ and all $u^i u^j$ for $i, j = 1, \ldots, p$.

Clearly, $B = A \otimes \mathbf{D}^1_p$, where $\mathbf{D}^1_p = \mathbf{D} \oplus_{\mathbf{R}} \cdots \oplus_{\mathbf{R}} \mathbf{D}$ (p times of $\mathbf{D}$), where $\mathbf{D}$ is the algebra of dual numbers. Then (by the theory of Weil functors) given a manifold M we have

$$T^B M = TT^A M \times_{T^A M} \cdots \times_{T^A M} TT^A M \quad (p \text{ times of } TT^A M),$$

where T^B is the Weil functor corresponding to the Weil algebra B.

We are going to describe all (not necessarily regular) $\mathcal{M}f_m$-natural operators $D : \wedge^q T^* \rightsquigarrow \wedge^p T^* T^A$ sending q-forms ω on an m-manifold M into p-forms $D(\omega)$ on $T^A M$.

Clearly, any such $\mathcal{M}f_m$-natural operator $D : \wedge^q T^* \rightsquigarrow \wedge^p T^* T^A$ can be treated (in an obvious way) as an excellent $\mathcal{M}f_m$-natural operator $D : \wedge^q T^* \rightsquigarrow \wedge^0 T^* T^B$ in the sense of Definition 3.1, and vice versa. So, in particular, any such D is regular because of the previous section.

Definition 3.1. An $\mathcal{M}f_m$-natural operator $D : \wedge^q T^* \rightsquigarrow \wedge^0 T^* T^B$ is *excellent* if, for any m-manifold M, any point $u \in T^A M$ and any q-form ω on M, the map $D(\omega)|_{T_u T^A M \times \cdots \times T_u T^A M} : T_u T^A M \times \cdots \times T_u T^A M \to \mathbf{R}$ is skew-symmetric p-linear.

So, we are going to describe all excellent $\mathcal{M}f_m$-natural operators $D : \wedge^q T^* \rightsquigarrow \wedge^0 T^* T^B$, where B is as above. Let D be one such operator. As in the previous section, let

$$Q^q_B := \Omega^q_0(\mathbf{R}^{k+p})/\underline{Q}^q_B, \quad \underline{Q}^q_B = \underline{B} \cdot \Omega^q_0(\mathbf{R}^{k+p}) + d\underline{B} \wedge \Omega^{q-1}_0(\mathbf{R}^{k+p}).$$

By Theorem 2.7, applied with B instead of A and D instead of C, if $m \geq k+p+1$, then $D = (-)^{\langle h_D\rangle}$ for a smooth map $h_D : Q^q_B \to \mathbf{R}$ uniquely determined by D. By the proof of Theorem 2.7,

$$h_D(w) := D(\sigma)(\kappa^B), \quad w = [\sigma]_{\underline{Q}^q_B}, \quad \sigma \in \Omega^q_0(\mathbf{R}^{k+p}) \subset \Omega^q_0(\mathbf{R}^m)$$

(the inclusion is given by the pull-back with respect to the obvious projection $\mathbf{R}^m = \mathbf{R}^{k+p} \times \mathbf{R}^{m-(k+p)} \to \mathbf{R}^{k+p}$), where $\kappa^B := j^B(\iota) \in T_0^B\mathbf{R}^m$, $\iota : \mathbf{R}^{k+p} \to \mathbf{R}^m$, $\iota(y_1, \dots, y_{k+p}) = (y_1, \dots, y_{k+p}, 0, \dots, 0)$, $y_1, \dots, y_{k+p} \in \mathbf{R}$.

Let $\mathbf{R}_+ = (\mathbf{R}_+, \cdot)$ be the multiplicative group of positive real numbers. We have the action of $(\mathbf{R}_+)^p$ on Q_B^q given by

$$t \diamond w := [(a_t)^*\sigma]_{\underline{Q}_B^q},$$

where $t = (t_1, \dots, t_p) \in (\mathbf{R}_+)^p$, $w = [\sigma]_{\underline{Q}_B^q} \in Q_B^q$, $\sigma \in \Omega_0^q(\mathbf{R}^{k+p})$ and $a_t := (y^1, \dots, y^k, t_1u^1, \dots, t_pu^p) : \mathbf{R}^{k+p} \to \mathbf{R}^{k+p}$, where $y^1, \dots, y^k, u^1, \dots, u^p : \mathbf{R}^{k+p} \to \mathbf{R}$ are the usual coordinates on $\mathbf{R}^{k+p}$. The action is well defined because $\underline{Q}_B^q$ is $(a_t)^*$-invariant.

Let S_p be the group of permutations of p elements $\{1, \dots, p\}$. We have the action of S_p on Q_B^q given by

$$s \,\square\, w := [(a_{s^{-1}})^*\sigma]_{\underline{Q}_B^q}, \quad s \in S_p, \quad w = [\sigma]_{\underline{Q}_B^q} \in Q_B^q, \quad \sigma \in \Omega_0^q(\mathbf{R}^{k+p}),$$

where $a_s := (y^1, \dots, y^k, u^{s(1)}, \dots, u^{s(p)})$. The action is well defined because $\underline{Q}_B^q$ is $(a_{s^{-1}})^*$-invariant.

Definition 3.2. A map $h : Q_B^q \to \mathbf{R}$ is *excellent* if

$$h(t \diamond w) = t_1 \cdot \ldots \cdot t_p \cdot h(w) \quad \text{and} \quad h(s \,\square\, w) = \mathrm{sign}(s) \cdot h(w)$$

for any $w \in Q_B^q$, any $t = (t_1, \dots, t_p) \in (\mathbf{R}_+)^p$ and any $s \in S_p$.

Lemma 3.3. *Assume $m \geq k + p + 1$. Let D be as above. Then the map $h_D : Q_B^q \to \mathbf{R}$ is excellent.*

Proof. The first formula from Definition 3.2 (for h_D instead of h) is a consequence of the invariance of D with respect to

$$\tilde{a}_t := (x^1, \dots, x^k, t_1x^{k+1}, \dots, t_px^{k+p}, x^{k+p+1}, \dots, x^m)$$

for $t = (t_1, \dots, t_p) \in (\mathbf{R}_+)^p$ and the p-linearity of $D(\omega)|_{T_{\kappa^A}T^A\mathbf{R}^m \times \cdots \times T_{\kappa^A}T^A\mathbf{R}^m}$ for any q-form ω on $\mathbf{R}^m$. More precisely, we can proceed as follows. Consider $w \in Q_B^q$ and $t = (t_1, \dots, t_p) \in (\mathbf{R}_+)^p$. We can write $\kappa^B = (v_1, \dots, v_p) \in T_{\kappa^A}T^A\mathbf{R}^m \times \cdots \times T_{\kappa^A}T^A\mathbf{R}^m$ and $w = [\sigma]_{\underline{Q}_B^q}$, where $\sigma \in \Omega_0^q(\mathbf{R}^{k+p}) \subset \Omega_0^q(\mathbf{R}^m)$. Then $T^B\tilde{a}_t(\kappa^B) = (t_1v_1, \dots, t_pv_p)$. Hence

$$\begin{aligned} h_D(t \diamond w) &= h_D([(a_t)^*\sigma]_{\underline{Q}_B^q}) = D((\tilde{a}_t)^*\sigma)(\kappa^B) = D(\sigma)(T^B\tilde{a}_t(\kappa^B)) \\ &= D(\sigma)(t_1v_1, \dots, t_pv_p) = t_1 \cdot \ldots \cdot t_p \cdot D(\sigma)(v_1, \dots, v_p) \\ &= t_1 \cdot \ldots \cdot t_p \cdot h_D(w), \end{aligned}$$

i.e., $h_D(t \diamond w) = t_1 \cdot \ldots \cdot t_p \cdot h_D(w)$ as well.

Similarly, the second formula from Definition 3.2 (for h_D instead of h) follows from the invariance of D with respect to

$$\tilde{a}_s := (x^1, \dots, x^k, x^{k+s(1)}, \dots, x^{k+s(p)}, x^{k+p+1}, \dots, x^m)$$

for $s \in S_p$ and the skew-symmetry of $D(\omega)|_{T_{\kappa^A}T^A\mathbf{R}^m \times \cdots \times T_{\kappa^A}T^A\mathbf{R}^m}$ for any q-form ω on $\mathbf{R}^m$. More precisely, we can proceed as follows. Consider $w \in Q_B^q$ and $s \in S_p$.

We can write $\kappa^B = (v_1, \dots, v_p) \in T_{\kappa^A} T^A \mathbf{R}^m \times \dots \times T_{\kappa^A} T^A \mathbf{R}^m$ and $w = [\sigma]_{\underline{Q}^q_B}$, where $\sigma \in \Omega^q_0(\mathbf{R}^{k+p}) \subset \Omega^q_0(\mathbf{R}^m)$. Then $T^B \tilde{a}_s(\kappa^B) = (v_{s(1)}, \dots, v_{s(p)})$. Hence

$$\begin{aligned} h_D(s \,\square\, w) &= h_D([(a_{s^{-1}})^* \sigma]_{\underline{Q}^q_B}) = D((\tilde{a}_{s^{-1}})^* \sigma)(\kappa^B) = D(\sigma)(T^B \tilde{a}_{s^{-1}}(\kappa^B)) \\ &= D(\sigma)(v_{s^{-1}(1)}, \dots, v_{s^{-1}(p)}) = \operatorname{sign}(s) \cdot D(\sigma)(v_1, \dots, v_p) \\ &= \operatorname{sign}(s) \cdot h_D(w), \end{aligned}$$

i.e., $h_D(s \,\square\, w) = \operatorname{sign}(s) \cdot h_D(w)$ as well. ∎

Lemma 3.4. *Assume $m \geq k + p + 1$. Let $h : Q^q_B \to \mathbf{R}$ be excellent. Then $(-)^{\langle h \rangle} : \wedge^q T^* \rightsquigarrow \wedge^0 T^* T^B$ can be treated as a natural operator $\wedge^q T^* \rightsquigarrow \wedge^p T^* T^A$.*

Proof. By the first formula from Definition 3.2, we get that $\omega^{\langle h \rangle}(w_1, \dots, w_p)$ is p-linear in $w_1, \dots, w_p \in T_u T^A M$ for any q-form ω on M and any $u \in T^A M$. More precisely, we can proceed as follows. Since the $\mathcal{M}f_m$-orbit of κ^A in $T^A M$ is dense (as $m \geq k$) and $(-)^{\langle h \rangle}$ is $\mathcal{M}f_m$-invariant, we can assume $M = \mathbf{R}^m$ and $u = \kappa^A \in T^A \mathbf{R}^m$. Let $\kappa^B = j^B \iota = (v_1, \dots, v_p)$. Let $t = (t_1, \dots, t_p) \in (\mathbf{R}_+)^p$ and let a_t and $\tilde{a}_t$ be as in the proof of the previous lemma, and let ω be a q-form on $\mathbf{R}^m$. Then

$$\begin{aligned} \omega^{\langle h \rangle}(t_1 v_1, \dots, t_p v_p) &= \omega^{\langle h \rangle}(T^B \tilde{a}_t(v_1, \dots, v_p)) = ((\tilde{a}_t)^* \omega)^{\langle h \rangle}(v_1, \dots, v_p) \\ &= h([\iota^* (\tilde{a}_t)^* \omega]_{\underline{Q}^q_B}) = h([(a_t)^* \iota^* \omega]_{\underline{Q}^q_B}) = h(t \diamond [\iota^* \omega]_{\underline{Q}^q_B}) \\ &= t_1 \cdot \ldots \cdot t_p \cdot h([\iota^* \omega]_{\underline{Q}^q_B}) = t_1 \cdot \ldots \cdot t_p \cdot \omega^{\langle h \rangle}(v_1, \dots, v_p), \end{aligned}$$

i.e., $\omega^{\langle h \rangle}(t_1 v_1, \dots, t_p v_p) = t_1 \cdot \ldots \cdot t_p \cdot \omega^{\langle h \rangle}(v_1, \dots, v_p)$. Then by the $\mathcal{M}f_m$-invariance of $(-)^{\langle h \rangle}$ and the fact that the $\mathcal{M}f_m$-orbit of κ^B is dense in $(T^B M)_{\kappa^A}$ (as $m \geq k + p$), we get

$$\omega^{\langle h \rangle}(t_1 w_1, \dots, t_p w_p) = t_1 \cdot \ldots \cdot t_p \cdot \omega^{\langle h \rangle}(w_1, \dots, w_p)$$

for any q-form ω on $\mathbf{R}^m$, any $w_1, \dots, w_p \in T_{\kappa^A} T^A \mathbf{R}^m$ and any $t_1 > 0, \dots, t_p > 0$. Now, the homogeneous function theorem implies that $\omega^{\langle h \rangle}(w_1, \dots, w_p)$ is p-linear in $w_1, \dots, w_p \in T_{\kappa^A} T^A \mathbf{R}^m$ for any q-form ω on $\mathbf{R}^m$ as well.

The second equality implies that $\omega^{\langle h \rangle}(w_1, \dots, w_p)$ is skew-symmetric in $w_1, \dots, w_p \in T_u T^A M$ for any q-form ω on M and any $u \in T^A M$. More precisely, we can proceed as follows. We can assume $M = \mathbf{R}^m$ and $u = \kappa^A \in T^A \mathbf{R}^m$. Let $\kappa^B = j^B \iota = (v_1, \dots, v_p)$. Let $s \in S_p$ and let $\tilde{a}_s$ be as in the proof of the previous lemma, and let ω be a q-form on $\mathbf{R}^m$. Then

$$\begin{aligned} \omega^{\langle h \rangle}(v_{s(1)}, \dots, v_{s(p)}) &= \omega^{\langle h \rangle}(T^B \tilde{a}_s(v_1, \dots, v_p)) = ((\tilde{a}_s)^* \omega)^{\langle h \rangle}(v_1, \dots, v_p) \\ &= h([\iota^* (\tilde{a}_s)^* \omega]_{\underline{Q}^q_B}) = h([(a_s)^* \iota^* \omega]_{\underline{Q}^q_B}) = h(s^{-1} \,\square\, [\iota^* \omega]_{\underline{Q}^q_B}) \\ &= \operatorname{sign}(s) \cdot h([\iota^* \omega]_{\underline{Q}^q_B}) = \operatorname{sign}(s) \cdot \omega^{\langle h \rangle}(v_1, \dots, v_p), \end{aligned}$$

i.e., $\omega^{\langle h \rangle}(v_{s(1)}, \dots, v_{s(p)}) = \operatorname{sign}(s) \cdot \omega^{\langle h \rangle}(v_1, \dots, v_p)$. Then by the $\mathcal{M}f_m$-invariance of $(-)^{\langle h \rangle}$ and the fact that the $\mathcal{M}f_m$-orbit of κ^B is dense in $(T^B M)_{\kappa^A}$, we get that

$$\omega^{\langle h \rangle}(w_{s(1)}, \dots, w_{s(p)}) = \operatorname{sign}(s) \cdot \omega^{\langle h \rangle}(w_1, \dots, w_p)$$

for any q-form ω on $\mathbf{R}^m$, any $w_1, \dots, w_p \in T_{\kappa^A} T^A \mathbf{R}^m$ and any $s \in S_p$ as well.

Then $\omega^{\langle h \rangle}$ is a p-form on $T^A M$ for any q-form ω on an m-manifold M. □

Summing up, we have proved

Theorem 3.5. *Assume $m \geq k+p+1$. The correspondence $D \mapsto h_D$ described above between the (not necessarily regular) $\mathcal{M}f_m$-natural operators $D : \wedge^q T^* \rightsquigarrow \wedge^p T^* T^A$ and the excellent maps $h_D : Q^q_B \to \mathbf{R}$ is one-to-one. The inverse correspondence is $h \mapsto (-)^{\langle h \rangle}$. In particular, any (not necessarily regular) $\mathcal{M}f_m$-natural operator $D : \wedge^q T^* \rightsquigarrow \wedge^p T^* T^A$ is regular.*

Corollary 3.6. *Assume additionally $q > k+p$. If $m \geq k+p+1$, then any $\mathcal{M}f_m$-natural operator $D : \wedge^q T^* \rightsquigarrow \wedge^p T^* T^A$ sending q-forms ω on an m-manifold M into p-forms $D(\omega) : T^A M \to \mathbf{R}$ is the 0 one.*

Proof. If $q > k+p$, then $\Omega^q_0(\mathbf{R}^{k+p}) = (0)$, and then $Q^q_B = (0)$. Then any excellent map $Q^q_B \to \mathbf{R}$ is 0. Now, the corollary is clear. □

Definition 3.7. A map $H : Q^q_B \to \mathbf{R}$ is *semi-excellent* if $H(t \diamond w) = t_1 \cdot \ldots \cdot t_p \cdot H(w)$ for any $w \in Q^q_B$ and any $t = (t_1, \dots, t_p) \in (\mathbf{R}_+)^p$.

Quite similarly one can prove (in fact, we have proved) the following

Theorem 3.8. *Assume $m \geq k+p+1$. There exists a one-to-one correspondence between the $\mathcal{M}f_m$-natural operators $\wedge^q T^* \rightsquigarrow \otimes^p T^* T^A$ and the semi-excellent maps $Q^q_B \to \mathbf{R}$.*

Corollary 3.9. *Assume additionally $q > k+p$. If $m \geq k+p+1$, then any $\mathcal{M}f_m$-natural operator $\wedge^q T^* \rightsquigarrow \otimes^p T^* T^A$ is the 0 one.*

Remark 3.10. Theorems 3.5 and 3.8 show that, to describe all $\mathcal{M}f_m$-natural operators $\wedge^q T^* \rightsquigarrow \otimes^p T^* T^A$ (resp., $\wedge^q T^* \rightsquigarrow \wedge^q T^* T^A$), it is sufficient to describe all semi-excellent (resp., excellent) maps $Q^q_B \to \mathbf{R}$. Such a description will be presented in the next section.

4. On semi-excellent maps

We will use the same notation as in the previous sections. In particular, let B and Q^q_B be as above.

We are going to present the full description of all semi-excellent maps $Q^q_B \to \mathbf{R}$.

Let $\mathbf{N}$ be the set of non-negative integers. Given $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbf{N}^p$ and $t = (t_1, \dots, t_p) \in (\mathbf{R}_+)^p$, we denote $t^\alpha := (t_1)^{\alpha_1} \dots (t_p)^{\alpha_p}$.

Definition 4.1. An element $w \in Q^q_B$ is *homogeneous of weight* $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbf{N}^p$ if

$$t \diamond w = t^\alpha \cdot w$$

for any $t = (t_1, \dots, t_p) \in (\mathbf{R}_+)^p$, where $\diamond$ and Q^q_B are as in the previous section.

Definition 4.2. An element $w \in Q^q_B$ is *adapted* if it is homogeneous of weight $\alpha(w)$ satisfying $\alpha(w) \in \{0,1\}^p$.

Definition 4.3. A basis $\mathcal{B} = \{w_1, \dots, w_K\}$ of the real vector space Q^q_B is called *adapted* if $w_1, \dots, w_K$ are adapted.

The description of all semi-excellent maps $Q^q_B \to \mathbf{R}$ is presented in the following

Theorem 4.4. (i) *We can choose a basis $\mathcal{B} = \{w_1, \dots, w_K\}$ in the real vector space Q^q_B and $K_o \le K$ such that $w_1, \dots, w_K$ are adapted and $w_1, \dots, w_{K_o}$ are precisely the elements of $\mathcal{B}$ of weight $(0, \dots, 0)$.*

(ii) *Let $\mathcal{B} = \{w_1, \dots, w_K\}$ and let K_o be as in part (i). Given $w \in \mathcal{B}$, let $\alpha(w)$ be the weight of w. Let $\lambda_1, \dots, \lambda_K$ be the dual basis to $\mathcal{B}$. Then any semi-excellent map $H : Q^q_B \to \mathbf{R}$ is a linear combination of the monomials*

$$(\lambda_{K_o+1})^{\gamma_{K_o+1}} \cdot \ldots \cdot (\lambda_K)^{\gamma_K}$$

for all $\gamma_{K_o+1}, \dots, \gamma_K \in \{0, 1\}$ such that

$$\sum_{j=K_o+1}^{K} \gamma_j \alpha(w_j) = (1, \dots, 1),$$

with uniquely determined coefficients given by smooth maps $Q^q_B \to \mathbf{R}$ depending on $\lambda_1, \dots, \lambda_{K_o}$ (i.e., of the form $a(\lambda_1, \dots, \lambda_{K_o})$ with $a \in C^\infty(\mathbf{R}^{K_o})$). Conversely, any such linear combination is semi-excellent.

Proof. Clearly, by the definition of Q^q_B, we can choose elements $v_1, \dots, v_{\tilde{K}} \in Q^q_B$ such that $Q^q_B = \mathrm{span}_{\mathbf{R}}\{v_1, \dots, v_{\tilde{K}}\}$ and, for $j = 1, \dots, \tilde{K}$, each v_j is of the form

$$\left[y^\alpha u^\beta dy^{\mu_1} \wedge \cdots \wedge dy^{\mu_{\tilde{k}}} \wedge du^{\nu_1} \wedge \cdots \wedge du^{\nu_{\tilde{p}}}\right]_{\underline{Q}^q_B}$$

for some $\alpha \in \mathbf{N}^k$ and $\beta = (\beta_1, \dots, \beta_p) \in \mathbf{N}^p$, and some $\tilde{k}, \tilde{p}, \mu_1, \dots, \mu_{\tilde{k}}, \nu_1, \dots, \nu_{\tilde{p}} \in \mathbf{N}$ satisfying $1 \le \mu_1 < \cdots < \mu_{\tilde{k}} \le k$, $1 \le \nu_1 < \cdots < \nu_{\tilde{p}} \le p$, $\tilde{k} + \tilde{p} = q$, $|\beta| = \beta_1 + \cdots + \beta_p \le 1$, and if $\beta_s = 1$, then $s \in \{1, \dots, p\} \setminus \{\nu_1, \dots, \nu_{\tilde{p}}\}$. From these generators we can choose the basis

$$\mathcal{B} = \{w_1, \dots, w_K\}$$

of the real vector space Q^q_B. Then $\mathcal{B}$ is adapted. Of course, we may assume that $w_1, \dots, w_{K_o}$ are precisely the elements of $\mathcal{B}$ of weight $(0, \dots, 0)$.

The proof of part (i) of the theorem is complete.

Let $H : Q^q_B \to \mathbf{R}$ be a semi-excellent map. Given $w \in \mathcal{B}$, let $\alpha(w)$ be the weight of w. Any element $w \in Q^q_B$ is of the form

$$w = \sum_{j=1}^{K} \lambda_j(w) w_j, \quad \text{where } \lambda_j : Q^q_B \to \mathbf{R} \text{ are the functionals dual to } w_1, \dots, w_K.$$

Then, for any $w = \sum_{j=1}^K \lambda_j(w) w_j \in Q^q_B$ and $t = (t_1, \dots, t_p) \in (\mathbf{R}_+)^p$, we have

$$t_1 \cdot \ldots \cdot t_p \cdot H\left(\sum_{j=1}^{K} \lambda_j(w) w_j\right) = H\left(t \diamond \sum_{j=1}^{K} \lambda_j(w) w_j\right) = H\left(\sum_{j=1}^{K} \lambda_j(w) t^{\alpha(w_j)} w_j\right).$$

Then, by the homogeneous function theorem [5], H is a linear combination of the monomials

$$(\lambda_1)^{\gamma_{K_o+1}} \cdot \ldots \cdot (\lambda_K)^{\gamma_K}$$

for all $\gamma_{K_o+1}, \ldots, \gamma_K \in \mathbf{N}$ such that

$$\sum_{j=K_o+1}^{K} \gamma_j \alpha(w_j) = (1, \ldots, 1),$$

with coefficients given by smooth maps depending on $\lambda_1, \ldots, \lambda_{K_o}$. Then $\gamma_{K_o+1}, \ldots, \gamma_K \in \{0, 1\}$ because $\alpha(w_j) \neq (0, \ldots, 0)$ for $j = K_o + 1, \ldots, K$.

The proof of part (ii) of the theorem is complete. □

Remark 4.5. The condition $\sum_{j=K_o+1}^{K} \gamma_j \alpha(w_j) = (1, \ldots, 1)$ from part (ii) of the theorem is a system of p linear equations with coefficients from $\{0, 1\}$ with $K - K_o$ unknowns γ_j from $\{0, 1\}$. So, we have reduced the description of regular $\mathcal{M}f_m$-natural operators $\wedge^q T^* \rightsquigarrow \otimes^p T^* T^A$ to a rather combinatoric problem.

5. Applications

To start, applying the general results obtained in the previous sections, if $m \geq q + 2 \geq 4$, we explicitly determine all $\mathcal{M}f_m$-natural operators $D : \wedge^q T^* \rightsquigarrow \wedge^q T^* T$ lifting q-forms ω on an m-manifold M to q-forms $D(\omega)$ on the tangent bundle TM. We prove

Proposition 5.1. *If $m \geq q + 2 \geq 4$, then any $\mathcal{M}f_m$-natural operator $D : \wedge^q T^* \rightsquigarrow \wedge^q T^* T$ is of the form*

$$\omega \mapsto a\omega^C + b\omega^V + cdi_L\omega^C$$

for real numbers a, b, c uniquely determined by D, where ω^C is the complete lift of ω from M to TM, ω^V is the vertical lift of ω from M to TM, i_L is the inner derivative with respect to the Liouville vector field L on TM, and d is the exterior derivative.

Proof. We prove this proposition for $m \geq q + 2 \geq 5$. We are going to apply the general results obtained above in the case $k = 1$, $p = q$, $A = C^\infty(\mathbf{R})/((u^0)^2)$ and $\underline{B} = \langle (u^0)^2, u^i u^j \mid i, j = 1, \ldots, q \rangle$, where $u^0 := y^1$. The collection consisting of $q+3$ classes

$$v_i^{(1)} := \left[du^0 \wedge du^1 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^q\right]_{\underline{Q}_B^q},$$

$$v^{(3)} := \left[u^1 du^0 \wedge du^2 \wedge \cdots \wedge du^q\right]_{\underline{Q}_B^q}$$

$$v^{(1)} := \left[du^1 \wedge \cdots \wedge du^q\right]_{\underline{Q}_B^q},$$

$$v^{(2)} := \left[u^o du^1 \wedge \cdots \wedge du^q\right]_{\underline{Q}_B^q}$$

for $i = 1, \ldots, q$ generates (over $\mathbf{R}$) the real vector space Q_B^q, where, of course, $\widehat{du^i}$ means that du^i is dropped. This will be proved in a more general situation in Lemma 5.4. So, we have the basis $w_1, \ldots, w_{K_1}, \ldots, w_K$ of Q_B^q such that $w_{K_1+1},\dots,w_K \in \{v^{(3)}, v^{(1)}, v^{(2)}\}$ and $w_1,\dots,w_{K_1} \in \{v_i^{(1)} \mid i=1,\dots,q\}$. We can see that this basis is adapted. Namely, $v_i^{(1)}$ is homogeneous of weight $(1,\dots,1,0,1,\dots,1) \in \{0,1\}^q$ (0 in ith position) for $i=1,\dots,q$, and the other elements are of weight $(1,\dots,1,\dots,1) \in \{0,1\}^q$. Let $\lambda_1,\dots,\lambda_K$ be the dual basis. Using Theorem 4.4, we can immediately see that any semi-excellent map $H: Q_B^q \to \mathbf{R}$ is of the form

$$H = a_{K_1+1}\lambda_{K_1+1} + \cdots + a_K\lambda_K,$$

where the coefficients are real numbers. Thus the vector space of all excellent maps $H: Q_B^q \to \mathbf{R}$ has dimension ≤ 3. Therefore the vector space of all $\mathcal{M}f_m$-natural operators $\wedge^q T^* \rightsquigarrow \wedge^q T^* T$ has dimension ≤ 3 because of Theorem 3.5. On the other hand, we have three linearly independent natural operators in question. Namely, ω^C and ω^V and $di_L\omega^C$. So, using the dimension argument we end the proof of the proposition in this case. That the operators ω^C and ω^V and $di_L\omega^C$ are linearly independent will be proved later in Lemma 5.6 (in a more general situation).

So, using the dimension argument we end the proof of the proposition.

In the case $m \geq q+2 = 4$, the proof is quite similar. □

Remark 5.2. In the description of $\mathcal{M}f_m$-natural operators $\wedge^q T^* \rightsquigarrow \wedge^q T^* T$ given in Proposition 5.1, we cannot see the operator

$$\omega \mapsto i_L d\omega^C.$$

Why? Because of $\omega^C = \mathcal{L}_L\omega^C = i_L d\omega^C + di_L\omega^C$.

In the rest of this section, we generalize Proposition 5.1. Namely, if $m \geq q+2 \geq 4$, we explicitly determine all $\mathcal{M}f_m$-natural operators $D: \wedge^q T^* \rightsquigarrow \wedge^q T^* T^r$ lifting q-forms ω on an m-manifold M to q-forms $D(\omega)$ on the r-tangent bundle $T^r M = J_0^r(\mathbf{R}, M)$.

At first, we consider the case $m \geq q+2 \geq 5$. We prove

Theorem 5.3. *If $m \geq q+2 \geq 5$, then any $\mathcal{M}f_m$-natural operator $D: \wedge^q T^* \rightsquigarrow \wedge^q T^* T^r$ is of the form*

$$\omega \mapsto \sum_{s=0}^{r} a_s \omega^{(s)} + \sum_{\sigma=1}^{r} b_\sigma di_L\omega^{(\sigma)}$$

for real numbers a_s, b_σ uniquely determined by D, where $\omega^{(s)}$ for $s=0,\dots,r$ are the classical (s)-lifts of ω to $T^r M$ in the sense of A. Morimoto [7]*, and L is the canonical vector field on $T^r M$ with the flow given by $\mathrm{Exp}(\tau L)(v) := j_0^r(\gamma \circ a_\tau)$, $\tau \in \mathbf{R}$, $v = j_0^r\gamma \in T^r M$, where $a_\tau: \mathbf{R} \to \mathbf{R}$, $a_\tau(t) = \exp(\tau)t$.*

To prove Theorem 5.3, we need a preparation. Now, $k=1$, $p=q$, $A = C^\infty(\mathbf{R})/((u^0)^{r+1})$ and $\underline{B} = \langle (u^0)^{r+1}, u^i u^j \mid i,j=1,\dots,q\rangle$, where $u^0 := y^1$. We prove some lemmas.

Lemma 5.4. *The collection consisting of classes*

$$v_i^{(1,s)} := \big[d(u^0)^s \wedge du^1 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^q\big]_{\underline{Q}^q_B},$$
$$v^{(3,s)} := \big[u^1 d(u^0)^s \wedge du^2 \wedge \cdots \wedge du^q\big]_{\underline{Q}^q_B},$$
$$v^{(1)} := \big[du^1 \wedge \cdots \wedge du^q\big]_{\underline{Q}^q_B},$$
$$v^{(2,s)} := \big[(u^o)^s du^1 \wedge \cdots \wedge du^q\big]_{\underline{Q}^q_B}$$

for $i = 1, \ldots, q$ and $s = 1, \ldots, r$ generates (over $\mathbf{R}$) the vector space Q^q_B. All elements of this collection are adapted. Namely, $v_i^{(1,s)}$ is homogeneous of weight $(1, \ldots, 1, 0, 1, \ldots, 1) \in \{0,1\}^q$ (0 in ith position) for $i = 1, \ldots, q$ and $s = 1, \ldots, r$, and the other elements are of weight $(1, \ldots, 1, \ldots, 1) \in \{0,1\}^q$.

Proof. Clearly, the classes $[u^\beta du^0 \wedge \cdots \wedge \widehat{du^j} \wedge \cdots \wedge du^q]_{\underline{Q}^q_B}$ for all $\beta \in \mathbf{N}^{q+1}$ and $j = 0, \ldots, q$ generate the vector space Q^q_B. Because of the construction of Q^q_B, all these classes are 0 except (eventually)

$$\big[(u^0)^k du^1 \wedge \cdots \wedge du^q\big]_{\underline{Q}^q_B},$$
$$\big[(u^0)^{k_1} du^0 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^q\big]_{\underline{Q}^q_B}$$

and

$$\big[(u^0)^{k_1} u^i du^0 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^q\big]_{\underline{Q}^q_B}$$

for $k = 0, \ldots, r$, $k_1 = 0, \ldots, r-1$, and $i = 1, \ldots, q$. Moreover,

$$\big[(u^0)^{k_1} du^0 \wedge d(u^1 u^i) \wedge du^2 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^q\big]_{\underline{Q}^q_B} = 0,$$

so

$$\big[(u^0)^{k_1} u^1 du^0 \wedge du^2 \wedge \cdots \wedge du^q\big]_{\underline{Q}^q_B} = \epsilon\big[(u^0)^{k_1} u^i du^0 \wedge du^1 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^q\big]_{\underline{Q}^q_B}$$

for some $\epsilon \in \{-1, 1\}$. Consequently, the classes $v_i^{(1,s)}$, $v^{(3,s)}$, $v^{(1)}$, $v^{(2,s)}$ for $i = 1, \ldots, q$ and $s = 1, \ldots, r$ generate (over $\mathbf{R}$) the vector space Q^q_B.

The rest of this lemma is a simple observation. □

Lemma 5.5. *Let $w_1, \ldots, w_K$ be the basis in Q^q_B such that $w_1, \ldots, w_{K_1} \in \{v_i^{(1,s)} \mid s = 1, \ldots, r;\ i = 1, \ldots, q\}$ and $w_{K_1+1}, \ldots, w_K \in \{v^{(3,s)}, v^{(2,s)}, v^{(1)} \mid s = 1, \ldots, r\}$. It is adapted. Let $\lambda_1, \ldots, \lambda_K$ be the dual basis. Then any semi-excellent map $Q^q_B \to \mathbf{R}$ is*

$$a_{K_1+1}\lambda_1 + \cdots + a_K \lambda_K$$

for any $a_{K_1+1}, \ldots, a_K \in \mathbf{R}$. Thus the vector space of excellent maps $Q^q_B \to \mathbf{R}$ has dimension $\leq 2r + 1$.

Proof. It follows immediately from Theorem 4.4 and the previous lemma. □

Lemma 5.6. *If $m \geq q + 2 \geq 4$, the collection of $\mathcal{M}f_m$-natural operators $\omega^{(s)}$ and $di_L\omega^{(\sigma)}$ for $s = 0, 1, \ldots, r$ and $\sigma = 1, \ldots, r$ is linearly independent.*

Proof. Suppose we have

$$\eta = \sum_{s=0}^{r} a_s \omega^{(s)} + \sum_{\sigma=1}^{r} b_\sigma d i_L \omega^{(\sigma)} = 0$$

for any $\omega \in \Omega^q(M)$. Then

$$\sum_{s=0}^{r} a_s (d\omega)^{(s)} = \sum_{s=0}^{r} a_s d(\omega^{(s)}) = d\eta = 0$$

for any $\omega \in \Omega^q(\mathbf{R}^m)$ because $d(\omega^{(s)}) = (d\omega)^{(s)}$. Putting $\omega = (x^1)^\lambda x^2 dx^3 \wedge \cdots \wedge dx^{q+2}$, we get

$$0 = \sum_{s=0}^{r} a_s (d\omega)^{(s)} \big(\partial_2{}^{(r)}, \dots, \partial_{q+2}^{(r)}\big) = \sum_{s=0}^{r} a_s (d\omega(\partial_2, \dots, \partial_{q+2}))^{(s)} = \sum_{s=0}^{r} a_s ((x^1)^\lambda)^{(s)}$$

for $\lambda = 0, 1, \dots, r$, where $\partial_i = \frac{\partial}{\partial x^i}$ for $i = 1, \dots, m$. Evaluating at $j_0^r(t, 0 \dots, 0) \in T^r\mathbf{R}^m$, we get

$$0 = \sum_{s=0}^{r} a_s \frac{1}{s!} \frac{d^s t^\lambda}{dt^s}\bigg|_{t=0} = a_\lambda$$

for $\lambda = 0, 1, \dots, r$. Then $a_0 = a_1 = \cdots = a_r = 0$, and then

$$\sum_{\sigma=1}^{r} b_\sigma d i_L \omega^{(\sigma)} = 0$$

for any $\omega \in \Omega^q(\mathbf{R}^m)$. Since the flow of L is a natural transformation $T^r \to T^r$ and the flow of $X^{(r)}$ is $\{T^r(\varphi_\tau)\}$, where $\{\varphi_\tau\}$ is the flow of X, then $\mathcal{L}_L X^{(r)} = 0$ for any vector field X on $\mathbf{R}^m$, where $\mathcal{L}_L$ is the Lie derivative with respect to L. Then

$$(\mathcal{L}_L \omega^{(\sigma)})(X_1^{(r)}, \dots, X_q^{(r)}) = L\omega^{(\sigma)}(X_1^{(r)}, \dots, X_q^{(r)}) = L\omega(X_1 \dots, X_q)^{(\sigma)},$$

and then

$$(d i_L \omega^{(\sigma)})(X_1^{(r)}, \dots, X_q^{(r)}) = L\omega(X_1 \dots, X_q)^{(\sigma)}$$

for any closed q-form ω on $\mathbf{R}^m$ and any vector fields $X_1, \dots, X_q$ on $\mathbf{R}^m$ (because $i_L d\omega^{(\sigma)} = 0$ if $d\omega = 0$) for $\sigma = 1, \dots, r$. Then, putting $\omega = (x^1)^\lambda dx^1 \wedge \cdots \wedge dx^q$ (it is closed) and $X_j = \partial_j$ for $j = 1, \dots, q$ and $\lambda = 0, \dots, r$, we get

$$\sum_{\sigma=1}^{r} b_\sigma L((x^1)^\lambda)^{(\sigma)} = 0$$

for $\lambda = 1, \dots, r$. Evaluating at $j_0^r(t, 0, \dots, 0) \in T^r M$, we get

$$0 = \sum_{\sigma=1}^{r} b_\sigma \frac{1}{\sigma!} \frac{d}{d\tau}\bigg|_{\tau=0} \frac{d^\sigma \exp(\tau\lambda) t^\lambda}{dt^\sigma}\bigg|_{t=0} = \lambda b_\lambda$$

for any $\lambda = 1, \dots, r$. Then $b_1 = \cdots = b_r = 0$. That is why the operators $\omega^{(s)}$ and $d i_L \omega^{(\sigma)}$ for $s = 0, \dots, r$ and $\sigma = 1, \dots, r$ are linearly independent as well.

The proof of the lemma is complete. □

We are now in a position to prove Theorem 5.3.

Proof of Theorem 5.3. By Lemma 5.5, the vector space of excellent maps $Q^q_B \to \mathbf{R}$ has dimension $\leq 2r+1$. Then, by Theorem 3.5, the vector space of all $\mathcal{M}f_m$-natural operators $\wedge^q T^* \rightsquigarrow \wedge^q T^* T^r$ has dimension $\leq 2r+1$ as well. Then using Lemma 5.6 and the dimension argument we complete the proof of the theorem. □

Now, we pass to the case $m \geq q+2 = 4$. We prove

Lemma 5.7. *If $m \geq q+2=4$, then the vector space of $\mathcal{M}f_m$-natural operators $\wedge^2 T^* \rightsquigarrow \wedge^2 T^* T^r$ has dimension $\leq 2r+1+\frac{r(r-1)}{2}$.*

Proof. Clearly, Lemma 5.4 is true also for $m \geq q+2=4$. Then we have the following version of Lemma 5.5:

Let $w_1, \dots, w_K$ be the basis in Q^q_B with $w_1, \dots, w_{K_1} \in \{v_1^{(1,s)} \mid s = 1, \dots, r\}$, $w_{K+1}, \dots, w_{K_2} \in \{v_2^{(1,s)} \mid s=1,\dots,r\}$, and $w_{K_2+1}, \dots, w_K \in \{v^{(3,s)}, v^{(2,s)}, v^{(1)} \mid s = 1, \dots, r\}$. It is adapted. Let $\lambda_1, \dots, \lambda_K$ be the dual basis. Then, by Theorem 4.4, any semi-excellent map $Q^q_B \to \mathbf{R}$ is of the form

$$H = \sum_{k_1=1}^{K_1} \sum_{k_2=K_1+1}^{K_2} a_{k_1k_2}\lambda_{k_1}\lambda_{k_2} + a_{K_2+1}\lambda_{K_2+1} + \dots + a_K\lambda_K$$

for real coefficients uniquely determined by H.

Then, because of the above version of Lemma 5.5, there is a linear monomorphism sending any such H into the collection of the sequence $(a_{K_1+1}, \dots, a_K)$ (corresponding to H) and the $(r\times r)$-matrix $[b_{s_1s_2}]$ such that $b_{s_1s_2} := H(v_1^{(1,s_1)} + v_2^{(1,s_2)})$ for $s_1, s_2 = 1, \dots, r$. If H is excellent, then

$$\begin{aligned} -b_{s_1s_2} &= -H(v_1^{(1,s_1)} + v_2^{(1,s_2)}) = H(s \,\square\, (v_1^{(1,s_1)} + v_2^{(1,s_2)})) \\ &= H(v_1^{(1,s_2)} + v_2^{(1,s_1)}) = b_{s_2s_1} \end{aligned}$$

for any $s_1, s_2 = 1, \dots, r$, where $s = (1,2) \in S_2$ is the cycle. Now, the lemma is clear because of Theorem 3.5. □

Theorem 5.8. *If $m \geq q+2=4$, then any $\mathcal{M}f_m$-natural operator $D : \wedge^2 T^* \rightsquigarrow \wedge^2 T^* T^r$ is of the form*

$$\omega \mapsto \sum_{1\leq\sigma_1<\sigma_2\leq r} a_{\sigma_1\sigma_2} i_L\omega^{(\sigma_1)} \wedge i_L\omega^{(\sigma_2)} + \sum_{s=0}^{r} a_s\omega^{(s)} + \sum_{\sigma=1}^{r} b_\sigma d i_L \omega^{(\sigma)}$$

for real numbers $a_{\sigma_1\sigma_2}, a_s, b_\sigma$ uniquely determined by D.

Proof. Suppose we have

$$\sum_{1\leq\sigma_1<\sigma_2\leq r} a_{\sigma_1\sigma_2} i_L\omega^{(\sigma_1)} \wedge i_L\omega^{(\sigma_2)} + \sum_{s=0}^{r} a_s\omega^{(s)} + \sum_{\sigma=1}^{r} b_\sigma d i_L \omega^{(\sigma)} = 0$$

for any q-form ω on $\mathbf{R}^m$. Putting $t\omega$ instead of ω, we get

$$t^2 \sum_{1\leq\sigma_1<\sigma_2\leq r} a_{\sigma_1\sigma_2} i_L\omega^{(\sigma_1)} \wedge i_L\omega^{(\sigma_2)} + t\Big(\sum_{s=0}^{r} a_s\omega^{(s)} + \sum_{\sigma=1}^{r} b_\sigma d i_L \omega^{(\sigma)}\Big) = 0$$

for any $t \in \mathbf{R}$. Then

$$\sum_{1\leq\sigma_1<\sigma_2\leq r} a_{\sigma_1\sigma_2} i_L\omega^{(\sigma_1)} \wedge i_L\omega^{(\sigma_2)} = 0 \quad \text{and} \quad \sum_{s=0}^{r} a_s\omega^{(s)} + \sum_{\sigma=1}^{r} b_\sigma d i_L\omega^{(\sigma)} = 0$$

for any q-form ω on $\mathbf{R}^m$. Then $a_0 = \cdots = a_r = 0$ and $b_1 = \cdots = b_r = 0$ (because of Lemma 5.6), and

$$\sum_{1\leq\sigma_1<\sigma_2\leq r} a_{\sigma_1\sigma_2} i_L\omega^{(\sigma_1)} \wedge i_L\omega^{(\sigma_2)} = 0$$

for any q-form ω on $\mathbf{R}^m$.

Put $\omega = dx^1 \wedge dx^2$. We can see that

$$L = \sum_{i=1}^{m}\sum_{\mu=1}^{r} \mu (x^i)^{(\mu)} \partial_i^{(r-\mu)},$$

because

$$(L((x^i)^{(\mu)}))\big|_{j_0^r\gamma} = \frac{1}{\mu!}\frac{d}{d\tau}\bigg|_{\tau=0}\frac{d^\mu}{dt^\mu}\bigg|_{t=0}(x^i(\gamma(\exp(\tau)t))) = \mu(x^i)^{(\mu)}\big|_{j_0^r\gamma}.$$

Then

$$\begin{aligned}(i_L\omega^{(\sigma)})(\partial_1^{(r)}) &= \sum_{i=1}^{m}\sum_{\mu=1}^{r} \mu(x^i)^{(\mu)}\omega^{(\sigma)}(\partial_i^{(r-\mu)}, \partial_1^{(r)}) \\ &= \sum_{i=1}^{m}\sum_{\mu=1}^{r} \mu(x^i)^{(\mu)}\omega(\partial_i, \partial_1)^{(\sigma-\mu)} = -\sigma(x^2)^{(\sigma)}.\end{aligned}$$

Similarly,

$$(i_L\omega^{(\sigma)})(\partial_2^{(r)}) = \sigma(x^1)^{(\sigma)}.$$

Then

$$(i_L\omega^{(\sigma_1)} \wedge i_L\omega^{(\sigma_2)})(\partial_1^{(r)}, \partial_2^{(r)}) = \sigma_1\sigma_2(-(x_2)^{(\sigma_1)}(x^1)^{(\sigma_2)} + (x^1)^{(\sigma_1)}(x^2)^{(\sigma_2)}).$$

Hence

$$\sum_{1\leq\sigma_1<\sigma_2\leq r} a_{\sigma_1\sigma_2}\sigma_1\sigma_2(-(x_2)^{(\sigma_1)}(x^1)^{(\sigma_2)} + (x^1)^{(\sigma_1)}(x^2)^{(\sigma_2)}) = 0.$$

Evaluating at $j_0^r(t^{s_1}, t^{s_2}, 0, \ldots, 0) \in T^r\mathbf{R}^m$, we get $a_{s_1s_2} = 0$ for $1 \leq s_1 < s_2 \leq r$. Then the dimension argument ends the proof of our theorem. □

Remark 5.9. Let $m \geq q + 2 = 3$. One can verify that

$$v_1^{(1,1)}, \ldots, v_1^{(1,r)}, v^{(1)}, v^{(3,1)}, \ldots, v^{(3,r)}, v^{(2,1)}, \ldots, v^{(2,r)}$$

(the collection of Lemma 5.4) is the (adapted) basis of Q_B^1. Let

$$\lambda_1^{(1,1)}, \ldots, \lambda_1^{(1,r)}, \lambda^{(1)}, \lambda^{(3,1)}, \ldots, \lambda^{(3,r)}, \lambda^{(2,1)}, \ldots, \lambda^{(2,r)}$$

be the dual basis. Then, by Theorem 4.4, any semi-excellent (and hence excellent) map $H : Q^1_B \to \mathbf{R}$ is of the form

$$a_1(\lambda_1^{(1,1)}, \dots, \lambda_1^{(1,r)})\lambda^{(1)} + \sum_{s=1}^{r} a_{2,s}(\lambda_1^{(1,1)}, \dots, \lambda_1^{(1,r)})\lambda^{(2,s)} + \sum_{s=1}^{r} a_{3,s}(\lambda_1^{(1,1)}, \dots, \lambda_1^{(1,r)})\lambda^{(3,s)},$$

where $a_1, a_{2,s}, a_{3,s} : \mathbf{R}^r \to \mathbf{R}$ are smooth maps. Thus, using Theorem 3.5, we reobtain the result of [6], which says that if $m \geq 3$, then the space of all $\mathcal{M}f_m$-natural operators $T^* \rightsquigarrow T^*T^r$ is a free $(2r+1)$-dimensional $C^\infty(\mathbf{R}^r)$-module. It is very likely that any $\mathcal{M}f_m$-natural operator $D : T^* \rightsquigarrow T^*T^r$ is of the form

$$\omega \mapsto \sum_{\nu=0}^{r} h_\nu(i_L\omega^{(1)}, \dots, i_L\omega^{(r)})\omega^{(\nu)} + \sum_{\sigma=1}^{r} g_\sigma(i_L\omega^{(1)}, \dots, i_L\omega^{(r)}) di_L\omega^{(\sigma)}$$

for smooth maps $h_\nu, g_\sigma : \mathbf{R}^r \to \mathbf{R}$ uniquely determined by D, where L is the canonical vector field on T^r mentioned above. We leave this as an open hypothesis.

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